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Semi-continuity of total dimension divisors for ℓ-adic sheaves (research announcement)

By

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Abstract

This is an announcement of the joint article [HY] with E. Yang, based on a talk given in December 2015 at RIMS. In this article, we give a generalization of Deligne and Laumon's lower semi-continuity property for Swan conductors of ℓ-adic sheaves on relative curves to higher relative dimensions in a geometric situation. We outline the main results and ideas in this report.

§ 1. Semi-continuity of Swan conductors

Let S be an excellent noetherian scheme, $f\colon X \to S$ a separated and smooth morphism of relative dimension 1, D a closed subscheme of X which is finite and flat over S, $U = X - D$ the complement and $j : U \to X$ the canonical injection. Let ℓ be a prime number invertible in S, Λ a finite field of characteristic ℓ and $\mathcal{F}$ a locally constant and constructible sheaf of Λ-modules on U.

Let s be a point of S. We say that a geometric point $\bar{s}$ of S above s is *algebraic* if $\bar{s}$ is the spectrum of an algebraic closure of $k(s)$. We denote by $X_{\bar{s}}$ and $D_{\bar{s}}$ the fibers of $f : X \to S$ and $f|_D : D \to S$ at an algebraic geometric point $\bar{s}$ of S, respectively. For each point $x \in D_{\bar{s}}$, we denote by $\mathrm{Sw}_x(j_!\mathcal{F}|_{X_{\bar{s}}})$ the classical Swan conductor of the sheaf $j_!\mathcal{F}|_{X_{\bar{s}}}$ at x, which is an integer number [6]. We define the *total dimension* of $j_!\mathcal{F}|_{X_{\bar{s}}}$ at x the sum of $\mathrm{Sw}_x(j_!\mathcal{F}|_{X_{\bar{s}}})$ and $\mathrm{rank}(\mathcal{F})$ and denote it by $\mathrm{dimtot}_x(j_!\mathcal{F}|_{X_{\bar{s}}})$. The sum

$$\sum_{x\in D_{\bar{s}}} \mathrm{dimtot}_x(j_!\mathcal{F}|_{X_{\bar{s}}}) \tag{1.1}$$

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is independent of the choice of $\bar{s}$ above s. It defines a function $\varphi\colon S \to \mathbb{Z}$. The semi-continuity property of Swan conductors of Deligne and Laumon is the following:

Theorem 1.1 ([3, 2.1.1]). *We take the notation and assumptions above.*

(1) *The function $\varphi\colon S \to \mathbb{Z}$ is constructible and lower semi-continuous.*

(2) *The morphism $f\colon X \to S$ is universally locally acyclic with respect to $j_!\mathcal{F}$ if $\varphi\colon S \to \mathbb{Z}$ is locally constant.*

If the morphism $f\colon X \to S$ has relative dimension ≥ 1, Saito generalized the semi-continuity property (Theorem 1.1) in terms of the total dimension of vanishing cycles in the following way:

Let k be a perfect field of characteristic $p > 0$, $f\colon X \to S$ a morphism of k-schemes of finite type that factors through a k-scheme Y, D a closed subset of X and $U = X - D$ the complement. We put $h\colon X \to Y$ and put $g\colon Y \to S$. We assume that $g\colon Y \to S$ is a smooth relative curve and that D is closed subset of X which is quasi-finite over S. Let ℓ be a prime number different form p, $\mathcal{K}$ a constructible sheaf of $\mathbb{F}_\ell$-modules on X. We assume that $f\colon X \to S$ is locally acyclic with respect to $\mathcal{K}$ and $h|_U\colon U \to Y$ is locally acyclic with respect to $\mathcal{K}|_U$. Let $\bar{s}$ be an algebraic geometric point of S and $h_{\bar{s}}\colon X_{\bar{s}} \to Y_{\bar{s}}$ the fiber of $h\colon X \to Y$ at $\bar{s}$. Notice that $Y_{\bar{s}}$ is a smooth curve over an algebraically closed field. Let x be a closed point of $D_{\bar{s}}$ and we denote by $\phi_x(\mathcal{K}|_{X_{\bar{s}}}, h_{\bar{s}})$ the stalk of the vanishing cycle complex at x. We denote by

$$\operatorname{dimtot}(\phi_x(\mathcal{K}|_{X_{\bar{s}}}, h_{\bar{s}})) \tag{1.2}$$

the alternating sum of the total dimension of every cohomology of $\phi_x(\mathcal{K}|_{X_{\bar{s}}}, h_{\bar{s}})$. It defines a function $\varphi_{\mathcal{K},h} : D \to \mathbb{Z}$.

Theorem 1.2 ([5, Proposition 2.16]). *The function $\varphi_{\mathcal{K},h} : D \to \mathbb{Z}$ is constructible. If $f|_D : D \to S$ is étale, the function*

$$h_*(\varphi_{\mathcal{K},h})\colon S \to \mathbb{Z}, \quad s \mapsto \sum_{x \in D_{\bar{s}}} \varphi_{\mathcal{K},h}(x) \tag{1.3}$$

is locally constant on S.

Theorem 1.2 is used to define the coefficients of characteristic cycles of ℓ-adic sheaves (cf. [5]). In [HY], we give another generalization of Theorem 1.1. We consider that D is Cartier divisor of X relative to S. When $f\colon X \to S$ has higher relative dimension, we replace the target of the function $\varphi : S \to \mathbb{Z}$ in Theorem 1.1 by a family of divisors on the fibers of $f : X \to S$ that we call the *total dimension divisors*. We formulate our main result by a semi-continuity property of this family of divisors. First of all, we recall the ramification theory of Abbes and Saito which is used to define the notion of total dimension divisor.

§ 2. Ramification theory of Abbes and Saito

In the following of this report, we fix a prime number $p > 0$, an algebraically closed field k of characteristic p and a prime number ℓ different from p.

In this section, let K be a complete discrete valuation field of characteristic $p > 0$, $\overline{K}$ a separable closure of K, G_K the Galois group of $\overline{K}$ over K, $\mathcal{O}_K$ the integer ring of K and F the residue field of $\mathcal{O}_K$. Abbes and Saito defined a decreasing filtration $\{G_K^r\}_{r\in\mathbb{Q}_{\geq 1}}$ of G_K which is called the *ramification filtration* [1]. For each $r \geq 1$, we put $G_K^{r+} = \overline{\bigcup_{s>r} G_K^s}$. Then $\{G_K^r\}_{r\in\mathbb{Q}_{\geq 1}}$ has the following properties (cf. [1, 2, 4]):

(i) G_K^1 is the inertia subgroup of G_K;

(ii) G_K^{1+} is wild inertia subgroup of G_K;

(iii) for each $r \in \mathbb{Q}_{>1}$, the quotient G_K^r/G_K^{r+} is abelian and killed by p;

(iv) if the residue field F is perfect, the ramification filtration coincides with the canonical upper numbering filtration shifted by one.

Let M be a finitely generated $\mathbb{F}_\ell$-module with a continuous G_K-action. Then, the module M has a decomposition $M = \bigoplus_{r\geq 1} M^{(r)}$, such that $M^{(1)} = M^{G_K^{1+}}$ and that, for each $r > 1$,

$$(M^{(r)})^{G_K^r} = 0 \quad \text{and} \quad (M^{(r)})^{G_K^{r+}} = M^{(r)}. \tag{2.1}$$

The decomposition is called the *slope* decomposition. The *total dimension* of M is defined by

$$\mathrm{dimtot}_K M = \sum_r r \cdot \dim_{\mathbb{F}_\ell} M^{(r)} \tag{2.2}$$

It coincides with the classical total dimension (§1) if the residue field F is perfect (cf. (iv)).

Let Y be a smooth k-scheme, E a reduced Cartier divisor of Y, $\{E_i\}_{i\in I}$ the set of irreducible components of E, $V = Y - E$ the complement and $u : V \to Y$ the canonical injection. We denote by ξ_i a geometric generic point of E_i $(i \in I)$, by $Y_{(\xi_i)}$ the strict localization of Y at ξ_i, by K_i the fraction field of $Y_{(\xi_i)}$, by $\overline{K}_i$ an separable closure of K_i and by η_i the generic point of $Y_{(\xi_i)}$. Notice that $Y_{(\xi_i)}$ is a spectrum of henselian discrete valuation ring. Let $\mathcal{G}$ be a locally constant and constructible sheaf of $\mathbb{F}_\ell$-modules on V. The restriction $\mathcal{G}|_{\eta_i}$ associates to a finitely generated $\mathbb{F}_\ell$-module with continuous $\mathrm{Gal}(\overline{K}_i/K_i)$-action. We define the *total dimension divisor* $\mathrm{DT}_Y(u_!\mathcal{G})$ of $u_!\mathcal{G}$ by

$$\mathrm{DT}_Y(u_!\mathcal{G}) = \sum_{i\in I} \mathrm{dimtot}_{K_i}(\mathcal{G}|_{\eta_i}) \cdot E_i \tag{2.3}$$

It has integer coefficients (cf. [4, Proposition 3.10]). When Y is a smooth curve, the degree of $\mathrm{DT}_Y(u_!\mathcal{G})$ is the sum of total dimensions of $\mathcal{G}$ at critical points (cf. (1.1)).

§ 3. Main results

Let S be an irreducible k-scheme of finite type, $f\colon X\to S$ a smooth morphism of finite type, $\{D\}_{i\in I}$ a set of effective Cartier divisors of X relative to S, D the sum of all D_i $(i\in I)$, $U=X-D$ the complement and $j\colon U\to X$ the canonical injection. For each $i\in I$, we assume that D_i is irreducible and $f|_{D_i}:D_i\to S$ is smooth. For an algebraic geometric point $\bar{t}\to S$, we denote by $D_{\bar{t}}$ the pull-back of the relative Cartier divisor D on the smooth scheme $X_{\bar{t}}$. Let $\mathcal{F}$ be a locally constant and constructible sheaf of $\mathbb{F}_\ell$-modules on U.

The main result of [HY] is the following theorem:

Theorem 3.1 ([HY, Theorem 4.5])**.** *Let $\bar{\eta}$ be an algebraic geometric generic point of S. Let R be the unique Cartier divisor of X supported on D such that $R_{\bar{\eta}}=\mathrm{DT}_{X_{\bar{\eta}}}(j_!\mathcal{F}|_{X_{\bar{\eta}}})$. Then,*

1. *For each algebraic geometric point $\bar{t}\to S$, the difference $R_{\bar{t}}-\mathrm{DT}_{X_{\bar{t}}}(j_!\mathcal{F}|_{X_{\bar{t}}})$ is an effective Cartier divisor on $X_{\bar{t}}$.*
2. *There exists an open dense subset W of S such that, for any algebraic geometric point $\bar{t}\to W$, we have $R_{\bar{t}}=\mathrm{DT}_{X_{\bar{t}}}(j_!\mathcal{F}|_{X_{\bar{t}}})$.*

The first step of proving Theorem 3.1 is the following proposition that compares the pull-back of the total dimension divisor of an ℓ-adic sheaf and the total dimension divisor of the pull-back of the sheaf.

Proposition 3.2 ([HY, Proposition 4.2])**.** *Let Y be a smooth k-scheme, E an reduced Cartier divisor on Y, $V=Y-E$ the complement, $u:V\to Y$ the canonical injection and $\mathcal{G}$ a locally constant and constructible sheaf of $\mathbb{F}_\ell$-modules on V. Let C be a smooth k-curve and $\iota:C\to Y$ an immersion. We assume that C intersects E properly at a closed point $y\in Y$. Then, we have*

$$(\mathrm{DT}_Y(u_!\mathcal{G}),C)_y\geq \mathrm{dimtot}_y(u_!\mathcal{G}|_C). \tag{3.1}$$

This inequality extends a similar one due to Saito [4, Proposition 3.9], where we need extra conditions that E is smooth at x and that the ramification of $\mathcal{G}$ along E behaves well at x. The second step is to reduce Theorem 3.1 to the relative curve case using Proposition 3.2 and a part of [4, Proposition 3.9] saying that (3.1) is an equality under some mild ramification and transversal conditions. Using Theorem 1.1, we obtain the following two properties:

(i) For any closed point $t \in S$, the difference $R_t - \mathrm{DT}_{X_t}(j_!\mathcal{F}|_{X_t})$ is an effective divisor on X_t;

(ii) There exists an open dense subset W of S such that, for any closed point $t \in W$, the difference $\mathrm{DT}_{X_t}(j_!\mathcal{F}|_{X_t}) - R_t$ is an effective divisor on X_t.

Finally, we obtain Theorem 3.1 as a consequence of properties (i) and (ii).

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