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The infinite base change lifting associated to an APF extension of a p-adic field

By

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Abstract

In this paper, we construct a base change lifting for an APF extension of a mixed characteristic local field. We do this by combining Arthur-Clozel's base change lifting with an operation coming from Kazhdan's theory of close local fields and Fontaine-Wintenberger's theory of fields of norms. Key facts are: (1) a compatibility of Deligne's and Kazhdan's theories of close local fields via the local Langlands correspondence, and (2) a coincidence of the restriction of the Galois groups with respect to a totally ramified extension and an operation coming from Deligne's theory.

Contents

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§ 1. Introduction

Let p be a prime number. In this paper, we shall construct a local base change lifting for an almost pro-p cyclic extension of infinite degree. The point is that the local base change lifting for a totally ramified extension coincides with an operation coming from the close local fields theory of Kazhdan under some conditions.

We state the result more precisely. In this article, the term "local field" means a complete discrete valuation field with finite residue field of characteristic p. For a local field L, we denote by $\mathscr{A}(\mathrm{GL}_N(L))$ the set of isomorphism classes of irreducible smooth representations of $\mathrm{GL}_N(L)$ over $\mathbb{C}$. We fix a separable closure $\overline{L}$ of L and denote the Weil group of L by W_L. We recall that an L-parameter of $\mathrm{GL}_N(L)$ is a group homomorphism $\phi\colon W_L \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C})$ such that $\phi|_{W_L}$ is semi-simple and smooth and $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ is algebraic. Let $\Phi(\mathrm{GL}_N(L))$ denote the set of isomorphism classes of L-parameters of $\mathrm{GL}_N(L)$. We note that $\Phi(\mathrm{GL}_1(L))$ is equal to the set $\mathrm{Hom}(L^\times, \mathbb{C}^\times)$ of smooth characters of $L^\times$. We denote by LLC_L the local Langlands correspondence (LLC) of GL_N over L, whose existence was firstly proven by [9] for L of positive characteristic and by [6] for L of characteristic zero. Let F be a finite extension of $\mathbb{Q}_p$. We fix an algebraic closure $\overline{F}$ of F. Let E be a subfield of $\overline{F}$ which is an *APF extension* of F, that is, for any $u \geq -1$, $\mathrm{Gal}(\overline{F}/E)\,\mathrm{Gal}(\overline{F}/F)^u$ is an open subgroup of $\mathrm{Gal}(\overline{F}/F)$, where $\mathrm{Gal}(\overline{F}/F)^u$ denotes the u-th ramification group in upper numbering. In particular E is an almost pro-p extension over F. Let F_∞ be the field of norms associated to E/F. We denote by Res_∞ the restriction map $\Phi(\mathrm{GL}_N(F)) \to \Phi(\mathrm{GL}_N(F_\infty))$ with respect to the natural injection $W_{F_\infty} \hookrightarrow W_F$.

Theorem 1.1. *Suppose that the extension E/F is procyclic. Then we can construct a map $\mathrm{BC}_\infty\colon \mathscr{A}(\mathrm{GL}_N(F)) \to \mathscr{A}(\mathrm{GL}_N(F_\infty))$ such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathscr{A}(\mathrm{GL}_N(F_\infty)) & \xrightarrow{\mathrm{LLC}_{F_\infty}} & \Phi(\mathrm{GL}_N(F_\infty)) \\ \uparrow{\scriptstyle \mathrm{BC}_\infty} & & \uparrow{\scriptstyle \mathrm{Res}_\infty} \\ \mathscr{A}(\mathrm{GL}_N(F)) & \xrightarrow{\mathrm{LLC}_F} & \Phi(\mathrm{GL}_N(F)). \end{array}$$

We shall call BC_∞ *the base change lifting of infinite degree.* We construct BC_∞ by using Arthur and Clozel's result [1] and close fields theory of Kazhdan [8]. Hence our construction is basically on the representation theory of p-adic groups, that is to say, the automorphic side. However, we use LLC when we prove compatibility of Arthur-Clozel's base change lifting and a bijection given by Kazhdan's theory by showing the corresponding statement in terms of L-parameters, that is, the Galois side. The author expects that in the future we will be able to avoid such arguments.

Furthermore, we study the structure of the fibers of BC_∞. Now we recall the Langlands sum following the exposition of [6, Chapter 1]. We take a partition $(N_1, \ldots, N_r)$ of N. Let $\pi_i \in \mathscr{A}(\mathrm{GL}_{N_i}(F))$ be an essentially square-integrable representation for each $1 \leq i \leq r$. Let s_i be the real number such that $|\cdot|^{s_i}$ is the absolute value of the central character of π_i. We reorder $\pi_1, \ldots, \pi_r$ so that $N_1^{-1}s_1 \geq \cdots \geq N_r^{-1}s_r$. We denote by $P(N_1, \ldots, N_r)$ the standard parabolic subgroup of $\mathrm{GL}_N(F)$ whose Levi component is $\mathrm{GL}_{N_1}(F) \times \cdots \times \mathrm{GL}_{N_r}(F)$. Then the normalized induction

$$\text{n-Ind}_{P(N_1,\ldots,N_r)}^{\mathrm{GL}_N(F)}(\pi_1 \boxtimes \cdots \boxtimes \pi_r)$$

has a unique irreducible quotient, which we denote by $\pi_1 \boxplus \cdots \boxplus \pi_r$ and call the Langlands sum of $\pi_1, \ldots, \pi_r$. Each $\pi \in \mathscr{A}(\mathrm{GL}_N(F))$ can be written as a Langlands sum and the $\pi_1, \ldots, \pi_r$ are uniquely determined up to a permutation.

We put $\Gamma = \mathrm{Gal}(E/F)$ and denote by $\widehat{\Gamma}$ the group of smooth characters of Γ with valued in $\mathbb{C}^\times$. By local class field theory, we identify an element of $\widehat{\Gamma}$ with a character $F^\times \to \mathbb{C}^\times$ which factors through $F^\times/N_{M/F}(M^\times)$ for some finite extension M/F contained in E. For a positive integer μ and $(\eta_1, \ldots, \eta_\mu) \in \widehat{\Gamma}^\mu$, we denote by $\widehat{\Gamma}(\eta_1, \ldots, \eta_r)$ the quotient of $\widehat{\Gamma}^\mu$ by the following equivalence relation: Two elements $(\xi_1, \ldots, \xi_\mu)$ and $(\theta_1, \ldots, \theta_\mu)$ in $\widehat{\Gamma}^\mu$ are equivalent if there exists a permutation σ of $\{1, \ldots, \mu\}$ such that $\eta_j\xi_j = \eta_{\sigma(j)}\theta_{\sigma(j)}$ for each j.

Theorem 1.2. *Let the notations and assumptions be as in Theorem 1.1. We suppose that* $(p, N) = 1$.

(i) *Let* $\pi \in \mathscr{A}(\mathrm{GL}_N(F))$ *be an essentially square-integrable representation. We put* $\pi_\infty = \mathrm{BC}_\infty(\pi)$. *Let* ω_∞ *denote the central character of* π_∞. *Then* $\mathrm{BC}_\infty^{-1}(\pi_\infty)$ *has a natural* $\widehat{\Gamma}$*-torsor structure and the map*

$$\omega\colon \mathrm{BC}_\infty^{-1}(\pi_\infty) \to \mathrm{BC}_\infty^{-1}(\omega_\infty)$$

which maps π' *to its central character* $\omega_{\pi'}$ *is bijective.*

(ii) (a) *Let* π *be any element of* $\mathscr{A}(\mathrm{GL}_N(F))$. *We suppose that* $p > N$. *There exist a positive integer* r, *positive integers* N_i, μ_i *and an essentially square-integrable representation* $\pi_i \in \mathscr{A}(\mathrm{GL}_{N_i}(F))$ *for each* $i = 1, 2, \ldots, r$, *and an element* $\eta_{i,j} \in \widehat{\Gamma}$ *for each* $1 \leq i \leq r$ *and* $2 \leq j \leq \mu_i$ *satisfying the following conditions:*

- *the equality* $\mu_1 N_1 + \cdots + \mu_r N_r = N$ *holds,*
- *the lifts* $\mathrm{BC}_\infty(\pi_1), \ldots, \mathrm{BC}_\infty(\pi_r)$ *are all distinct, and*
- *we can write*

$$\begin{aligned}\pi = &\pi_1 \boxplus (\pi_1 \otimes \eta_{1,2}) \boxplus \cdots \boxplus (\pi_1 \otimes \eta_{1,\mu_1}) \\ &\boxplus \cdots \\ &\boxplus \pi_r \boxplus (\pi_r \otimes \eta_{r,2}) \boxplus \cdots \boxplus (\pi_r \otimes \eta_{r,\mu_r}).\end{aligned}$$

(b) *Under the notation of* (a), *the group* $\widehat{\Gamma}(\pi) = \widehat{\Gamma}^{\mu_1} \times \cdots \times \widehat{\Gamma}^{\mu_r}$ *transitively acts on* $\mathrm{BC}_\infty^{-1}(\pi_\infty)$. *As a homogeneous space of* $\widehat{\Gamma}(\pi)$, *this is isomorphic to*

$$\widehat{\Gamma}(1, \eta_{1,2}, \ldots, \eta_{1,\mu_1}) \times \cdots \times \widehat{\Gamma}(1, \eta_{r,2}, \ldots, \eta_{r,\mu_r}).$$

Remark 1.3. We denote the local reciprocity map of F by $\mathrm{rec}_F \colon W_F \to F^\times$. For $\phi \in \Phi(\mathrm{GL}_N(F))$, let χ_ϕ denote the determinant character of ϕ. If $p > N$, then Theorem 1.2 shows that, using LLC_{F_∞}, we can characterize LLC_F as the unique map which makes the diagram

$$\begin{array}{ccccc} \mathrm{Hom}(F^\times, \mathbb{C}^\times) & \xleftarrow{\omega} & \mathscr{A}(\mathrm{GL}_N(F)) & \xrightarrow{\mathrm{BC}_\infty} & \mathscr{A}(\mathrm{GL}_N(F_\infty)) \\ \downarrow{\scriptstyle \mathrm{rec}_F^*} & & \downarrow{\scriptstyle \mathrm{LLC}_F} & & \downarrow{\scriptstyle \mathrm{LLC}_{F_\infty}} \\ \mathrm{Hom}(W_F, \mathbb{C}^\times) & \xleftarrow{\chi} & \Phi(\mathrm{GL}_N(F)) & \xrightarrow{\mathrm{Res}_{\infty/0}} & \Phi(\mathrm{GL}_N(F_\infty)) \end{array}$$

commute and has the following properties:

- a Steinberg representation $\mathrm{St}_m(\sigma)$ maps to the outer tensor product

  $$\mathrm{LLC}_F(\sigma) \boxtimes \mathrm{Sym}^{m-1}\mathbf{Std},$$

  where **Std** is the standard representation of $\mathrm{SL}_2(\mathbb{C})$, and
- a Langlands sum maps to the corresponding direct sum.

The outline of this article is as follows: First we prepare some notations which appear frequently in this article. In Section 2 we briefly review two theories of close fields. One is on the Galois side and due to Deligne [5] and the other is on the automorphic side and due to Kazhdan [8]. We devote §§2.1 to Deligne's theory and §§2.2 to Kazhdan's theory. The two are compatible via LLC in a certain sense, which we see in §§2.3. In Section 3 we prove key lemmas, which say that the restriction functor of Galois groups with respect to a totally ramified extension L/K coinsides with Deligne's correspondence under some conditions. We devote §§3.1 to prove the lemmas for the case L/K is finite. In §§3.2, we briefly review the theory of fields of norms due to Fontaine and Wintenberger [12] and prove the lemmas for the case L/K is infinite. In Section 4 we prove Theorem 1.1 as Theorem 4.3. Finally, in Section 5, we prove Theorem 1.2 as Theorem 5.1.

Notation

- Let p be a prime number.

- In this article, the term "local field" means a complete discrete valuation field with finite residue field of characteristic p. Let K denote a local field and F a local field of characteristic 0, i.e. a finite extension of $\mathbb{Q}_p$. A separable closure $\overline{K}$ of K and an algebraic closure $\overline{F}$ of F are fixed.

- We write $\mathcal{O}_K$ for the ring of integers of K, $\mathfrak{p}_K$ for the maximal ideal of $\mathcal{O}_K$ and k_K for the residue field of K. We denote by v_K the normalized additive valuation of K. We write $\mathcal{G}_K = \mathrm{Gal}(\overline{K}/K)$ and W_K for the Weil group of K.

- For any $u \geq -1$, let $\mathcal{G}_K^u$ denote the u-th ramification group in upper numbering. Let L be an extension of K contained in $\overline{K}$. We put $\mathcal{G}_L = \mathrm{Gal}(\overline{K}/L)$. The extension L is called APF if $\mathcal{G}_L\mathcal{G}_K^u$ is open in $\mathcal{G}_K$ for any $u \geq -1$. If so, we put $\mathcal{G}_L^0 = \mathcal{G}_K^0 \cap \mathcal{G}_L$. For a real number $u \geq -1$, we define

$$\psi_{L/K}(u) = \begin{cases} \displaystyle\int_0^u (\mathcal{G}_K^0 : \mathcal{G}_L^0\mathcal{G}_K^v)dv & \text{if } u \geq 0, \text{ or} \\ u & \text{otherwise,} \end{cases}$$

which is independent of the choice of $\overline{K}$. This is Wintenberger's definition [12, 1.2.1]. If L/K is finite and Galois, then it coinsides with that of Serre [11]. We denote by $i(L/K)$ the largest i satisfying $\mathcal{G}_L\mathcal{G}_K^i = \mathcal{G}_K$. A real number $b \geq -1$ is called a ramification break of L/K if $\mathcal{G}_L\mathcal{G}_K^b \supsetneq \mathcal{G}_L\mathcal{G}_K^{b+\varepsilon}$ for any $\varepsilon > 0$. Note that, if L/K is inifinite, then the set of the ramification breaks is a countably infinite set. Let $b_0 < b_1 < b_2 < \cdots$ be the ramification breaks of L/K. We put

$$K_n = \overline{K}^{\mathcal{G}_L\mathcal{G}_K^{b_n}}.$$

Here, we recall some properties on APF extensions [12, 1.4]:

(1) For any integer $n \geq 0$, K_n/K is finite. Furthermore, K_0/K is unramified, K_1/K_0 is tamely totally ramified, and if $n \geq 1$ then K_{n+1}/K_n is totally ramified of degree a power of p.

(2) For any $n \geq 1$ and any subextension M of L/K_{n+1}, we have

$$\psi_{L/K_n}(b_n) = i(L/K_n) = i(M/K_n) = i(K_{n+1}/K_n).$$

(3) If L/K is infinite, then $i(L/K_n) \to \infty$ as $n \to \infty$.

- Let L/K be an infinite APF extension. We denote by $X(L/K)$ or K_∞ the field of norms associated to L/K (for the definition, see §§2.3 of this article or [12, §2]). We put $i(K_\infty/K) = i(L/K)$. For any integer $n \geq 0$, we put $i(K_\infty/K_n) = i(L/K_n)$.

- We denote by $\mathscr{A}(\mathrm{GL}_N(K))$ the set of isomorphism classes of irreducible smooth representations of $\mathrm{GL}_N(K)$ over $\mathbb{C}$. Let $\mathbb{K}_l(\mathrm{GL}_N(K))$, or $\mathbb{K}_l(K)$ for short, denote the principal congruence subgroup of level l of $\mathrm{GL}_N(K)$:

$$\mathbb{K}_l(K) = \mathbb{K}_l(\mathrm{GL}_N(K)) = \mathrm{Ker}(\mathrm{GL}_N(\mathcal{O}_K) \to \mathrm{GL}_N(\mathcal{O}/\mathfrak{p}_K^l)).$$

  We denote by $\mathscr{A}_l(\mathrm{GL}_N(K))$ the subset of $\mathscr{A}(\mathrm{GL}_N(K))$ consisting of representations which have a non-trivial $\mathbb{K}_l(K)$-fixed vector.

- An L-parameter of $\mathrm{GL}_N(K)$ is a group homomorphism $\phi\colon W_K \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C})$ such that $\phi|_{W_K}$ is semi-simple and smooth and $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ is algebraic. Let $\Phi(\mathrm{GL}_N(K))$ denote the set of isomorphism classes of L-parameters of $\mathrm{GL}_N(K)$. We write

$$\Phi_l(\mathrm{GL}_N(K)) = \{\phi \in \Phi(\mathrm{GL}_N(K)) \mid \mathcal{G}_K^l \subset \mathrm{Ker}\,\phi\}.$$

- For a real number r, we write $\lceil r \rceil$ for the least integer λ satisfying $r \leq \lambda$.

§ 2. Theory of close fields

We recall Deligne's and Kazhdan's theory of close fields. Two local fields $K(1)$ and $K(2)$ are called *l-close* if there exists an isomorphism of rings $\alpha\colon \mathcal{O}_{K(2)}/\mathfrak{p}_{K(2)}^l \xrightarrow{\sim} \mathcal{O}_{K(1)}/\mathfrak{p}_{K(1)}^l$.

Notation 2.1. If two local fields $K(1)$ and $K(2)$ are l-close by $\alpha\colon \mathcal{O}_{K(2)}/\mathfrak{p}_{K(2)}^l \xrightarrow{\sim} \mathcal{O}_{K(1)}/\mathfrak{p}_{K(1)}^l$, then we fix a uniformizer ϖ_2 of $K(2)$ and choose a lift $\varpi_1 \in \mathfrak{p}_{K(1)}$ of $\alpha(\varpi_2 \bmod \mathfrak{p}_{K(2)}^l)$. We put $\beta = (\alpha, \varpi_2, \varpi_1)$.

As we see below, the datum β gives bijections $(\gamma_\beta)_l^*$ and $A_{\beta,l}^*$, which will appear in Deligne's and Kazhdan's theory, respectively.

§ 2.1. Deligne's theory

First, we briefly review Deligne's theory ([5]). Let R be a local ring and $\mathfrak{m}$ the maximal ideal of R. Suppose that the residue field $R/\mathfrak{m}$ is finite of characteristic p. Then R is called a *truncated discrete valuation ring*, or *tdvr* for short, if $\mathfrak{m}$ is monogenic and nilpotent. Note that R is a tdvr if and only if there exists a local field K such that R is isomorphic to $\mathcal{O}_K/\mathfrak{p}_K^l$ for some positive integer l. A *triple* in Deligne's theory is a triple (R, M, ε), where R is a tdvr, M is a free R-module of rank 1, and $\varepsilon\colon M \to \mathfrak{m}$ is a surjection of R-modules. Let $((R', \mathfrak{m}'), M', \varepsilon')$ be another triple. A *morphism of triples* $(R, M, \varepsilon) \to (R', M', \varepsilon')$ is a triple (r, φ, η), where $r \geq 1$ is an integer, $\varphi\colon R \to R'$ is a local homomorphism and $\eta\colon M \to M'^{\otimes r}$ is an R-linear homomorphism which makes the diagram

$$\begin{array}{ccc} M & \xrightarrow{\eta} & M'^{\otimes r} \\ \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon'_{0,r}} \\ \mathfrak{m} & \xrightarrow{\varphi} & \mathfrak{m}' \end{array}$$

commute. Here, we fix a generator x' of M' and define $\varepsilon'_{0,r}$ to be the R'-linear homomorphism given by $a'_1x' \otimes \cdots \otimes a'_rx' \mapsto a'_1 \cdots a'_r\varepsilon'(x')^r$. The $\varepsilon'_{0,r}$ is independent of the choice of x'. The integer r is called *the ramification index* of the morphism (r, φ, η). The morphism (r, φ, η) is said to be *flat* if $\lg(R') = r\lg(R)$, where $\lg(R)$ (resp. $\lg(R')$) denotes the length of R (resp. R') as an R-module (resp. R'-module). The morphism (r, φ, η) is said to be *finite* if φ is finite. For any tripe $T = (R, M, \varepsilon)$, we denote by $\mathrm{Ext}(T)^l$ the category whose objects are finite flat triples over T which satisfy the condition C^l in [5, 1.5.4], and morphisms are $R(l)$-equivalence classes ([5, 2.3]) of morphisms of triples over T.

Now take a positive integer $l' \leq l$. We denote by $\overline{\varepsilon}$ the natural surjection $M/\mathfrak{m}^{l'}M \to \mathfrak{m}/\mathfrak{m}^{l'}$ induced by ε. Then $\overline{T} = (R/\mathfrak{m}^{l'}, M/\mathfrak{m}^{l'}M, \overline{\varepsilon})$ is a triple in Deligne's theory, which is called *the reduction mod* $\mathfrak{m}^{l'}$ *of* T. Let $\mathrm{Ext}(T)^{l,l'}$ denote the full subcategory of $\mathrm{Ext}(T)^l$ consisting of objects satisfying the condition $\mathrm{C}^{l'}$. The canonical functor $\mathrm{Red}_T^{l,l'}\colon \mathrm{Ext}(T)^{l,l'} \to \mathrm{Ext}(\overline{T})^{l'}$ which maps an object S of $\mathrm{Ext}(T)^{l,l'}$ to the reduction of S mod $\mathfrak{m}^{rl'}$, where r is the ramification index of S/T, gives an equivalence of categories [5, Corollaire 2.9].

Let K be a local field and l a positive integer. We denote by $\mathrm{Tr}_l(K)$ the triple $(\mathcal{O}_K/\mathfrak{p}_K^l, \mathfrak{p}_K/\mathfrak{p}_K^{l+1}, \varepsilon)$ attached to K, where ε is the composite of the natural maps $\mathfrak{p}_K/\mathfrak{p}_K^{l+1} \to \mathfrak{p}_K/\mathfrak{p}_K^l \to \mathcal{O}_K/\mathfrak{p}_K^l$. We fix a separable closure $\overline{K}$ of K. Let $\mathrm{Ext}(K)$ denote the category of finite separable field extensions of K contained in $\overline{K}$ and $\mathrm{Ext}(K)^l$ the full subcategory of $\mathrm{Ext}(K)$ consisting of K' such that $\mathcal{G}_{K'} \supset \mathcal{G}_K^l$. Note that the natural morphism

$$\mathcal{G}_K \to \varprojlim_{\substack{K' \in \mathrm{Ext}(K)^l \\ K'/K\colon \text{Galois}}} \mathrm{Gal}(K'/K)$$

induces an isomorphism

$$\mathcal{G}_K/\mathcal{G}_K^l \xrightarrow{\sim} \varprojlim_{\substack{K' \in \mathrm{Ext}(K)^l \\ K'/K\colon \text{Galois}}} \mathrm{Gal}(K'/K). \tag{2.1}$$

We can construct a functor

$$T_K^l\colon \mathrm{Ext}(K)^l \to \mathrm{Ext}(\mathrm{Tr}_l(K))^l$$

such that, for an object K' of $\mathrm{Ext}(K)^l$, $T_K^l(K')$ is the extension of triples $\mathrm{Tr}_l(K) \to \mathrm{Tr}_{lr}(K')$ attached to the field extension K'/K, where r is the ramification index of K'/K. Deligne has proved that the T_K^l is an equivalence of categories ([5, Théorème 2.8]).

Let $K(1)$ and $K(2)$ be l-close local fields by α. We fix a separable closure $\overline{K(i)}$ of $K(i)$ and a datum $\beta = (\alpha, \varpi_2, \varpi_1)$ as in Notation 2.1. We have an $\mathcal{O}_{K(2)}/\mathfrak{p}_{K(2)}^l$-linear isomorphism $\eta(\beta)\colon \mathfrak{p}_{K(2)}/\mathfrak{p}_{K(2)}^{l+1} \xrightarrow{\sim} \mathfrak{p}_{K(1)}/\mathfrak{p}_{K(1)}^{l+1}$ by $(\varpi_2 \bmod \mathfrak{p}_{K(2)}^{l+1}) \mapsto (\varpi_1 \bmod \mathfrak{p}_{K(1)}^{l+1})$. Then $\gamma_\beta = (1, \alpha, \eta(\beta))$ define an isomorphism of triples $\mathrm{Tr}_l(K(2)) \xrightarrow{\sim} \mathrm{Tr}_l(K(1))$. By mapping an extension $\mathrm{Tr}_l(K(1)) \to X$ to $\mathrm{Tr}_l(K(1)) \xrightarrow{\gamma_\beta} \mathrm{Tr}_l(K(2)) \to X$ of $\mathrm{Tr}_l(K(2))$, we obtain an equivalence of categories

$$\gamma_\beta^* = (\gamma_\beta)_l^*\colon \mathrm{Ext}(\mathrm{Tr}_l(K(1)))^l \to \mathrm{Ext}(\mathrm{Tr}_l(K(2)))^l.$$

Now we fix a quasi-inverse T^{-1} of $T_{K(2)}^l$. For any object K' of $\mathrm{Ext}(K(1))^l$ which is Galois over $K(1)$, the composite

$$\begin{aligned}\mathrm{Gal}(K'/K(1)) &\xrightarrow{T_{K(1)}^l} \mathrm{Aut}_{\mathrm{Tr}_l(K(1))}(T_{K(1)}^l(K')) \xrightarrow{\gamma_\beta^*} \mathrm{Aut}_{\mathrm{Tr}_l(K(2))}(\gamma_\beta^* T_{K(1)}^l(K')) \\ &\xrightarrow{T^{-1}} \mathrm{Gal}(T^{-1}\gamma_\beta^* T_{K(1)}^l(K')/K(2))\end{aligned}$$

is an isomorphism. Since the set

$$\{T^{-1}\gamma_\beta^* T_{K(1)}^l(K') \mid K' \in \mathrm{Ext}(K(1))^l \text{ and } K'/K(1)\colon \text{Galois}\}$$

is cofinal in $\mathrm{Ext}(K(2))^l$, the inverse of the composite and the isomorphisms (2.1) for $K = K(1), K(2)$ give an isomorphism

$$(\gamma_\beta)_* = (\gamma_\beta)_{l,*}\colon \mathcal{G}_{K(2)}/\mathcal{G}_{K(2)}^l \xrightarrow{\sim} \mathcal{G}_{K(1)}/\mathcal{G}_{K(1)}^l,$$

which is uniquely determined by β up to inner isomorphisms. Hence we can define a bijection

$$\gamma_\beta^* = (\gamma_\beta)_l^*\colon \Phi_l(\mathrm{GL}_N(K(1))) \xrightarrow{\sim} \Phi_l(\mathrm{GL}_N(K(2)))$$

by $\phi_1 \mapsto \phi_1 \circ ((\gamma_\beta)_* \times \mathrm{id}_{\mathrm{SL}_2(\mathbb{C})})$ for any $\phi_1 \in \Phi_l(\mathrm{GL}_N(K(1)))$.

Now we take a positive integer $l' \le l$. There exists a unique ring isomorphism $\alpha'\colon \mathcal{O}_{K(2)}/\mathfrak{p}_{K(2)}^{l'} \xrightarrow{\sim} \mathcal{O}_{K(1)}/\mathfrak{p}_{K(1)}^{l'}$ which makes the diagram

$$\begin{array}{ccc}\mathcal{O}_{K(2)}/\mathfrak{p}_{K(2)}^l & \xrightarrow{\alpha} & \mathcal{O}_{K(1)}/\mathfrak{p}_{K(1)}^l \\ \downarrow & & \downarrow \\ \mathcal{O}_{K(2)}/\mathfrak{p}_{K(2)}^{l'} & \xrightarrow{\alpha'} & \mathcal{O}_{K(1)}/\mathfrak{p}_{K(1)}^{l'}\end{array}$$

commute, where the vertical arrows are the natural surjections. Thus the datum β induces an isomorphism of triples $\mathrm{Tr}_{l'}(K(2)) \to \mathrm{Tr}_{l'}(K(1))$, which induces an equivalence of categories $(\gamma_\beta)^*_{l'}\colon \mathrm{Ext}(\mathrm{Tr}_{l'}(K(1)))^{l'} \to \mathrm{Ext}(\mathrm{Tr}_{l'}(K(2)))^{l'}$. For $i = 1, 2$, we put $\mathrm{Red}^{l,l'}_{K(i)} = \mathrm{Red}^{l,l'}_{\mathrm{Tr}_l(K(i))}$. There exists a canonical isomorphism

$$\mathrm{Red}^{l,l'}_{K(i)} \mathrm{Tr}_l(K(i)) \xrightarrow{\sim} \mathrm{Tr}_{l'}(K(i)).$$

For any $K' \in \mathrm{Ext}(K(1))^{l'}$ the triple $T^l_{K(1)}(K')$ satisfies the condition $\mathrm{C}^{l'}$, and thus is an object of $\mathrm{Ext}(\mathrm{Tr}_l(K(1)))^{l,l'}$. We have natural isomorphisms

$$\mathrm{Red}^{l,l'}_{K(1)} T^l_{K(1)}(K') \xrightarrow{\sim} T^{l'}_{K(1)}(K') \quad \text{and} \quad \mathrm{Red}^{l,l'}_{K(2)} (\gamma_\beta)^*_l T^l_{K(1)}(K') \xrightarrow{\sim} (\gamma_\beta)^*_{l'} T^{l'}_{K(1)}(K').$$

Via these isomorphisms, we identify the left-hand sides with the corresponding right-hand sides. We take a quasi-inverse $\lambda\colon \mathrm{Ext}(\mathrm{Tr}_{l'}(K(2)))^{l'} \xrightarrow{\sim} \mathrm{Ext}(\mathrm{Tr}_l(K(2)))^{l,l'}$ of $\mathrm{Red}^{l,l'}_{K(2)}$. For any object $S(2)$ of $\mathrm{Ext}(\mathrm{Tr}_l(K(2)))^{l,l'}$, $T^{-1}(S(2))$ is in $\mathrm{Ext}(K(2))^{l'}$ and $T^{-1} \circ \lambda$ is a quasi-inverse of $T^{l'}_{K(2)}$. By using the quasi-inverse, we can construct a group isomorphism $(\gamma_\beta)_{l',*}\colon \mathcal{G}_{K(2)}/\mathcal{G}^{l'}_{K(2)} \xrightarrow{\sim} \mathcal{G}_{K(1)}/\mathcal{G}^{l'}_{K(1)}$ by the similar method to that of $(\gamma_\beta)_{l,*}$. For any object K' of $\mathrm{Ext}(K(1))^{l'}$ which is Galois over $K(1)$, the diagrams

$$\begin{array}{ccc} \mathrm{Gal}(K'/K(1)) & \xrightarrow{T^l_{K(1)}} & \mathrm{Aut}_{\mathrm{Tr}_l(K(1))}(T^l_{K(1)}(K')) \\ & \searrow^{T^{l'}_{K(1)}} & \downarrow \mathrm{Red}^{l,l'}_{K(1)} \\ & & \mathrm{Aut}_{\mathrm{Tr}_{l'}(K(1))}(T^{l'}_{K(1)}(K')), \end{array}$$

$$\begin{array}{ccc} \mathrm{Aut}_{\mathrm{Tr}_l(K(1))}(T^l_{K(1)}(K')) & \xrightarrow{(\gamma_\beta)^*_l} & \mathrm{Aut}_{\mathrm{Tr}_l(K(2))}((\gamma_\beta)^*_l T^l_{K(1)}(K')) \\ \downarrow \mathrm{Red}^{l,l'}_{K(1)} & & \downarrow \mathrm{Red}^{l,l'}_{K(2)} \\ \mathrm{Aut}_{\mathrm{Tr}_{l'}(K(1))}(T^{l'}_{K(1)}(K')) & \xrightarrow{(\gamma_\beta)^*_{l'}} & \mathrm{Aut}_{\mathrm{Tr}_{l'}(K(2))}((\gamma_\beta)^*_{l'} T^{l'}_{K(1)}(K')), \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Aut}_{\mathrm{Tr}_l(K(2))}((\gamma_\beta)^*_l T^l_{K(1)}(K')) & \xrightarrow{T^{-1}} & \mathrm{Gal}(T^{-1}(\gamma_\beta)^*_l T^l_{K(1)}(K')/K(2)) \\ \downarrow \mathrm{Red}^{l,l'}_{K(2)} & & \downarrow (\dagger) \\ \mathrm{Aut}_{\mathrm{Tr}_{l'}(K(2))}((\gamma_\beta)^*_{l'} T^{l'}_{K(1)}(K')) & \xrightarrow{T^{-1}\circ\lambda} & \mathrm{Gal}(T^{-1}\lambda(\gamma_\beta)^*_{l'} T^{l'}_{K(1)}(K')/K(2)) \end{array}$$

commute, where the arrow $(\dagger)$ is the isomorphism induced by a natural isomorphism $\mathrm{id}_{\mathrm{Ext}(\mathrm{Tr}_l(K(2)))^{l,l'}} \xrightarrow{\sim} \lambda \circ \mathrm{Red}^{l,l'}_{K(2)}$. Hence the isomorphism $\mathcal{G}_{K(2)}/\mathcal{G}^{l'}_{K(2)} \xrightarrow{\sim} \mathcal{G}_{K(1)}/\mathcal{G}^{l'}_{K(1)}$ induced by $(\gamma_\beta)_{l,*}$ coincides with $(\gamma_\beta)_{l',*}$ up to inner isomorphisms.

Therefore we have the following:

Lemma 2.2. *Let l be a positive integer and $K(1)$ and $K(2)$ l-close local fields. We fix a datum β as in Notation 2.1. Then, for any positive integer $l' \leq l$, we can construct a bijection $(\gamma_\beta)^*_{l'} \colon \Phi_{l'}(\mathrm{GL}_N(K(1))) \xrightarrow{\sim} \Phi_{l'}(\mathrm{GL}_N(K(2)))$. The bijection $(\gamma_\beta)^*_{l'}$ coincides with the restriction of $(\gamma_\beta)^*_l$ on $\Phi_{l'}(\mathrm{GL}_N(K(1)))$.*

§ 2.2. Kazhdan's theory

To review Kazhdan's theory of close local fields [8], we first recall an equivalence of $\mathrm{Rep}_l(\mathrm{GL}_N(L))$ and the category of representations of some Hecke algebra, where L is a local field.

We put $\mathbb{K}_l(L) = \mathrm{Ker}(\mathrm{GL}_N(\mathcal{O}_L) \to \mathrm{GL}_N(\mathcal{O}/\mathfrak{p}_L^l))$. We denote by $\mathrm{Rep}(\mathrm{GL}_N(L))$ the category of admissible smooth representations of $\mathrm{GL}_N(L)$ and by $\mathrm{Rep}_l(\mathrm{GL}_N(L))$ the full subcategory of $\mathrm{Rep}(\mathrm{GL}_N(L))$ consisting of representations generated by their $\mathbb{K}_l(L)$-fixed vectors. We denote by $\mathscr{A}_l(\mathrm{GL}_N(L))$ the subset of $\mathscr{A}(\mathrm{GL}_N(L))$ consisting of representations which have a non-trivial $\mathbb{K}_l(L)$-fixed vector. We denote by $\mathscr{H}_l(\mathrm{GL}_N(L))$ the algebra of compactly supported $\mathbb{K}_l(L)$-bi-invariant functions on $\mathrm{GL}_N(L)$ with values in $\mathbb{C}$ whose product is the convolution $*_l$ with respect to the Haar measure $\mu_{\mathrm{GL}_N(L),l}$ on $\mathrm{GL}_N(L)$ normalized by

$$\mu_{\mathrm{GL}_N(L),l}(\mathbb{K}_l(L)) = 1.$$

The characteristic function $e_{\mathbb{K}_l(L)}$ of $\mathbb{K}_l(L)$ is the unity of $\mathscr{H}_l(\mathrm{GL}_N(L))$. The category of left $\mathscr{H}_l(\mathrm{GL}_N(L))$-modules is denoted by $\mathrm{Mod}(\mathscr{H}_l(\mathrm{GL}_N(L)))$.

Lemma 2.3 ([4, Corollaire 3.9 (ii)])**.** *The functor $V \mapsto V^{\mathbb{K}_l(L)}$ gives an equivalence of categories*

$$\mathrm{Rep}_l(\mathrm{GL}_N(L)) \to \mathrm{Mod}(\mathscr{H}_l(\mathrm{GL}_N(L))).$$

By using this, we can prove the following:

Lemma 2.4. *For $l \leq l'$, the functor*

$$\begin{aligned}\mathrm{Mod}(\mathscr{H}_l(\mathrm{GL}_N(L))) &\to \mathrm{Mod}(\mathscr{H}_{l'}(\mathrm{GL}_N(L)))\\ W &\mapsto (\mathscr{H}_{l'}(\mathrm{GL}_N(L)) *_{l'} e_{\mathbb{K}_l(L)}) \underset{\mathscr{H}_l(\mathrm{GL}_N(L))}{\otimes} W\end{aligned}$$

makes the diagram

$$\begin{array}{ccc}\mathrm{Rep}_l(\mathrm{GL}_N(L)) & \hookrightarrow & \mathrm{Rep}_{l'}(\mathrm{GL}_N(L))\\ \downarrow & & \downarrow\\ \mathrm{Mod}(\mathscr{H}_l(\mathrm{GL}_N(L))) & \longrightarrow & \mathrm{Mod}(\mathscr{H}_{l'}(\mathrm{GL}_N(L)))\end{array}$$

commute up to natural equivalences, where the two vertical arrows are the equivalences in Lemma 2.3 and the top horizontal arrow is the natural injection.

Proof. The following proof is similar to that of [4, Corollaire 3.9 (ii)]. Throughout this proof, we put $G = \mathrm{GL}_N(L)$, $K_l = \mathbb{K}_l(L)$, $\mathscr{H}_l = \mathscr{H}_l(G)$ and $e_l = e_{K_l}$. Note that the $\mathbb{C}$-vector space $\mathscr{H}_{l'} *_{l'} e_l$ has an $\mathscr{H}_{l'}$-$\mathscr{H}_l$-bimodule structure via $(h_{l'}, h'_{l'} *_{l'} e_l, h_l) \mapsto h_{l'} *_{l'} h'_{l'} *_{l'} h_l$ for any $h_{l'}, h'_{l'} \in \mathscr{H}_{l'}$ and $h_l \in \mathscr{H}_l$. Let (π, V) be any object of $\mathrm{Rep}_l(G)$. The map

$$(\mathscr{H}_{l'} *_{l'} e_l) \underset{\mathscr{H}_l}{\otimes} V^{K_l} \to V^{K_{l'}}$$

defined by

$$(h *_{l'} e_l) \otimes v \mapsto \int_G (h *_{l'} e_l)(g)\pi(g)v d\mu_{G,l'}(g)$$

is a well-defined left $\mathscr{H}_{l'}$-module homomorphism. It suffices to show that this is an isomorphism. This is surjective since π is an object of $\mathrm{Rep}_l(G)$. We denote by $\mathcal{N}$ the kernel of the above homomorphism. Now let $\mathrm{Mod}_l(\mathscr{H}_{l'})$ denote the full subcategory of $\mathrm{Mod}(\mathscr{H}_{l'})$ consisting of objects W which are generated by $e_l * W$. Then the equivalence of categories of Lemma 2.3 induces that of $\mathrm{Rep}_l(G)$ and $\mathrm{Mod}_l(\mathscr{H}_{l'})$. This equivalence and Lemma 2.3 imply that the latter is equivalent to $\mathrm{Mod}(\mathscr{H}_l)$ and stable by subquotient. Since $\mathscr{H}_{l'} *_{l'} e_l$ and $V^{K_{l'}}$ are object of $\mathrm{Mod}_l(\mathscr{H}_{l'})$, so is $\mathcal{N}$. In addition, there is no non-trivial vectors on $\mathcal{N}$ which is fixed by the left action of e_l. Therefore $\mathcal{N} = 0$ and the above homomorphism is an isomorphism. □

Next we recall Kazhdan's theory. Let $K(1)$ and $K(2)$ be l-close local fields. We fix a datum $\beta = (\alpha, \varpi_2, \varpi_1)$ as in Notation 2.1. By the Cartan decomposition, this gives a $\mathbb{C}$-linear isomorphism

$$\beta^* \colon \mathscr{H}_l(\mathrm{GL}_N(K(1))) \xrightarrow{\sim} \mathscr{H}_l(\mathrm{GL}_N(K(2)))$$

(see [8]). In Kazhdan's original paper, he showed that if $K(1)$ and $K(2)$ are sufficiently close then β^* is compatible with the convolution products. Lemaire showed a more precise result for GL_N:

Lemma 2.5 ([10, Proposition 3.1.1])**.** *If $K(1)$ and $K(2)$ are l-close, the isomorphism β^* is compatible with the convolution products. Hence it is a $\mathbb{C}$-algebra isomorphism.*

Hence we can define an equivalence of categories

$$\mathrm{Mod}(\mathscr{H}_l(\mathrm{GL}_N(K(1)))) \xrightarrow{\sim} \mathrm{Mod}(\mathscr{H}_l(\mathrm{GL}_N(K(2))))$$

by base change via $\beta^* \colon \mathscr{H}_l(\mathrm{GL}_N(K(1))) \xrightarrow{\sim} \mathscr{H}_l(\mathrm{GL}_N(K(2)))$. By Lemma 2.3, we obtain an equivalence of categories

$$A_{\beta,l} \colon \mathrm{Rep}_l(\mathrm{GL}_N(K(1))) \xrightarrow{\sim} \mathrm{Rep}_l(\mathrm{GL}_N(K(2))).$$

This induces a bijection $\mathscr{A}_l(\mathrm{GL}_N(K(1))) \xrightarrow{\sim} \mathscr{A}_l(\mathrm{GL}_N(K(2)))$, which we also denote by $A_{\beta,l}$.

§ 2.3. Compatibility of Deligne's and Kazhdan's theories via LLC

Let K be a local field. Let $\mathrm{LLC}_K\colon \mathscr{A}(\mathrm{GL}_N(K)) \xrightarrow{\sim} \Phi(\mathrm{GL}_N(K))$ denote the local Langlands correspondence of GL_N over K. We will see a result of Aubert, Baum, Plymen and Solleveld in [2] as Theorem 2.8, which is a compatibility of $(\gamma_\beta)_l^*$ and $A_{\beta,l}$ via the local Langlands correspondence. Before that, we shall prove the following lemma:

Lemma 2.6. *Let K be a local field and l a positive integer. Then we have*

$$\mathrm{LLC}_K(\mathscr{A}_l(\mathrm{GL}_N(K))) = \Phi_l(\mathrm{GL}_N(K)).$$

To prove it, we need Lemma 2.7 as below. We prepare some notations. Let $N_1, \dots, N_r$ be positive integers satisfying $N_1 + \cdots + N_r = N$. We denote by $P(N_1, \dots, N_r)$ the standard parabolic subgroup of $\mathrm{GL}_N(K)$ whose Levi component is $\mathrm{GL}_{N_1}(K) \times \cdots \times \mathrm{GL}_{N_r}(K)$. We put $\mathbb{K}_l(N) = \mathbb{K}_l(\mathrm{GL}_N(K))$.

Lemma 2.7. *For each $i = 1, \dots, r$, we take any $\pi_i \in \mathscr{A}(\mathrm{GL}_{N_i}(K))$. Then*

$$\left(\text{n-Ind}_{P(N_1,\dots,N_r)}^{\mathrm{GL}_N(K)}(\pi_1 \boxtimes \cdots \boxtimes \pi_r)\right)^{\mathbb{K}_l(N)} = 0$$

holds if and only if there exists an $i = 1, \dots, r$ such that $\pi_i^{\mathbb{K}_l(N_i)} = 0$.

Proof. We put $P = P(N_1, \dots, N_r)$. By the Iwasawa decomposition of $\mathrm{GL}_N(K)$, we can choose a complete set of representatives Ω of $P\backslash \mathrm{GL}_N(K)$ contained in $\mathrm{GL}_N(\mathcal{O}_K)$. Since $\mathbb{K}_l(N)$ is a normal subgroup of $\mathrm{GL}_N(\mathcal{O}_K)$, we have $g\mathbb{K}_l(N)g^{-1} = \mathbb{K}_l(N)$ for any $g \in \Omega$ and the map

$$\left(\text{n-Ind}_P^{\mathrm{GL}_N(K)}(\pi_1 \boxtimes \cdots \boxtimes \pi_r)\right)^{\mathbb{K}_l(N)} \to \bigoplus_{g\in\Omega} (\pi_1 \boxtimes \cdots \boxtimes \pi_r)^{P\cap\mathbb{K}_l(N)}$$

given by $f \mapsto (f(g))_{g\in\Omega}$ is a well-defined $\mathbb{C}$-linear isomorphism. Thus the left-hand side is zero if and only if $(\pi_1 \boxtimes \cdots \boxtimes \pi_r)^{P\cap\mathbb{K}_l(N)} = 0$. Since the action of P on $\pi_1 \boxtimes \cdots \boxtimes \pi_r$ factors through the natural surjection $P \to \mathrm{GL}_{N_1}(K) \times \cdots \times \mathrm{GL}_{N_r}(K)$ and the image of $P \cap \mathbb{K}_l(N)$ under the surjection is $\mathbb{K}_l(N_1) \times \cdots \times \mathbb{K}_l(N_r)$, the equality is equivalent to $(\pi_1 \boxtimes \cdots \boxtimes \pi_r)^{\mathbb{K}_l(N_1)\times\cdots\times\mathbb{K}_l(N_r)} = 0$. The left-hand side is isomorphic to $\pi_1^{\mathbb{K}_l(N_1)} \boxtimes \cdots \boxtimes \pi_r^{\mathbb{K}_l(N_r)}$. This completes our proof. □

Proof of Lemma 2.6. Take any $\pi \in \mathscr{A}(\mathrm{GL}_N(K))$. We put $\phi = \mathrm{LLC}_K(\pi)$. First assume that π is supercuspidal. Let $d(\pi)$ be the depth of π defined by the equation (32) in [2, §4] and $d(\phi)$ the unique rational number such that

$$\phi \notin \Phi_{d(\phi)}(\mathrm{GL}_N(K)) \text{ and } \phi \in \Phi_{l'}(\mathrm{GL}_N(K)) \text{ for any } l' > d(\phi).$$

By [2, Lemma 4.3], the statement $\pi \in \mathscr{A}_l(\mathrm{GL}_N(K))$ is equivalent to $d(\pi) \leq l-1$. Since $d(\pi) = d(\phi)$ ([2, Proposition 4.2]), the inequality is equivalent to $d(\phi) \leq l-1$. By the characterization of $d(\phi)$, this is equivalent to $\phi \in \Phi_l(\mathrm{GL}_N(K))$, and we have proved the case π is supercuspidal.

We assume π is essentially square-integrable. Then there exist a unique divisor m of N and a unique supercuspidal representation $\sigma \in \mathscr{A}(\mathrm{GL}_{N/m}(K))$ such that π is equivalent to the unique irreducible quotient $\mathrm{St}_m(\sigma)$ of

$$\text{n-Ind}_{P(N/m,\ldots,N/m)}^{\mathrm{GL}_N(K)}(\sigma \otimes |\det|^{(1-m)/2} \boxtimes \cdots \boxtimes \sigma \otimes |\det|^{(m-1)/2})$$

([13, Theorem 9.3]). By the characterization of π, we have $\pi^{\mathbb{K}_l(N)} = 0$ if and only if

$$\text{n-Ind}_{P(N/m,\ldots,N/m)}^{\mathrm{GL}_N(K)}(\sigma \otimes |\det|^{(1-m)/2} \boxtimes \cdots \boxtimes \sigma \otimes |\det|^{(m-1)/2})^{\mathbb{K}_l(N)} = 0,$$

which is equivalent to $\sigma^{\mathbb{K}_l(N/m)} = 0$ by Lemma 2.7. We put $\phi(\sigma) = \mathrm{LLC}_K(\sigma)$. Note that $\phi|_{\mathcal{G}_K^l} = \phi(\sigma)^{\oplus m}|_{\mathcal{G}_K^l}$. Since σ is supercuspidal, we have $\sigma^{\mathbb{K}_l(N/m)} = 0$ if and only if $\phi(\sigma)|_{\mathcal{G}_K^l}$ is non-trivial, which is equivalent to saying that $\phi(\sigma)^{\oplus m}|_{\mathcal{G}_K^l}$ is non-trivial. This completes the proof for the case π is essentially square-integrable.

The same argument shows the general case since any $\pi \in \mathscr{A}(\mathrm{GL}_N(K))$ is uniquely written by a Langlands sum of several essentially square-integrable representations and LLC_K maps it to the direct sum of the corresponding L-parameters. ☐

Let $K(1)$ and $K(2)$ be local fields which are l-close. We choose a datum $\beta = (\alpha, \varpi_2, \varpi_1)$ as in Notation 2.1. We have obtained Deligne's bijection

$$(\gamma_\beta)^* = (\gamma_\beta)_l^* \colon \Phi_l(\mathrm{GL}_N(K(2))) \xrightarrow{\sim} \Phi_l(\mathrm{GL}_N(K(1)))$$

in §§2.1 and Kazhdan's bijection

$$A_{\beta,l} \colon \mathscr{A}_l(\mathrm{GL}_N(K(1))) \xrightarrow{\sim} \mathscr{A}_l(\mathrm{GL}_N(K(2)))$$

in §§2.2. Then we have the following compatibility:

Theorem 2.8 ([2, Theorem 6.1]). *Let l' be any integer which satisfies the inequality $0 < l' < 2^{-N+1}l$. Then the following diagram is commutative:*

$$\begin{array}{ccc} \mathscr{A}_{l'}(\mathrm{GL}_N(K(1))) & \xrightarrow{A_{\beta,l'}} & \mathscr{A}_{l'}(\mathrm{GL}_N(K(2))) \\ \downarrow{\scriptstyle \mathrm{LLC}_{K(1)}} & & \downarrow{\scriptstyle \mathrm{LLC}_{K(2)}} \\ \Phi_{l'}(\mathrm{GL}_N(K(1))) & \xrightarrow{(\gamma_\beta)_{l'}^*} & \Phi_{l'}(\mathrm{GL}_N(K(2))). \end{array}$$

Remark 2.9. In [2, Theorem 6.1], the bottom part of the above square is

$$\Phi_l(\mathrm{GL}_N(K(1))) \xrightarrow{(\gamma_\beta)_l^*} \Phi_l(\mathrm{GL}_N(K(2))).$$

Since we have Lemma 2.2 and Lemma 2.6, we can replace l by l'.

§ 3. Key lemmas

In this section, we prove important lemmas, which show some compatibility of the restriction functor of Galois groups with respect to a totally ramified extension and Deligne's theory.

§ 3.1. The case L/K is finite

In this subsection, we show the key lemmas for finite totally ramified extensions. Let K be a local field, $\overline{K}$ a separable closure of K, and $L \subset \overline{K}$ a finite totally ramified extension of K of degree a power of p. We have

$$\mathcal{G}_L \cap \mathcal{G}_K^u = W_L \cap \mathcal{G}_K^u = \mathcal{G}_L^{\psi_{L/K}(u)} \tag{3.1}$$

[12, 1.1.2]. For any separable subextension M of $\overline{K}/K$, we denote by $i(M/K)$ the largest i satisfying $\mathcal{G}_M\mathcal{G}_K^i = \mathcal{G}_K$. Then for any integer $l \leq \lceil p^{-1}(p-1)i(L/K)\rceil$, the norm map $N_{L/K}$ induces an isomorphism of rings $\alpha_{L/K}\colon \mathcal{O}_L/\mathfrak{p}_L^l \xrightarrow{\sim} \mathcal{O}_K/\mathfrak{p}_K^l$ (see [12, Proposition 2.2.1]). In particular, K and L are l-close. We take a uniformizer ϖ_L of L. By the assumption L/K is totally ramified, $N_{L/K}(\varpi_L)$ is a uniformizer of K. Hence, as we have seen in §§2.1, the datum $\beta = (\alpha_{L/K}, \varpi_L, N_{L/K}(\varpi_L))$ gives an isomorphism of triples $\gamma_\beta\colon \mathrm{Tr}_l(L) \xrightarrow{\sim} \mathrm{Tr}_l(K)$. We can show that the γ_β is independent of the choice of ϖ_L. We put $\mathfrak{N}_{L/K} = \gamma_\beta$. We have an equivalence of categories

$$\mathfrak{N}_{L/K}^* = \gamma_\beta^*\colon\ \mathrm{Ext}(\mathrm{Tr}_l(K))^l \xrightarrow{\sim} \mathrm{Ext}(\mathrm{Tr}_l(L))^l.$$

On the other hand, we have a functor $\rho\colon \mathrm{Ext}(K) \to \mathrm{Ext}(L)$ which maps an extension K' of K to the composite $K'L$. If K' is an object of $\mathrm{Ext}(K)^l$, then by the equalities (3.1), we have

$$\mathcal{G}_{K'L} = \mathcal{G}_{K'} \cap \mathcal{G}_L \supset \mathcal{G}_K^l \cap \mathcal{G}_L = \mathcal{G}_L^{\psi_{L/K}(l)} = \mathcal{G}_L^l.$$

(We use the assumption $l \leq \lceil p^{-1}(p-1)i(L/K)\rceil$ when we prove the last equality $\mathcal{G}_L^{\psi_{L/K}(l)} = \mathcal{G}_L^l$.) Thus $K'L$ is in $\mathrm{Ext}(L)^l$ and we obtain the functor $\rho^l\colon \mathrm{Ext}(K)^l \to \mathrm{Ext}(L)^l$ induced by ρ.

Now we can prove the following lemma:

Lemma 3.1. *Let K be a local field, $L \subset \overline{K}$ a finite totally ramified extension of K of degree a power of p. Let l be a positive integer satisfying*

$$l \leq \left\lceil \frac{p-1}{2p} i(L/K) \right\rceil .$$

Then we have the commutative diagram (up to natural equivalences)

$$\begin{array}{ccc} \mathrm{Ext}(K)^l & \xrightarrow{\quad \rho^l \quad} & \mathrm{Ext}(L)^l \\ \big\downarrow {\scriptstyle T_K^l} & & \big\downarrow {\scriptstyle T_L^l} \\ \mathrm{Ext}(\mathrm{Tr}_l(K))^l & \xrightarrow{\mathfrak{N}^*_{L/K}} & \mathrm{Ext}(\mathrm{Tr}_l(L))^l. \end{array}$$

Therefore the group isomorphism

$$(\mathfrak{N}_{L/K})_* \colon \mathcal{G}_L/\mathcal{G}_L^l \to \mathcal{G}_K/\mathcal{G}_K^l$$

induced by $\mathfrak{N}_{L/K}$ coincides with the homomorphism which comes from the natural injection $\mathcal{G}_L \hookrightarrow \mathcal{G}_K$.

Proof. We take a Galois object K' of $\mathrm{Ext}(K)^l$. We put $L' = K'L$. We shall construct an isomorphism

$$\mathfrak{N}' \colon T_L^l(L') \xrightarrow{\sim} \mathfrak{N}^*_{L/K} T_K^l(K')$$

in $\mathrm{Ext}(\mathrm{Tr}_l(L))^l$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Gal}(L'/L) & \xrightarrow{\quad \cdot|_{K'} \quad} & \mathrm{Gal}(K'/K) \\ \big\downarrow {\scriptstyle T_L^l} & & \big\downarrow {\scriptstyle T_K^l} \\ \mathrm{Aut}_{\mathrm{Tr}_l(L)}(T_L^l(L')) & & \mathrm{Aut}_{\mathrm{Tr}_l(K)}(T_K^l(K')) \\ & \searrow^{\mathrm{ad}(\mathfrak{N}')} & \Big\| \\ & & \mathrm{Aut}_{\mathrm{Tr}_l(L)}(\mathfrak{N}^*_{L/K} T_K^l(K')), \end{array} \tag{3.2}$$

where we define $\mathrm{ad}(\mathfrak{N}')$ by $\mathrm{ad}(\mathfrak{N}')(\sigma) = \mathfrak{N}' \circ \sigma \circ \mathfrak{N}'^{-1}$ for any $\sigma \in \mathrm{Aut}_{\mathrm{Tr}_l(L)}(T_L^l(L'))$. Let r denote the ramification index of K'/K. We have $l \leq 2^{-1} i(L/K)$ and

$$\mathcal{G}_{K'} \supset \mathcal{G}_K^l \supset \mathcal{G}_K^{2^{-1} i(L/K)}.$$

Hence we obtain inequalities

$$\psi_{K'/K}^{-1}\left(\frac{1}{2} i(L/K) r\right) \leq \frac{1}{2} i(L/K) + \frac{r-1}{r} \cdot \frac{1}{2} i(L/K) \leq i(L/K).$$

Taking account of $\mathcal{G}_L\mathcal{G}_K^{i(L/K)} = \mathcal{G}_K$, we have

$$\begin{aligned}\mathcal{G}_{L'}\mathcal{G}_{K'}^{2^{-1}i(L/K)r} =& \mathcal{G}_{L'}(\mathcal{G}_K^{\psi^{-1}_{K'/K}(2^{-1}i(L/K)r)} \cap \mathcal{G}_{K'})\\ \supset& \mathcal{G}_{L'}(\mathcal{G}_K^{i(L/K)} \cap \mathcal{G}_{K'})\\ =& \mathcal{G}_{L'}\mathcal{G}_K^{i(L/K)}\\ =& \mathcal{G}_{K'} \cap (\mathcal{G}_K^{i(L/K)}\mathcal{G}_L)\\ =& \mathcal{G}_{K'}.\end{aligned}$$

Hence we obtain $\mathcal{G}_{L'}\mathcal{G}_{K'}^{2^{-1}i(L/K)r} = \mathcal{G}_{K'}$. Thus we have $2^{-1}i(L/K)r \leq i(L'/K')$ and the norm map $N_{L'/K'}$ provides an isomorphism $\mathfrak{N}_{L'/K'}\colon \mathrm{Tr}_{lr}(L') \xrightarrow{\sim} \mathrm{Tr}_{lr}(K')$, which makes the diagram

$$\begin{array}{ccc}\mathrm{Tr}_l(L) & \longrightarrow & \mathrm{Tr}_{lr}(L')\\ \downarrow{\scriptstyle \mathfrak{N}_{L/K}} & & \downarrow{\scriptstyle \mathfrak{N}_{L'/K'}}\\ \mathrm{Tr}_l(K) & \longrightarrow & \mathrm{Tr}_{lr}(K')\end{array}$$

commute. Thus $\mathfrak{N}_{L'/K'}$ is in fact an isomorphism in $\mathrm{Ext}(\mathrm{Tr}_l(L))^l$. We put $\mathfrak{N}' = \mathfrak{N}_{L'/K'}$.

The commutativity of the diagram (3.2) follows from the equality $N_{L'/K'} \circ \sigma = \sigma \circ N_{L'/K'}$ for any $\sigma \in \mathrm{Gal}(L'/L)$. Lemma 3.1 follows from the diagram (3.2). □

For any real number $l \geq 0$, we define

$$\Phi_l(\mathrm{GL}_N(K)) = \{\phi \in \Phi(\mathrm{GL}_N(K)) \mid \mathcal{G}_K^l \subset \mathrm{Ker}\,\phi\}.$$

By Lemma 3.1, we obtain the following lemma:

Lemma 3.2. *Let K be a local field, $L \subset \overline{K}$ a finite totally ramified extension of K of degree a power of p. Let l be a positive integer satisfying*

$$l \leq \left\lceil \frac{p-1}{2p} i(L/K) \right\rceil.$$

Then the restriction of L-parameters

$$\begin{aligned}\Phi_l(\mathrm{GL}_N(K)) &\to \Phi_l(\mathrm{GL}_N(L))\\ \phi &\mapsto \phi|_{W_L \times \mathrm{SL}_2(\mathbb{C})}\end{aligned}$$

coincides with the map

$$\begin{aligned}(\mathfrak{N}_{L/K})_l^*\colon\ \Phi_l(\mathrm{GL}_N(K)) &\to \Phi_l(\mathrm{GL}_N(L))\\ \phi &\mapsto \phi \circ ((\mathfrak{N}_{L/K})_* \times \mathrm{id}_{\mathrm{SL}_2(\mathbb{C})}).\end{aligned}$$

§ 3.2. The case L/K is infinite: fields of norms

Similar results to Lemma 3.1 and Lemma 3.2 also hold if L/K is an APF extension of infinite degree. To explain this, we recall the theory of fields of norms by Fontaine-Wintenberger [12]. Let K be a local field, $\overline{K}$ a separable closure of K, and L an infinite APF extension of K contained in $\overline{K}$. Let $b_0 < b_1 < b_2 < \cdots$ be the ramification breaks of L/K. We put

$$K_n = \overline{K}^{\mathcal{G}_L \mathcal{G}_K^{b_n}}.$$

For any integer $m \geq n \geq 0$, we denote by $N_{m/n}$ (resp. N_n) the norm map from $K_m^\times$ to $K_n^\times$ (resp. from $K_n^\times$ to $K^\times$). Let $\mathscr{E}_{L/K}$ be the filtered ordered set consisting of finite extensions M/K contained in L. We put

$$X(L/K)^\times = \varprojlim_{M \in \mathscr{E}_{L/K}} M^\times,$$

whose transitive maps are the norm maps. Since $\{K_n\}_{n=1}^\infty$ is cofinal in $\mathscr{E}_{L/K}$, the natural map

$$X(L/K)^\times = \varprojlim_{M \in \mathscr{E}_{L/K}} M^\times \to \varprojlim_{n \geq 1} K_n^\times$$

gives an isomorphism. By adding the zero element to $X(L/K)^\times$, we obtain a field $X(L/K)$, which we call *the field of norms associated to L/K.* Here, its addition is defined as follows. We take $x = (x_n)_{n=1}^\infty$ and $y = (y_n)_{n=1}^\infty$ in $X(L/K)^\times$, where $x_n, y_n \in K_n^\times$. If there exists an integer $m_0 \geq 1$ such that for any integer $m \geq m_0$ we have $x_m + y_m \neq 0$, then we define $x + y$ as

$$(x+y)_n = \lim_{m \to \infty} N_{K_m/K_n}(x_m + y_m).$$

Otherwise, we put $x + y = 0$. The characteristic of $X(L/K)$ is p and $X(L/K)$ becomes a local field by being endowed with the additive valuation defined by $v_{X(L/K)}(x) = v_{K_n}(x_n)$ for some/any $n \geq 1$ (the right-hand side is independent of the choice of n). We also write K_∞ for the local field $X(L/K)$. For any $M \in \mathscr{E}_{L/K}$, we put $i(K_\infty/M) = i(L/M)$. The natural projection $K_\infty^\times \to K^\times$ (resp. $K_\infty^\times \to K_n^\times$) is denoted by N_∞ (resp. $N_{\infty/n}$). We extend N_∞ (resp. $N_{\infty/n}$) to all of K_∞ by setting $N_\infty(0) = 0$ (resp. $N_{\infty/n}(0) = 0$). Now we can prove similar statements to [12, Proposition 2.2.1]:

Lemma 3.3. *We use the above notation. Suppose $i(K_\infty/K) > 0$.*

(i) *For any x_∞ and y_∞ in $\mathcal{O}_{K_\infty}$, we have*

$$v_K(N_\infty(x_\infty + y_\infty) - N_\infty(x_\infty) - N_\infty(y_\infty)) \geq \frac{p-1}{p} i(K_\infty/K).$$

(ii) *For any* $x \in \mathcal{O}_K$*, there exists* $x_\infty \in \mathcal{O}_{K_\infty}$ *such that*

$$v_K(N_\infty(x_\infty) - x) \geq \frac{p-1}{p} i(K_\infty/K).$$

Proof. First, we remark that, by assumption and [12, 1.4.1 (b)], we have $K = K_0 = K_1$ and the extension K_m/K is totally ramified of degree a power of p.

(i) Let $x_\infty = (x_n)_{n=1}^\infty$ and $y_\infty = (y_n)_{n=1}^\infty$ with $x_n, y_n \in K_n$. By the definition of N_∞, we have

$$N_\infty(x_\infty + y_\infty) = \lim_{m \to \infty} N_{K_m/K}(x_m + y_m),$$
$$N_\infty(x_\infty) = N_{K_m/K}(x_m), \text{ and } N_\infty(y_\infty) = N_{K_m/K}(y_m).$$

By [12, Proposition 2.2.1 (i)], the inequality

$$v_K(N_{K_m/K}(x_m + y_m) - N_{K_m/K}(x_m) - N_m(y_m)) \geq \frac{p-1}{p} i(K_m/K)$$

holds for any $m \geq 1$. Since $i(K_m/K) \geq i(K_\infty/K)$, we have

$$v_K(N_{K_m/K}(x_m + y_m) - N_{K_m/K}(x_m) - N_m(y_m)) \geq \frac{p-1}{p} i(K_\infty/K)$$

for any $m \geq 1$. By taking the limit of the above inequality as $m \to \infty$, we obtain the required inequality.

(ii) The following proof is similar to that of [12, Proposition 2.2.1 (ii)]. We choose a uniformizer $\varpi_\infty \in \mathcal{O}_{K_\infty}$ and put $\varpi = N_\infty(\varpi_\infty)$. Then ϖ is a uniformizer of K. For any $\overline{x} \in k_K$, let $[\overline{x}]$ denote the multiplicative representative of $\overline{x}$ in $\mathcal{O}_K$. Then there exist $x(0), x(1), x(2), \ldots \in k_K$ such that

$$x = \sum_{j=0}^{\infty} [x(j)] \varpi^j.$$

For any integer $j \geq 0$, we put $x_\infty(j) = ([x(j)^{1/[K_n:K]}])_{n=1}^\infty$, which is an element of K_∞. We now define $x_\infty \in \mathcal{O}_{K_\infty}$ as

$$x_\infty = \sum_{j=0}^{\infty} x_\infty(j) \varpi_\infty^j.$$

Then by (i) we have

$$v_K(N_\infty(x_\infty) - x) \geq \frac{p-1}{p} i(K_\infty/K).$$

Therefore the x_∞ satisfies the required inequality. □

By Lemma 3.3, we find that, for any positive integer $l \leq \lceil p^{-1}(p-1)i(L/K)\rceil$, the map N_∞ gives an isomorphism

$$\alpha_{K_\infty/K}\colon \mathcal{O}_{K_\infty}/\mathfrak{p}^l_{K_\infty} \to \mathcal{O}_K/\mathfrak{p}^l_K.$$

Hence we can define an isomorphism

$$\mathfrak{N}_{K_\infty/K}\colon \mathrm{Tr}_l(K_\infty) \xrightarrow{\sim} \mathrm{Tr}_l(K)$$

and an equivalence of categories

$$\mathfrak{N}^*_{K_\infty/K} = (\mathfrak{N}_{K_\infty/K})^*_l\colon\ \mathrm{Ext}(\mathrm{Tr}_l(K))^l \to \mathrm{Ext}(\mathrm{Tr}_l(K_\infty))^l$$

in the same way as L/K is finite.

Now suppose we have an extension L'/K of infinite degree and there exists a finite separable morphism $\tau\colon L \to L'$ of K-algebras. Note that L'/K is also APF by [12, Proposition 1.2.3 (ii)]. Let $\mathscr{E}'_\tau$ denote the set of $M' \in \mathscr{E}_{L'/K}$ such that the canonical morphism

$$M' \otimes_{M'\cap\tau(L)} \tau(L) \to L'$$

is an isomorphism, which is cofinal in $\mathscr{E}_{L'/K}$. Then we can define a homomorphism

$$X_K(\tau)\colon X(L/K) \to X(L'/K)$$

by

$$(X_K(\tau)(x_\infty))_{M'\in\mathscr{E}'_\tau} = \tau(x_{\tau^{-1}(M')})$$

for any $x_\infty = (x_M)_{M\in\mathscr{E}_{L/K}} \in X(L/K)$. We can show $X_K(\tau)$ is also finite and separable [12, Théorème 3.1.2]. Now a functor $X_{L/K}$ from the category of separable extensions of L to that of K_∞ is defined as follows. For a separable extension L'/L, we put

$$X_{L/K}(L') = \varinjlim_{L''/L\colon \text{finite},\ L''\subset L'} X(L''/K),$$

where the transition map is given by $X_K(\tau)\colon X(L''_1/K) \to X(L''_2/K)$ for any homomorphism $\tau\colon L''_1 \to L''_2$ of finite separable extensions of L contained in L'. If L'/L is finite, then $X_{L/K}(L')$ is canonically isomorphic to $X_K(L')$. For separable extensions L'_1 and L'_2 of L, the map from $\mathrm{Hom}_L(L'_1, L'_2)$ to $\mathrm{Hom}_{K_\infty}(X_{L/K}(L'_1), X_{L/K}(L'_2))$ is also given by X_K. Then the functor $X_{L/K}$ gives an equivalence of categories [12, Théorème 3.2.2] and $X_{L/K}(\overline{K})$ is a separable closure of K_∞ [12, Corollaire 3.2.3], which we denote by $\overline{K}_\infty$. We put $\mathcal{G}_{K_\infty} = \mathrm{Gal}(\overline{K}_\infty/K_\infty)$ and identify $\mathcal{G}_{K_\infty}$ with the subgroup $\mathcal{G}_L$ of $\mathcal{G}_K$ via $X_{L/K}$. By [12, Corollaire 3.3.6], we have

$$\mathcal{G}_{K_\infty} \cap \mathcal{G}^u_K = W_{K_\infty} \cap \mathcal{G}^u_K = \mathcal{G}^{\psi_{L/K}(u)}_{K_\infty},$$

which is similar to the equality (3.1). By using this, we can show that the functor $\rho^l\colon \mathrm{Ext}(K)^l \to \mathrm{Ext}(K_\infty)^l$ defined by $K' \mapsto X_{L/K}(K'L)$ is well-defined in the same way as §§3.1. Now we can state the following lemma similar to Lemma 3.1:

Lemma 3.4. *Let K be a local field, $L \subset \overline{K}$ an infinite APF extension of K satisfying $i(L/K) > 0$, and l a positive integer satisfying*

$$l \leq \left\lceil \frac{p-1}{2p} i(L/K) \right\rceil .$$

Then we have the commutative diagram (up to natural equivalences)

$$\begin{array}{ccc} \mathrm{Ext}(K)^l & \xrightarrow{\ \rho^l\ } & \mathrm{Ext}(K_\infty)^l \\ \downarrow{\scriptstyle T_K^l} & & \downarrow{\scriptstyle T_{K_\infty}^l} \\ \mathrm{Ext}(\mathrm{Tr}_l(K))^l & \xrightarrow{\ \mathfrak{N}^*_{K_\infty/K}\ } & \mathrm{Ext}(\mathrm{Tr}_l(K_\infty))^l. \end{array}$$

Hence the group isomorphism

$$(\mathfrak{N}_{K_\infty/K})_*\colon \mathcal{G}_{K_\infty}/\mathcal{G}_{K_\infty}^l \to \mathcal{G}_K/\mathcal{G}_K^l$$

induced by $\mathfrak{N}_{K_\infty/K}$ coincides with the homomorphism which comes from the natural injection $\mathcal{G}_{K_\infty} \hookrightarrow \mathcal{G}_K$.

Proof. Take a finite Galois extension K'/K with $\mathrm{Gal}(K'/K)^l = \{1\}$. We put $L' = LK$ and $K'_\infty = X_{L/K}(L')$. As is the case with Lemma 3.1, we can show that $2^{-1}i(L/K)r \leq i(K'_\infty/K_\infty)$ and the projection $(K'_\infty)^\times \to (K')^\times$ gives an isomorphism of triples

$$\mathfrak{N}'\colon T^l_{K_\infty}(K'_\infty) \to \mathfrak{N}^*_{K_\infty/K} T_K(K')$$

over $\mathrm{Tr}_l(K_\infty)$, which makes the diagram

$$\begin{array}{ccccc} \mathrm{Gal}(K'_\infty/K_\infty) & \xleftarrow{\ X_K\ } & \mathrm{Gal}(L'/L) & \hookrightarrow & \mathrm{Gal}(K'/K) \\ \downarrow{\scriptstyle T^l_{K_\infty}} & & & & \downarrow{\scriptstyle T^l_K} \\ \mathrm{Aut}_{\mathrm{Tr}_l(K_\infty)}(T^l_{K_\infty}(K'_\infty)) & & & & \mathrm{Aut}_{\mathrm{Tr}_l(K)}(T^l_K(K')) \\ & \searrow^{\mathrm{ad}(\mathfrak{N}')} & & & \parallel \\ & & & & \mathrm{Aut}_{\mathrm{Tr}_l(K_\infty)}(\mathfrak{N}^*_{K_\infty/K} T_K(K')) \end{array}$$

commute. This completes our proof. □

By Lemma 3.4, a statement similar to Lemma 3.2 follows:

Lemma 3.5. *Let K be a local field, $L \subset \overline{K}$ an infinite APF extension of K satisfying $i(L/K) > 0$, and l a positive integer satisfying*

$$l \leq \left\lceil \frac{p-1}{2p} i(L/K) \right\rceil .$$

Then the restriction of L-parameters

$$\begin{aligned} \Phi_l(\mathrm{GL}_N(K)) &\to \Phi_l(\mathrm{GL}_N(K_\infty)) \\ \phi &\mapsto \phi|_{W_{K_\infty} \times \mathrm{SL}_2(\mathbb{C})} \end{aligned}$$

coincides with the map

$$\begin{aligned} (\mathfrak{N}_{K_\infty/K})_l^* \colon \ \Phi_l(\mathrm{GL}_N(K)) &\to \Phi_l(\mathrm{GL}_N(K_\infty)) \\ \phi &\mapsto \phi \circ ((\mathfrak{N}_{K_\infty/K})_* \times \mathrm{id}_{\mathrm{SL}_2(\mathbb{C})}). \end{aligned}$$

§ 4. Proof of Theorem 1.1

In this section, we shall prove the main theorem of this article (Theorem 1.1) as Theorem 4.3.

Let F be a finite extension of $\mathbb{Q}_p$ and E/F an APF extension of infinite degree. We fix an algebraic closure $\overline{F}$ of E. Let $b_0 < b_1 < b_2 < \cdots$ be the ramification breaks of E/F. For an integer $r \geq 0$, we put $F_r = \overline{F}^{\mathcal{G}_E \mathcal{G}_F^{b_r}}$. Let F_∞ denote the field of norms associated to E/F. Let n and m be positive integers or ∞ satisfying $n \leq m$. First we give a bijection from a subset of $\mathscr{A}(\mathrm{GL}_N(F_n))$ to that of $\mathscr{A}(\mathrm{GL}_N(F_m))$ as follows:

Theorem 4.1. *We take a non-decreasing sequence $\{l_n\}_{n=0}^\infty$ of non-negative integers such that $l_n \to \infty$ $(n \to \infty)$ and $l_n \leq \lceil p^{-1}(p-1) i(E/F_n) \rceil$.*

(i) *For any indices $1 \leq n \leq m \leq \infty$, there exists a natural equivalence of categories*

$$A_{m/n,l_n} \colon \ \mathrm{Rep}_{l_n}(\mathrm{GL}_N(F_n)) \xrightarrow{\sim} \mathrm{Rep}_{l_n}(\mathrm{GL}_N(F_m)).$$

This $\{A_{m/n,l_n} \mid 1 \leq n \leq m \leq \infty\}$ makes the diagram

$$\begin{array}{ccc} \mathrm{Rep}_{l_n}(\mathrm{GL}_N(F_n)) & \xrightarrow{A_{m'/n,l_n}} & \mathrm{Rep}_{l_n}(\mathrm{GL}_N(F_{m'})) \\ \downarrow{\scriptstyle A_{m/n,l_n}} & & \\ \mathrm{Rep}_{l_n}(\mathrm{GL}_N(F_m)) & & \downarrow \\ \downarrow & & \\ \mathrm{Rep}_{l_m}(\mathrm{GL}_N(F_m)) & \xrightarrow[A_{m'/m,l_m}]{} & \mathrm{Rep}_{l_m}(\mathrm{GL}_N(F_{m'})) \end{array} \tag{4.1}$$

commute (*up to natural equivalences*) *for any* $n \leq m \leq m'$. *We also denote by* $A_{m/n,l_n}$ *the bijection* $\mathscr{A}_{l_n}(\mathrm{GL}_N(F_n)) \xrightarrow{\sim} \mathscr{A}_{l_n}(\mathrm{GL}_N(F_m))$ *induced by the equivalence* $A_{m/n,l_n}$.

(ii) *If we take another sequence* $\{l'_n\}_{n=1}^{\infty}$ *satisfying the same condition on* $\{l_n\}$, *then the restrictions of* $A_{m/n,l_n}$ *and* $A_{m/n,l'_n}$ *to* $\mathscr{A}_{l_n}(\mathrm{GL}_N(F_n)) \cap \mathscr{A}_{l'_n}(\mathrm{GL}_N(F_n))$ *coincide.*

(iii) *We can take the direct limit of* $\{A_{\infty/n,l_n}\}_n$:

$$\varinjlim_n A_{\infty/n,l_n} \colon \varinjlim_n \mathscr{A}_{l_n}(\mathrm{GL}_N(F_n)) \xrightarrow{\sim} \mathscr{A}(\mathrm{GL}_N(F_\infty)),$$

which is also bijective.

Proof. (i) If $m < \infty$, we put $N_{m/n} = N_{F_m/F_n}$. Otherwise, let $N_{m/n} = N_{\infty/n}$ denote the natural projection $F_\infty^\times \to F_n^\times$ as in §§3.2. We have seen in §3 that, by the assumption $l_n \leq \lceil p^{-1}(p-1)i(E/F_n)\rceil$, the map $N_{m/n}$ induces an isomorphism of rings

$$\alpha_{m/n} \colon \mathcal{O}_{F_m}/\mathfrak{p}_{F_m}^{l_n} \xrightarrow{\sim} \mathcal{O}_{F_n}/\mathfrak{p}_{F_n}^{l_n}.$$

We fix a uniformizer ϖ_m of F_m. As we have seen in §§2.2, from the datum $\beta_{m/n} = (\alpha_{m/n}, \varpi_m, N_{m/n}(\varpi_m))$, we can construct an isomorphism of $\mathbb{C}$-algebras

$$\beta_{m/n}^* = (\alpha_{m/n}, \varpi_m, N_{m/n}(\varpi_m))^* \colon \mathscr{H}_{l_n}(\mathrm{GL}_N(F_n)) \xrightarrow{\sim} \mathscr{H}_{l_n}(\mathrm{GL}_N(F_m)),$$

which induces an equivalence of categories

$$A_{\beta_{m/n},l_n} \colon \mathrm{Rep}_{l_n}(\mathrm{GL}_N(F_n)) \xrightarrow{\sim} \mathrm{Rep}_{l_n}(\mathrm{GL}_N(F_m)).$$

We remark that $\beta_{m/n}^*$ and $A_{\beta_{m/n},l_n}$ are independent of the choice of ϖ_m. We put $A_{m/n,l_n} = A_{\beta_{m/n},l_n}$. Now we shall show the commutativity of the diagram (4.1). We take a uniformizer $\varpi_{m'}$ of $F_{m'}$. Since $\beta_{m/n}^*, \beta_{m'/n}^*$ and $\beta_{m'/m}^*$ are independent of the choice of uniformizers, we have

$$\begin{aligned}
\beta_{m/n}^* &= (\alpha_{m/n}, N_{m'/m}(\varpi_{m'}), N_{m'/n}(\varpi_{m'}))^*,\\
\beta_{m'/n}^* &= (\alpha_{m'/n}, \varpi_{m'}, N_{m'/n}(\varpi_{m'}))^* \text{ and}\\
\beta_{m'/m}^* &= (\alpha_{m'/m}, \varpi_{m'}, N_{m'/m}(\varpi_{m'}))^*.
\end{aligned}$$

Then we can show that the diagram

$$\begin{array}{ccc}
\mathscr{H}_{l_n}(\mathrm{GL}_N(F_n)) & \xrightarrow{\beta_{m/n}^*} & \mathscr{H}_{l_n}(\mathrm{GL}_N(F_m)) \\
& \searrow^{\beta_{m'/n}^*} & \downarrow \beta_{m'/m}^* \\
& & \mathscr{H}_{l_n}(\mathrm{GL}_N(F_{m'})).
\end{array}$$

is commutative. This and Lemma 2.4 show the commutativity of the diagram (4.1).

(ii) We put $l = l_n$ and $l' = l'_n$. For any s and t, we put $\mathscr{H}_s(F_t) = \mathscr{H}_s(\mathrm{GL}_N(F_t))$. We may assume $l \leq l'$. Let e_t denote the unit element of $\mathscr{H}_l(F_t)$. By Lemma 2.4, it suffices to show the diagram

$$\begin{array}{ccc} \mathrm{Mod}(\mathscr{H}_l(F_n)) & \longrightarrow & \mathrm{Mod}(\mathscr{H}_{l'}(F_n)) \\ \Big\downarrow{\scriptstyle \beta^*_{m/n,l}} & & \Big\downarrow{\scriptstyle \beta^*_{m/n,l'}} \\ \mathrm{Mod}(\mathscr{H}_l(F_m)) & \longrightarrow & \mathrm{Mod}(\mathscr{H}_{l'}(F_m)) \end{array}$$

is commutative up to natural equivalences, where the horizontal arrows are the functors given in Lemma 2.4. This follows from the isomorphism

$$\begin{aligned} &\mathscr{H}_{l'}(F_m) \otimes_{\mathscr{H}_{l'}(F_n)} \big((\mathscr{H}_{l'}(F_n) *_{l'} e_n) \otimes_{\mathscr{H}_l(F_n)} W_n\big) \\ \xrightarrow{\sim}\ & (\mathscr{H}_{l'}(F_m) *_{l'} e_m) \otimes_{\mathscr{H}_l(F_m)} \big(\mathscr{H}_l(F_m) \otimes_{\mathscr{H}_l(F_n)} W_n\big) \end{aligned}$$

given by

$$h_{l',m} \otimes ((h_{l',n} *_{l'} e_n) \otimes x) \mapsto (h_{l',m} *_{l'} \beta^*_{m/n,l'}(h_{l',n}) *_{l'} e_n) \otimes (e_m \otimes x)$$

for any left $\mathscr{H}_l(F_n)$-module W_n, $h_{l',m} \in \mathscr{H}_{l'}(F_m), h_{l',n} \in \mathscr{H}_{l'}(F_n)$, and $x \in W_n$.

(iii) The map $\varinjlim_n A_{\infty/n,l_n}$ is clearly injective. Since $l_n \to \infty$ as $n \to \infty$, we have the equality $\mathscr{A}(F_\infty) = \cup_n \mathscr{A}_{l_n}(F_\infty)$. Thus $\varinjlim_n A_{\infty/n,l_n}$ is surjective.

□

Now let E/F be a procyclic APF extension of infinite degree. For a cyclic extension F'/F of prime degree, let

$$\mathrm{BC}_{F'/F} \colon \mathscr{A}(\mathrm{GL}_N(F)) \to \mathscr{A}(\mathrm{GL}_N(F'))$$

be the base change lifting in the sense of [1, Chapter 1, Section 6]. For a general cyclic extension F'/F of finite degree, we define $\mathrm{BC}_{F'/F}$ as the composite of the base changes attached to intermediate extensions of F'/F of prime degree. We write $\mathrm{BC}_{F_m/F_n} = \mathrm{BC}_{m/n}$ and $\mathrm{BC}_n = \mathrm{BC}_{F_n/F}$.

Theorem 4.2. *We take a sequence $\{l_n\}_{n=0}^{\infty}$ satisfying the condition in Theorem 4.1. Further assume that there exists a positive integer n_0 such that, for any $n \geq n_0$,*

$$l_n < \frac{1}{2^{N-1}} \left\lceil \frac{p-1}{p} i(E/F_n) \right\rceil.$$

(i) *For any indices $n_0 \leq n \leq m < \infty$, the bijection*

$$A_{m/n,l_n} \colon \mathscr{A}_{l_n}(\mathrm{GL}_N(F_n)) \xrightarrow{\sim} \mathscr{A}_{l_n}(\mathrm{GL}_N(F_m))$$

coincides with the restriction of $\mathrm{BC}_{m/n}$ *to* $\mathscr{A}_{l_n}(\mathrm{GL}_N(F_n))$.

(ii) *For any* $\pi \in \mathscr{A}(\mathrm{GL}_N(F))$, *there exists an integer* $n \geq 1$ *such that*

$$\mathrm{BC}_n(\pi) \in \mathscr{A}_{l_n}(\mathrm{GL}_N(F_n)).$$

Proof. (i) The case $N = 1$ follows by the definition of $A_{m/n,l_n}$. We assume $N \geq 2$. Take a uniformizer ϖ_∞ of F_∞. Note that, the maps $A_{m/n,l_n}$ in Theorem 4.1 and $(\mathfrak{N}_{F_m/F_n})^*_{l_n}$ in §§3.1 are induced by the same datum

$$\beta_{m/n} = (\alpha_{m/n}, N_{\infty/m}(\varpi_\infty), N_{\infty/n}(\varpi_\infty)),$$

where $\alpha_{m/n}$ is the same as in the proof of Theorem 4.1 (i). By applying Theorem 2.8 to

$$(K(1), K(2), l, l', \beta) = (F_n, F_m, \lceil p^{-1}(p-1)i(E/F_n)\rceil, l_n, \beta_{m/n}),$$

we see that $A^*_{m/n,l_n}$ is compatible with $(\mathfrak{N}_{F_m/F_n})^*_{l_n}$ via LLC. Now we have inequalities $l_n < 2^{-N+1}\lceil p^{-1}(p-1)i(E/F_n)\rceil \leq \lceil (2p)^{-1}(p-1)i(E/F_n)\rceil$. Hence, by Lemma 3.2, the map $(\mathfrak{N}_{F_m/F_n})^*_{l_n}$ coincides with the map induced by the restriction $W_{F_m} \hookrightarrow W_{F_n}$. Since the latter map is compatible with $\mathrm{BC}_{m/n}$ via LLC, we have completed the proof.

(ii) By Lemma 2.6, this is also reduced to showing the corresponding assertion on Galois representations. Thus we shall show that for any $\phi \in \Phi(\mathrm{GL}_N(F))$ there exists n such that $\phi|_{W_{F_n}} \in \Phi_{l_n}(\mathrm{GL}_N(F_n))$. Take any $\phi \in \Phi(\mathrm{GL}_N(F))$. Then we have $\phi \in \Phi_l(\mathrm{GL}_N(F))$ for some l. By the equality (3.1) in Section 3, we have

$$W_{F_n} \cap \mathcal{G}_F^l = \mathcal{G}_{F_n}^{\psi_{F_n/F}(l)}$$

for any n. Since $\psi_{F_n/F}(l) \leq \psi_{E/F_n}(\psi_{F_n/F}(l)) = \psi_{E/F}(l)$, we have

$$\mathcal{G}_{F_n}^{\psi_{F_n/F}(l)} \supset \mathcal{G}_{F_n}^{\psi_{E/F}(l)}.$$

Hence we obtain

$$W_{F_n} \cap \mathcal{G}_F^l \supset \mathcal{G}_{F_n}^{\psi_{E/F}(l)}.$$

Since $l_n \to \infty$ as $n \to \infty$, there exists an integer n such that $\psi_{E/F}(l) \leq l_n$. Thus $\phi|_{W_{F_n}}$ is trivial on $\mathcal{G}_{F_n}^{l_n}$ i.e. $\phi|_{W_{F_n} \times \mathrm{SL}_2(\mathbb{C})} \in \Phi_{l_n}(\mathrm{GL}_N(F_n))$, as claimed. □

Now we prove the following main theorem:

Theorem 4.3. *Let* F *be a finite extension of* $\mathbb{Q}_p$ *and* E/F *a procyclic APF extension. Then we can construct a map* $\mathrm{BC}_\infty\colon \mathscr{A}(\mathrm{GL}_N(F)) \to \mathscr{A}(\mathrm{GL}_N(F_\infty))$ *such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathscr{A}(\mathrm{GL}_N(F_\infty)) & \xrightarrow{\mathrm{LLC}_{F_\infty}} & \Phi(\mathrm{GL}_N(F_\infty)) \\ \uparrow{\scriptstyle \mathrm{BC}_\infty} & & \uparrow{\scriptstyle \mathrm{Res}_\infty} \\ \mathscr{A}(\mathrm{GL}_N(F)) & \xrightarrow{\mathrm{LLC}_F} & \Phi(\mathrm{GL}_N(F)). \end{array} \tag{4.2}$$

Proof. First assume $N \geq 2$. We take $\{l_n\}$ as in Theorem 4.2. We define

$$\mathrm{BC}_\infty \colon \mathscr{A}(\mathrm{GL}_N(F)) \to \mathscr{A}(\mathrm{GL}_N(F_\infty))$$

by mapping π to $A_{\infty/n,l_n}(\mathrm{BC}_n(\pi))$, where the n is as in Theorem 4.2 (ii). By Theorem 4.2 (i), $\mathrm{BC}_\infty(\pi)$ is independent of the choice of n.

We shall show BC_∞ is also independent of the choice of $\{l_n\}$. We take another sequence $\{l'_n\}_{n=0}^\infty$ which satisfies the conditions in Theorem 4.2. Then there exists an positive integer n such that $\mathrm{BC}_n(\pi) \in \mathscr{A}_{l_n}(\mathrm{GL}_N(F_n)) \cap \mathscr{A}_{l'_n}(\mathrm{GL}_N(F_n))$. By Theorem 4.1 (ii), we have $A_{\infty/n,l_n} \circ \mathrm{BC}_n(\pi) = A_{\infty/n,l'_n} \circ \mathrm{BC}_n(\pi)$. As a consequence, we have proved that $\mathrm{BC}_\infty(\pi) = A_{\infty/n,l_n}(\mathrm{BC}_n(\pi))$ is independent both of n and $\{l_n\}$.

We shall prove the commutativity of the diagram (4.2). Let F' be a finite extension of F contained in $\overline{F}$. We define $\mathrm{Res}_{F'/F} \colon \Phi(\mathrm{GL}_N(F)) \to \Phi(\mathrm{GL}_N(F'))$ by $\mathrm{Res}_{F'/F}(\phi) = \phi|_{W_{F'} \times \mathrm{SL}_2(\mathbb{C})}$ for any $\phi \in \Phi(\mathrm{GL}_N(F))$. If F' is contained in E, we define $\mathrm{Res}_{F_\infty/F'} \colon \Phi(\mathrm{GL}_N(F')) \to \Phi(\mathrm{GL}_N(F_\infty))$ by $\mathrm{Res}_{F_\infty/F'}(\phi') = \phi'|_{W_{F_\infty} \times \mathrm{SL}_2(\mathbb{C})}$ for any $\phi' \in \Phi(\mathrm{GL}_N(F'))$.

Now take a uniformizer ϖ_∞ of F_∞. Note that, the maps $A_{\infty/n,l_n}$ in Theorem 4.1 and $(\mathfrak{N}_{F_\infty/F_n})^*_{l_n}$ in §§3.2 are induced by the same datum $(\alpha_{\infty/n}, \varpi_\infty, N_{\infty/n}(\varpi_\infty))$, which we denote by $\beta_{\infty/n}$, where $\alpha_{\infty/n}$ is the same as in the proof of Theorem 4.1 (i). By applying Theorem 2.8 to

$$(K(1), K(2), l, l', \beta) = (F_n, F_\infty, \lceil p^{-1}(p-1)i(E/F_n)\rceil, l_n, \beta_{\infty/n}),$$

we have

$$\begin{aligned}\mathrm{LLC}_{F_\infty}(\mathrm{BC}_\infty(\pi)) &= \mathrm{LLC}_{F_\infty}(A_{\infty/n,l_n}(\mathrm{BC}_n(\pi))) \\ &= (\mathfrak{N}_{F_\infty/F_n})^*_{l_n}(\mathrm{LLC}_{F_n}(\mathrm{BC}_n(\pi))).\end{aligned}$$

Since $(\mathfrak{N}_{F_\infty/F_n})^*_{l_n}$ and $\mathrm{Res}_{F_\infty/F_n}$ coincide on $\Phi_{l_n}(\mathrm{GL}_N(F_n))$ by Lemma 3.5, we have

$$\begin{aligned}(\mathfrak{N}_{F_\infty/F_n})^*_{l_n}(\mathrm{LLC}_{F_n}(\mathrm{BC}_n(\pi))) &= \mathrm{Res}_{F_\infty/F_n}(\mathrm{LLC}_{F_n}(\mathrm{BC}_n(\pi))) \\ &= \mathrm{Res}_{F_\infty/F_n} \circ \mathrm{Res}_{F_n/F}(\mathrm{LLC}_F(\pi)) \\ &= \mathrm{Res}_{F_\infty/F}(\mathrm{LLC}_F(\pi)).\end{aligned}$$

Hence we obtain $\mathrm{LLC}_{F_\infty}(\mathrm{BC}_\infty(\pi)) = \mathrm{Res}_{F_\infty/F}(\mathrm{LLC}_F(\pi))$, which shows the commutativity of the diagram (4.2).

The case $N = 1$ is similar to the above, except that we must choose $\{l_n\}_{n=0}^\infty$ satisfying the conditions in Theorem 4.1 and $l_n < \lceil (p-1)(2p)^{-1}i(E/F_n)\rceil$ for all sufficiently large n in order to apply Lemma 3.5. □

Remark 4.4. As we have noted after Theorem 1.1, the proof of the independence of the choice of n relies on LLC. However, that of $\{l_n\}$ does not need LLC.

§ 5. Proof of Theorem 1.2

Finally, we prove Theorem 1.2. We recall some notation. For a procyclic group Γ, we denote by $\widehat{\Gamma}$ the group of smooth characters of Γ with valued in $\mathbb{C}^\times$. For a positive integer μ and $(\eta_1, \dots, \eta_\mu) \in \widehat{\Gamma}^\mu$, we denote by $\widehat{\Gamma}(\eta_1, \dots, \eta_r)$ the quotient of $\widehat{\Gamma}^\mu$ by the following equivalence relation: Two elements $(\xi_1, \dots, \xi_\mu)$ and $(\theta_1, \dots, \theta_\mu)$ in $\widehat{\Gamma}^\mu$ are equivalent if there exists a permutation σ of $\{1, \dots, \mu\}$ such that $\eta_j \xi_j = \eta_{\sigma(j)} \theta_{\sigma(j)}$ for each j. Now we recall the statement of Theorem 1.2 as follows:

Theorem 5.1. *Let F be a finite extension of $\mathbb{Q}_p$ and E/F a procyclic APF extension of infinite degree with the Galois group Γ. We suppose that $(p, N) = 1$.*

(i) *Let $\pi \in \mathscr{A}(\mathrm{GL}_N(F))$ be an essentially square-integrable representation. We put $\pi_\infty = \mathrm{BC}_\infty(\pi)$. Let ω_∞ denote the central character of π_∞. Then $\mathrm{BC}_\infty^{-1}(\pi_\infty)$ has a natural $\widehat{\Gamma}$-torsor structure and the map*

$$\omega \colon \mathrm{BC}_\infty^{-1}(\pi_\infty) \to \mathrm{BC}_\infty^{-1}(\omega_\infty)$$

which maps π' to its central character $\omega_{\pi'}$ is bijective.

(ii) (a) *Let π be any element of $\mathscr{A}(\mathrm{GL}_N(F))$. We suppose that $p > N$. There exist a positive integer r, positive integers N_i, μ_i and an essentially square-integrable representation $\pi_i \in \mathscr{A}(\mathrm{GL}_{N_i}(F))$ for each $i = 1, 2, \dots, r$, and an element $\eta_{i,j} \in \widehat{\Gamma}$ for each $1 \le i \le r$ and $2 \le j \le \mu_i$ satisfying the following conditions:*

* *the equality $\mu_1 N_1 + \cdots + \mu_r N_r = N$ holds,*
* *the lifts $\mathrm{BC}_\infty(\pi_1), \dots, \mathrm{BC}_\infty(\pi_r)$ are all distinct, and*
* *we can write*

$$\begin{aligned} \pi = &\pi_1 \boxplus (\pi_1 \otimes \eta_{1,2}) \boxplus \cdots \boxplus (\pi_1 \otimes \eta_{1,\mu_1}) \\ &\boxplus \cdots \\ &\boxplus \pi_r \boxplus (\pi_r \otimes \eta_{r,2}) \boxplus \cdots \boxplus (\pi_r \otimes \eta_{r,\mu_r}). \end{aligned}$$

(b) *Under the notation of (a), the group $\widehat{\Gamma}(\pi) = \widehat{\Gamma}^{\mu_1} \times \cdots \times \widehat{\Gamma}^{\mu_r}$ transitively acts on $\mathrm{BC}_\infty^{-1}(\pi_\infty)$. As a homogeneous space of $\widehat{\Gamma}(\pi)$, this is isomorphic to*

$$\widehat{\Gamma}(1, \eta_{1,2}, \dots, \eta_{1,\mu_1}) \times \cdots \times \widehat{\Gamma}(1, \eta_{r,2}, \dots, \eta_{r,\mu_r}).$$

Proof. We take $\{l_n\}_{n=1}^\infty$ as in Theorem 4.2. First, we show (i) for a supercuspidal π. We put $\pi_\infty = \mathrm{BC}_\infty(\pi)$. The fiber $\mathrm{BC}_\infty^{-1}(\pi_\infty)$ has a $\widehat{\Gamma}$-set structure via $\pi' \mapsto \pi' \otimes \eta$, where $\pi' \in \mathrm{BC}_\infty^{-1}(\pi_\infty)$ and $\eta \in \widehat{\Gamma}$. We shall show that this action is simply transitive. The assumption $(p, N) = 1$ shows that the $\widehat{\Gamma}$-action is simple. Let us prove the transitivity. We take any $\pi' \in \mathrm{BC}_\infty^{-1}(\pi_\infty)$. By Theorem 4.2 (ii), we can take an integer n such that both $\mathrm{BC}_n(\pi)$ and $\mathrm{BC}_n(\pi')$ belong to $\mathscr{A}_{l_n}(\mathrm{GL}_N(F_n))$. Since $\mathrm{BC}_\infty = A_{\infty/n,l_n} \circ \mathrm{BC}_n$ and $A_{\infty/n,l_n}$ is injective, we have $\mathrm{BC}_n(\pi) = \mathrm{BC}_n(\pi')$. It suffices to show that there exists a smooth character $\eta\colon F^\times \to \mathbb{C}^\times$ which factors through $F^\times/N_{F_n/F}(F_n^\times)$ such that $\pi' \simeq \pi \otimes \eta$. We show this by induction on n. The case $n = 1$ is [1, Chapter 1, Proposition 6.7]. We assume that the assertion holds for $n-1$. By the case $n = 1$, we can find a smooth character $\eta_1\colon F_{n-1}^\times \to \mathbb{C}^\times$ which factors through $F_{n-1}^\times/N_{F_n/F_{n-1}}(F_n^\times)$ and satisfies $\mathrm{BC}_{n-1}(\pi') \simeq \mathrm{BC}_{n-1}(\pi) \otimes \eta_1$. Let ω_π (resp. $\omega_{\pi'}$) denote the central character of π (resp. π'). Then we have

$$\omega_{\pi'} \circ N_{F_{n-1}/F} = (\omega_\pi \circ N_{F_{n-1}/F})\eta_1^N.$$

Thus we obtain

$$\eta_1^N = (\omega_{\pi'}\omega_\pi^{-1}) \circ N_{F_{n-1}/F}.$$

By the assumption that $(p, N) = 1$, we find a character η_1' on $F^\times$ such that

$$\eta_1 = \eta_1' \circ N_{F_{n-1}/F}.$$

Hence we have $\mathrm{BC}_{n-1}(\pi') \simeq \mathrm{BC}_{n-1}(\pi \otimes \eta_1')$ and by the induction hypothesis there exists a smooth character η_{n-1} on $F^\times$ which is trivial on $N_{F_{n-1}/F}(F_{n-1}^\times)$ and satisfies $\pi' \simeq \pi \otimes (\eta_1'\eta_{n-1})$. Then $\eta = \eta_1'\eta_{n-1}$ is the requested character.

We define a map $\mathrm{BC}_\infty^{-1}(\pi_\infty) \to \mathrm{BC}_\infty^{-1}(\omega_\infty)$ by taking the central character. This maps $\pi \otimes \eta$ to $\omega_\pi \eta^N$. By the assumption $(p, N) = 1$, it is bijective.

Now we show (i) for any essentially square-integrable π. Then there exist a unique divisor m of N and a unique supercuspidal representation $\sigma \in \mathscr{A}(\mathrm{GL}_{N/m}(F))$ such that π is equivalent to the unique irreducible quotient $\mathrm{St}_m(\sigma)$ of

$$\text{n-Ind}_{P(N/m,\dots,N/m)}^{\mathrm{GL}_N(F)}(\sigma \otimes |\det|^{(1-m)/2} \boxtimes \cdots \boxtimes \sigma \otimes |\det|^{(m-1)/2})$$

([13, Theorem 9.3]). We put $\sigma_\infty = \mathrm{BC}_\infty(\sigma)$. Let us show that the map

$$\begin{aligned}\mathrm{BC}_\infty^{-1}(\sigma_\infty) &\to \mathrm{BC}_\infty^{-1}(\pi_\infty)\\ \sigma' &\mapsto \mathrm{St}_m(\sigma')\end{aligned}$$

is bijective. Its well-definedness follows from [1, Lemma 6.12], [3, Théorème 2.17 (c)] and [7, Proposition A.4.1]. Its injectivity follows from the uniqueness of the expression $\mathrm{St}_m(\sigma')$. We show its surjectivity. Take any $\pi' \in \mathrm{BC}_\infty^{-1}(\pi_\infty)$. Then by [3, Théorème 2.17 (c)] and [7, Proposition A.4.1], we have $\mathrm{BC}_n(\pi') = \mathrm{St}_m(\mathrm{BC}_n(\sigma))$ for some n. Since π' is essentially square-integrable, there exists a divisor m' of N and a supercuspidal representation $\sigma' \in \mathscr{A}(\mathrm{GL}_{N/m'}(F))$ such that $\pi' = \mathrm{St}_{m'}(\sigma')$. The assumption $(p, N) =$

1 and [1, Lemma 6.12] show that $\mathrm{BC}_n(\pi') = \mathrm{St}_{m'}(\mathrm{BC}_n(\sigma'))$. Hence we have $m' = m$ and $\sigma' \in \mathrm{BC}_n^{-1}(\mathrm{BC}_n(\sigma)) \subset \mathrm{BC}_\infty^{-1}(\sigma_\infty)$. Therefore the surjectivity follows, as claimed.

The statement (ii) follows from the uniqueness of the Langlands sum and the fact that the functor $A_{\infty/n,l_n}$ preserves the Langlands sum ([7, Proposition A.4.1]). □

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