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# Affinoids in the Lubin-Tate perfectoid space and special cases of the local Langlands correspondence in positive characteristic (announcement)

By

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## Abstract

This is a research announcement of [To16], where, following the work of Weinstein, Boyarchenko-Weinstein and Imai-Tsushima, we construct a family of affinoids in the Lubin-Tate perfectoid space and formal models such that the $\ell$-adic cohomology groups of the reductions of the formal models realize the local Langlands correspondence for $\mathrm{GL}(n)$ and the local Jacquet-Langlands correspondence for certain representations related to totally tamely ramified extensions of the base field of degree $n$. Unlike the preceding work, the base field is assumed to be of positive characteristic in [To16].

## Introduction

This is a research announcement of [To16] and is based on the author's talk *Affinoids in the Lubin-Tate perfectoid space and some cases of the local Langlands correspondence* at the RIMS Workshop Algebraic Number Theory and Related Topics 2015.

Let $K$ be a non-archimedean local field with finite residue field $k$. (For now, we make no assumption on the characteristic of $K$.) Let $n$ be a positive integer. Denote by $D$ the central division algebra over $K$ of invariant $1/n$ and by $W_K$ the Weil group of $K$. A projective system of $p$-adic analytic spaces, called *the Lubin-Tate spaces*, admits a natural action of (a large subgroup $G^0$ of) $G = \mathrm{GL}_n(K) \times D^\times \times W_K$ and therefore (the induction of) the inductive limit of the $\ell$-adic cohomology groups of the Lubin-Tate spaces affords a representation of $G$. The non-abelian Lubin-Tate theory

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asserts that the decomposition of this representation is described by *the local Langlands correspondence* for $\mathrm{GL}_n(K)$ and *the local Jacquet-Langlands correspondence.* This is often simply expressed as "the $\ell$-adic cohomology of the Lubin-Tate tower realizes the local Langlands correspondence and the local Jacquet-Langlands correspondence."

However, the known proofs of the non-abelian Lubin-Tate theory, due to [Bo99] and [HT01], are of global nature, and thus the geometry of the Lubin-Tate spaces and its relation to representations are not yet fully understood. The main result of [To16] is concerned with a question related to detailed studies of the geometry of the Lubin-Tate spaces.

A result [Yo10] of Yoshida is an example of such studies. There he constructed a semistable model of a Lubin-Tate space and showed that an open subscheme of the reduction is isomorphic to a Deligne-Lusztig variety of $\mathrm{GL}_n(k)$. Since the $\ell$-adic cohomology group of the Deligne-Lusztig variety produces irreducible cuspidal representations of $\mathrm{GL}_n(k)$ (the Deligne-Lusztig theory) and irreducible supercuspidal representations of $\mathrm{GL}_n(K)$ *of depth zero* are, in turn, known to be constructed from such representations of $\mathrm{GL}_n(k)$ via a representation-theoretic procedure (a special case of the theory of types), this result can be seen as revealing a relation between the geometry of the Lubin-Tate tower and the representation theory, for representations of $\mathrm{GL}_n(K)$ of depth zero.

While the main theorem deals with a similar subject for more complicated representations, the setting is different from that of Yoshida in two ways.

One is that we construct affinoids (and formal models) instead of semistable models. In fact, the Deligne-Lusztig variety studied in [Yo10] can also be obtained as the reduction of (the canonical formal model of) an affinoid subspace of the Lubin-Tate space. It is often easier to construct affinoids with interesting smooth reductions than constructing a semistable model of the whole space. See [We10], [IT17a], [IT17b] for such studies of Lubin-Tate spaces in various different settings.[1]

The other difference is the use of *the Lubin-Tate perfectoid space*, which is a certain limit space of the Lubin-Tate tower. In this limit the defining equation simplifies and the group actions can be made more explicit.

Thus, in rough terms, the main theorem of [To16] asserts the existence of a family of affinoids[2] in the Lubin-Tate perfectoid space and formal models such that the $\ell$-adic cohomology groups of the reductions of the formal models realize the local Langlands and Jacquet-Langlands correspondences for certain representations related to totally tamely ramified extensions of $K$ of degree $n$. For some technical reason, the author had to assume, unlike preceding work [We16], [BW16], [IT15, IT16], that $K$ is of positive

[1] Note, however, that (under the assumption that $n = 2$) Imai and Tsushima construct not only affinoids, but also a stable model in [IT17a], using the theory of semistable coverings.

[2] The Lubin-Tate perfectoid space is an adic space and accordingly we work with adic spaces. Here and in the rest of the paper, by "an affinoid" we mean "an open affinoid adic subspace." (See [Hu94, 2 Definition.] for the definition of affinoid adic spaces.)

characteristic. So far, he has not been able to extend the result to include mixed-characteristic cases.

In Section 1 we briefly review some facts on the Lubin-Tate perfectoid space and formulate the question with which the main theorem is concerned. In Section 2 we state the main theorem and also compare it with the preceding results [BW16], [IT15, IT16], [We16]. In Section 3 we give some remarks on the (largely computational) proof.

**Notation and convention** As in the introduction above, we fix a non-archimedean local field $K$ with finite residue field $k$.[3] We denote by $p$ the characteristic of $k$ and take a prime number $\ell \neq p$. In the following, we simply write "cohomology" to mean $\ell$-adic cohomology. Also, we fix an isomorphism $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$ and always identify representations over $\overline{\mathbb{Q}}_\ell$ with those over $\mathbb{C}$ via this isomorphism.

Let $L$ be a non-archimedean local field. We denote by $\mathcal{O}_L$ the ring of integers of $L$ and by $\mathfrak{p}_L \subset \mathcal{O}_L$ the maximal ideal. We put $U_L = U_L^0 = \mathcal{O}_L^\times$ and $U_L^i = 1 + \mathfrak{p}_L^i$ for $i \geq 1$. The Weil group of $L$ is denoted by $W_L$. The Artin reciprocity map $\mathrm{Art}_L \colon L^\times \xrightarrow{\sim} W_L^{\mathrm{ab}}$ is normalized so that uniformizers are sent to geometric Frobenius elements. We abusively denote by $\mathrm{Art}_L^{-1}$ the composite of the natural surjection $W_L \to W_L^{\mathrm{ab}}$ with the inverse map of $\mathrm{Art}_L$.

More generally, for a non-archimedean valuation field $L$, we denote by $\mathcal{O}_L$ the valuation ring of $L$.

For a finite abelian group $A$, we write $A^\vee = \mathrm{Hom}(A, \overline{\mathbb{Q}}_\ell^\times)$ for the character group. For a closed subgroup $H'$ of a locally profinite group $H$ and a smooth representation $\sigma$ of $H'$, $\mathrm{Ind}_{H'}^H \sigma$ (resp. $\text{c-}\mathrm{Ind}_{H'}^H \sigma$) denotes the smooth induction (resp. the compact induction) of $\sigma$ from $H'$ to $H$. (See [BH06, 2.4, 2.5] for the definitions of these two sorts of inductions.)

## § 1. Formulation of the question

**Lubin-Tate perfectoid space** Here we briefly review some facts on the Lubin-Tate perfectoid space. The author learned a great deal from the exposition in [IT15] as well as that in [We16].

We denote the cardinality of $k$ by $q$. Take a uniformizer $\varpi \in K$. We fix an algebraic closure $\overline{K}$ of $K$ and denote by $\overline{k}$ the residue field of $\overline{K}$. We also fix the completion $\mathbf{C}$ of $\overline{K}$.

Let $n$ be a positive integer. Let $\Sigma_0$ be a one-dimensional formal $\mathcal{O}_K$-module over $\overline{k}$ of height $n$, which is unique up to isomorphism. Let $K^{\mathrm{ur}} \subset \overline{K}$ be the maximal

[3] To remark on the preceding work, we allow $K$ to be of any characteristic in our general setting. In the discussion of a simple expression of $\delta$ after Theorem 1.1, in Theorem 2.2 and in Section 3, we assume $K$ to be of positive characteristic.

unramified extension of $K$ and $\widehat{K}^{\mathrm{ur}} \subset \mathbf{C}$ its completion.

We denote by $\mathcal{C}$ the category of complete Noetherian local $\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}$-algebras with residue field $\overline{k}$. Let $R \in \mathcal{C}$. For a formal $\mathcal{O}_K$-module $\Sigma$ over $R$ and an integer $m \geq 0$, we mean by "a Drinfeld level $\varpi^m$-structure on $\Sigma$" what is called "a structure of level $m$ on $\Sigma$" in [Dr74, p.572 Definition]. We define a functor $\mathcal{C} \to$ Sets by associating to $R \in \mathcal{C}$ the set of isomorphism classes of triples $(\Sigma, \iota, \varphi)$ in which $\Sigma$ is a formal $\mathcal{O}_K$-module over $R$, $\iota\colon \Sigma_0 \to \Sigma \otimes_R \overline{k}$ is an isomorphism of formal $\mathcal{O}_K$-modules over $\overline{k}$ and $\varphi\colon (\varpi^{-m}\mathcal{O}_K/\mathcal{O}_K)^n \to \Sigma[\varpi^m](R)$ is a Drinfeld level $\varpi^m$-structure on $\Sigma$. Then it is shown in [Dr74, Propositions 4.2, 4.3.] that this functor is representable by an $n$-dimensional regular local ring $R_m$. These rings $R_m$ naturally form an inductive system $\{R_m\}$. We write $R_\infty = (\varinjlim R_m)^\wedge$ for the completion of the inductive limit with respect to the ideal generated by the maximal ideal of $R_0$.

Let $K^{\mathrm{ab}} \subset \overline{K}$ be the maximal abelian extension of $K$ and $\widehat{K}^{\mathrm{ab}} \subset \mathbf{C}$ its completion. We denote by $\mathcal{O}_{\mathbf{C}}[[X_1^{q^{-\infty}}, \ldots, X_n^{q^{-\infty}}]]$ the $(\varpi, X_1, \ldots, X_n)$-adic completion of $\mathcal{O}_{\mathbf{C}}[X_1^{q^{-\infty}}, \ldots, X_n^{q^{-\infty}}] = \varinjlim_{X_i \mapsto X_i^q} \mathcal{O}_{\mathbf{C}}[X_1, \ldots, X_n]$. Let $\mathrm{Map}(U_K, \mathcal{O}_{\mathbf{C}})$ denote the $\mathcal{O}_{\mathbf{C}}$-algebra of continuous maps from $U_K$ to $\mathcal{O}_{\mathbf{C}}$.

**Theorem 1.1.**

(1) *There exists a continuous $\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}$-homomorphism*

$$\mathcal{O}_{\widehat{K}^{\mathrm{ab}}} \to R_\infty, \tag{1.1}$$

*$q$-th power projective systems $\left(\delta(X_1, \ldots, X_n)^{q^{-m}}\right)_{m \geq 0}$ and $(t^{q^{-m}})_{m \geq 0}$ of topologically nilpotent elements in $\mathcal{O}_{\mathbf{C}}[[X_1^{q^{-\infty}}, \ldots, X_n^{q^{-\infty}}]]$ (resp. in $\mathcal{O}_{\mathbf{C}}$) and an isomorphism*

$$R_\infty \widehat{\otimes}_{\mathcal{O}_{\widehat{K}^{\mathrm{ab}}}} \mathcal{O}_{\mathbf{C}} \simeq \mathcal{O}_{\mathbf{C}}[[X_1^{q^{-\infty}}, \ldots, X_n^{q^{-\infty}}]] / \overline{(\delta(X_1, \ldots, X_n)^{q^{-m}} - t^{q^{-m}})_{m \geq 0}},$$

*where, on the left hand side, $\mathcal{O}_{\widehat{K}^{\mathrm{ab}}}$-structure is induced by the homomorphism* (1.1) *and $\overline{(\cdot)}$ denotes the closure.*

(2) *We put*

$$R_{\infty, \mathcal{O}_{\mathbf{C}}} = R_\infty \widehat{\otimes}_{\mathcal{O}_{\widehat{K}^{\mathrm{ab}}}} \mathrm{Map}(U_K, \mathcal{O}_{\mathbf{C}}),$$

*where the right factor is considered as an $\mathcal{O}_{\widehat{K}^{\mathrm{ab}}}$-algebra via*

$$\mathcal{O}_{\widehat{K}^{\mathrm{ab}}} \to \mathrm{Map}(U_K, \mathcal{O}_{\mathbf{C}});\ a \mapsto (\mathrm{Art}_K(u)(a))_{u \in U_K}.$$

*Then*

$$\mathcal{M}^{\mathrm{ad}}_{\Sigma_0, \infty, \overline{\eta}} = \{|\cdot| \in \mathrm{Spa}(R_{\infty, \mathcal{O}_{\mathbf{C}}}, R_{\infty, \mathcal{O}_{\mathbf{C}}}) \mid |\varpi| \neq 0\}$$

*is a perfectoid space.*

*Proof.* These assertions follow from [We16, 2.3–2.10] (see also [IT15, 1.1]). □

We call $\mathcal{M}^{\mathrm{ad}}_{\Sigma_0,\infty,\overline{\eta}}$ the Lubin-Tate perfectoid space.

*Remark.*

- The constructions of the homomorphism (1.1) and the projective system $(t^{q^{-m}})_{m\geq 0}$ depend on a choice of projective system of non-trivial $\varpi^m$-torsions of the Lubin-Tate module (see [We16, 2.3, 2.5] for details).
- The projective system $\left(\delta(X_1,\ldots,X_n)^{q^{-m}}\right)_{m\geq 0}$ can be made explicit (see [IT15, 1.1] or [BW16, proof of Theorem 2.10.3], [We16, 5.3]). If $K$ is of equal-characteristic, then it is particularly simple. Putting

$$\Delta(X_1,\ldots,X_n)=\det(X_i^{q^{j-1}})_{1\leq i,j\leq n}\in\mathbb{Z}[X_1,\ldots,X_n],$$

we have

$$\delta(X_1,\ldots,X_n)=\sum_{m_1+\cdots+m_n=0}\Delta(X_1^{q^{nm_1}},\ldots,X_n^{q^{nm_n}})\in\mathcal{O}_{\mathbf{C}}[[X_1^{q^{-\infty}},\ldots,X_n^{q^{-\infty}}]].$$

Note that $\delta(X_1,\ldots,X_n)^{q^{-m}}$ is automatically determined by $\delta(X_1,\ldots,X_n)$ in this case.

We set $\mathcal{O}_D=\operatorname{End}\Sigma_0$ and $D=\mathcal{O}_D\otimes_{\mathcal{O}_K}K$. Then $D$ is a central division algebra over $K$ of invariant $1/n$. Put $G=\mathrm{GL}_n(K)\times D^\times\times W_K$. We define $N_G$ by

$$N_G\colon \mathrm{GL}_n(K)\times D^\times\times W_K\to K^\times;\ (g,d,\sigma)\mapsto(\det g^{-1})(\operatorname{Nrd}d)(\operatorname{Art}_K^{-1}\sigma),$$

where $\operatorname{Nrd}\colon D^\times\to K^\times$ is the reduced norm. Let $v$ denote the normalized valuation of $K$. Then, as in the usual non-abelian Lubin-Tate theory, $\mathcal{M}^{\mathrm{ad}}_{\Sigma_0,\infty,\overline{\eta}}$ admits a natural action of $G^0=\operatorname{Ker}(v\circ N_G)$.

**Formulation of the question** With the Lubin-Tate perfectoid space introduced above, one may ask the following question.

**Question 1.2.** Do there exist affinoids $\mathcal{A}\subset\mathcal{M}^{\mathrm{ad}}_{\Sigma_0,\infty,\overline{\eta}}$ and formal models $\mathscr{A}$ with the following properties?

(1) The stabilizer $S_{\mathcal{A}}$ of $\mathcal{A}$ in $G^0$ naturally acts on the formal model $\mathscr{A}$ of $\mathcal{A}$. In particular, $S_{\mathcal{A}}$ acts on the reduction $\overline{\mathscr{A}}$ of $\mathscr{A}$ and the cohomology groups $H_c^i\left(\overline{\mathscr{A}},\overline{\mathbb{Q}}_\ell\right)$.

(2) For an irreducible supercuspidal representation $\pi$ of $\mathrm{GL}_n(K)$, we have an isomorphism

$$\operatorname{Hom}_{\mathrm{GL}_n(K)}\left(\text{c-Ind}_{S_{\mathcal{A}}}^G H_c^{n-1}\left(\overline{\mathscr{A}},\overline{\mathbb{Q}}_\ell\right)((n-1)/2),\pi\right)\simeq\mathrm{LJ}(\pi)\boxtimes\mathrm{LL}(\pi)$$

of representations of $D^\times \times W_K$ whenever the left hand side is non-zero. Here $\mathrm{LJ}(\pi)$ (resp. $\mathrm{LL}(\pi)$) denotes the image of $\pi$ under the local Jacquet-Langlands (resp. Langlands) correspondence.

In short, the question asks the existence of affinoids and formal models such that the cohomology groups of the reductions realize the two correspondences for some representations.

*Remark.*

(1) There are several different normalizations for the local Langlands correspondence. Here $\mathrm{LL}(\pi)$ is the same as $\mathrm{rec}_K(\pi)$ in [HT01, p.2]. Equivalently, if $\sigma = \mathrm{LL}(\pi)$, then $\pi = \pi(\sigma)$ in the notation of [He06, 5. Theorem.]. Also, $\mathrm{LJ}(\pi) = \pi'$ in the notation of [He06, 3. Theorem 2.].

(2) This question does not ask anything about the relation between the cohomology groups of the reductions $\overline{\mathscr{A}}$ and the cohomology of the Lubin-Tate tower usually studied in the non-abelian Lubin-Tate theory. However, we expect that each reduction $\overline{\mathscr{A}}$ is closely related to the special fiber of a semistable model of some (finite-level) Lubin-Tate space.[4]

(3) In fact, in the main theorem to be stated below, $\pi$ can be an irreducible smooth (rather than supercuspidal) representation. One can show that if $\pi$ occurs in the compact induction then $\pi$ is supercuspidal.

Given an affinoid and a formal model in the main theorem (or in the preceding results discussed below), we can state the condition for an irreducible supercuspidal representation $\pi$ to occur in the compact induction in a purely representation-theoretic manner. In this paper, we do so in terms of $\mathrm{LL}(\pi)$ instead of $\pi$.

## § 2. Main theorem

In order to state the main theorem and also to compare the result with the preceding ones, we introduce the following notions.

**Definition 2.1.** Let $F/K$ be a tamely ramified extension of degree $n$ and $\chi$ a smooth character of $F^\times$.

Let $i \geq 0$ be an integer. The character $\chi$ is said to be *minimal with the jump at $i$* if the following hold:

- $\chi|_{U_F^{i+1}}$ factors through the norm map $N_{F/K}\colon F^\times \to K^\times$,

[4] Results related to this remark are obtained in recent preprints [Mi16], [Ts16] of Mieda and Tsushima.

- $\chi|_{U_F^i}$ does not factor through $N_{F/E}$ for any subextensions $K \subset E \subsetneq F$, and
- if $i = 0$, then $F/K$ is unramified.

We say that $\chi$ is *minimal* if it is minimal with the jump at $i$ for some $i \geq 0$.

*Remark.*

(1) If $\xi$ is a minimal character of $F^\times$, then the induced representation $\mathrm{Ind}_{F/K}\,\xi = \mathrm{Ind}_{W_F}^{W_K}(\xi \circ \mathrm{Art}_F^{-1})$ is irreducible. (This is essentially an application of Mackey's criterion. See [BH05a, A.2. Proposition] for a proof.)

More generally, in the work [BH05a, BH05b, BH10] of Bushnell-Henniart, an $n$-dimensional irreducible smooth representation of $W_K$ is said to be *essentially tame* if it is induced from a character of $F^\times$ for some tamely ramified extension $F/K$ of degree $n$. In *loc. cit.*, essentially tame representations of $W_K$ are parametrized by the $K$-isomorphism classes of *admissible pairs* $(F/K, \xi)$ and minimal admissible pairs are discussed as particularly simple examples of admissible pairs. Our definition of minimal characters is a straightforward adaptation.

(2) If $\xi$ is a minimal character of $F^\times$ with the jump at $i$, then the ramification index $e$ of $F/K$ is coprime to $i$. (To see this, one may assume that $F/K$ is totally ramified and $i \geq 1$. If $d = \gcd(e, i)$ and $F/E$ is a subextension of degree $d$, then $\xi|_{U_F^i}$ necessarily factors through $N_{F/E}$ because $N_{F/E}$ induces an isomorphism $U_F^i/U_F^{i+1} \xrightarrow{\sim} U_E^{i/d}/U_E^{(i/d)+1}$ by the tameness assumption. This shows that $d = 1$. See [BH10, (8.2.3)] for a more general statement.)

The main theorem of [To16] is the following.

**Theorem 2.2.** *Suppose that $K$ is of equal-characteristic and that $p$ does not divide $n$. Let $\nu > 0$ be a positive integer. Let $L/K$ be a totally ramified extension of degree $n$. Then there exists an affinoid $\mathcal{Z}_\nu$ in $\mathcal{M}^{\mathrm{ad}}_{\Sigma_0,\infty,\overline{\eta}}$ and a formal model $\mathscr{Z}_\nu$ of $\mathcal{Z}_\nu$ such that the following hold.*

(1) *The stabilizer $\mathrm{Stab}_\nu$ of $\mathcal{Z}_\nu$ in $G^0$ naturally acts on $\mathscr{Z}_\nu$.*

(2) *For $\nu$ coprime to $n$ and an irreducible smooth representation $\pi$ of $\mathrm{GL}_n(K)$, we have*

$$\mathrm{Hom}_{\mathrm{GL}_n(K)}\left(\text{c-Ind}_{\mathrm{Stab}_\nu}^{G} H_c^{n-1}\left(\overline{\mathscr{Z}}_\nu, \overline{\mathbb{Q}}_\ell\right)((n-1)/2), \pi\right)$$
$$\simeq \begin{cases} \mathrm{LJ}(\pi) \boxtimes \mathrm{LL}(\pi) & \text{if } \mathrm{LL}(\pi) \text{ is induced from a minimal character of } L^\times \\ & \quad \text{with the jump at } \nu \\ 0 & \text{otherwise.} \end{cases}$$

*Here, $((n-1)/2)$ denotes the twist by the unramified character of $W_K$ sending geometric Frobenius elements to $q^{(1-n)/2} \in \mathbb{C}^\times \simeq \overline{\mathbb{Q}}_\ell^\times$.*

*Remark.* There are several preceding results concerning Question 1.2;

- In [BW16, Theorem 3.6.1], an affinoid in the Lubin-Tate perfectoid space and a formal model are constructed for each positive integer (denoted by $m$ in *loc. cit.*). If $L/K$ is the unramified extension of degree $n$, then they give an answer to Question 1.2 for representations of $W_K$ induced from a minimal character of $L^\times$ with the jump at $\nu = m$.

- Similarly, if $L/K$ is a totally tamely ramified extension of degree $n$ as in Theorem 2.2, then the affinoid and the formal model in [IT15, Theorem] give an answer to Question 1.2 for representations of $W_K$ induced from a minimal character of $L^\times$ with the jump at one. (Under the assumption that $p \nmid n$,) such representations can be characterized as character twists of representations of exponential Swan conductor one, which they call *essentially simple epipelagic representations* in *loc. cit.*

  The work [IT16] settles Question 1.2 for essentially simple epipelagic representations in the remaining case, i.e. when $p$ divides $n$. In this case, essentially simple epipelagic representations are never essentially tame.

- The affinoids (and natural formal models) constructed in [We16] in the course of proving the main theorem give an answer to Question 1.2 in the case where $p \neq 2$ and $n = 2$. (See [We16, Theorem 4.11, Theorem 5.2 and Remark 5.3.].) Note that in this case every relevant representation of $W_K$ is induced from a minimal character of $L^\times$ for some quadratic extension $L/K$.

In all of the above results, $K$ may be of mixed-characteristic or equal-characteristic. Thus, Theorem 2.2 is a "totally ramified version" of [BW16], and generalizes [IT15] and the "totally ramified part" of [We16], under the assumption that $K$ is of equal-characteristic.

## § 3. Some remarks on the proof

The proof of Theorem 2.2 naturally consists of the following three steps:

(i) constructing affinoids and formal models,

(ii) computing the reduction and the cohomology and

(iii) comparing the resulting representation with the local Langlands and Jacquet-Langlands correspondences.

**Step (i)** The construction of affinoids and formal models involves an explicit computation, building on the defining equation $\delta$ of (the formal model of) the Lubin-Tate perfectoid space. It is modeled on the constructions in [We16] and [IT15]; the affinoids are centered at a point $\xi$ of $\mathcal{M}^{\mathrm{ad}}_{\Sigma_0,\infty,\overline{\eta}}$ with CM by $L$ exactly as in *loc. cit.* and the coordinate around $\xi$ used to define the affinoids is obtained by closely comparing the coordinates in [We16] and [IT15].

**Step (ii)** The computation of the reduction $\overline{\mathscr{X}}_\nu$ as well as the stabilizer $\mathrm{Stab}_\nu$ and its action is done through a detailed analysis of $\delta$ under the change of coordinate.

Here we only indicate some aspects of the argument that are particularly easy to state.

**Proposition 3.1.** *Suppose that $n \nmid \nu$. We denote the Artin-Schreier polynomial with respect to $q$-th power by $\wp(x) = x^q - x \in k[x]$.*

*Then each connected component of $\overline{\mathscr{X}}_\nu$ is isomorphic to the perfection of a smooth affine algebraic variety $Z_\nu$ defined in $\mathbb{A}^{n+1}_{\overline{k}} = \operatorname{Spec} \overline{k}[z, y_1, \ldots, y_n]$ by*

$$\begin{cases} y_1 + \cdots + y_n = 0 \\ \wp(z) = P_\nu(y_1, \ldots, y_n), \end{cases}$$

*where $P_\nu \in k[y_1, \ldots, y_n]$ depends only on $\nu \mod 2n$, and for odd $\nu = 2\mu + 1$*

$$P_\nu(y_1, \ldots, y_n) = \begin{cases} -\displaystyle\sum_{\mu < j-i < n-\mu} y_i y_j & \text{if } 0 < \nu < n \\ \displaystyle\sum_{n-\mu \le j-i \le \mu} y_i y_j & \text{if } n < \nu < 2n, \end{cases}$$

*and for even $\nu = 2\mu$*

$$P_\nu(y_1, \ldots, y_n) = \begin{cases} \displaystyle\sum_{\mu < j-i < n-\mu} \wp(y_i)\wp(y_j) + \sum_{j-i=\mu} \wp(y_i) y_j^q + \sum_{j-i=n-\mu} y_i^q \wp(y_j) \\ \hfill \text{if } 0 < \nu < n \\ -\displaystyle\sum_{n-\mu < j-i < \mu} \wp(y_i)\wp(y_j) + \sum_{j-i=\mu} \wp(y_i) y_j + \sum_{j-i=n-\mu} y_i \wp(y_j) \\ \hfill \text{if } n < \nu < 2n. \end{cases}$$

The cohomology group of $\overline{\mathscr{X}}_\nu$ is closely related to that of $Z_\nu$ and the computation is reduced to the latter.

The algebraic variety $Z_\nu$ has several obvious automorphisms;

- For any $a \in k$, the translation $z \mapsto z + a$ defines an automorphism (induced from the Artin-Schreier covering). We regard the additive group $k$ as a subgroup of $\mathrm{Aut}(Z_\nu)$ by this action.

- The permutation
$$\begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & 1 & \cdots & n-2 & n-1 \end{pmatrix}$$
of indices of $\{y_i\}$ induces an automorphism $\gamma$.

- If $\nu$ is odd, then
$$z \mapsto z, \quad y_i \mapsto -y_i \ \text{ for } 1 \leq i \leq n$$
defines an automorphism $\iota$.

- Regarding the above defining equations of $Z_\nu$ as elements of $k[z, y_1, \dots, y_n]$, one obtains a model $Z_{\nu,0}$ of $Z_\nu$ over $k$. This induces the action of $\Omega = \mathrm{Gal}(\overline{k}/k)$ on $Z_\nu$. In particular the geometric Frobenius element $\mathrm{Frob}_q \in \Omega$ acts on $Z_\nu$.

It can be proved that the restriction to $\mathrm{Stab}_\nu \cap \mathrm{GL}_n(K)$ of the natural action of $\mathrm{Stab}_\nu$ on $\overline{\mathscr{X}}_\nu$ comes from an action of $\mathrm{Stab}_\nu \cap \mathrm{GL}_n(K)$ on $Z_\nu$ and similarly for $\mathrm{Stab}_\nu \cap D^\times$. The following proposition is concerned with the actions of these groups.

**Proposition 3.2.** *We continue to assume that $n \nmid \nu$. Take $k_n/k$ to be the field extension of degree $n$. Put $S_{1,\nu} = \mathrm{Im}(\mathrm{Stab}_\nu \cap \mathrm{GL}_n(K) \to \mathrm{Aut}(Z_\nu))$ and $S_{2,\nu} = \mathrm{Im}(\mathrm{Stab}_\nu \cap D^\times \to \mathrm{Aut}(Z_\nu))$. Then we have the following.*

(1) *If $\nu$ is odd, then $S_{1,\nu} = S_{2,\nu} = k \subset \mathrm{Aut}(Z_\nu)$.*

(2) *Assume that $\nu$ is even. Then $S_{1,\nu}$ (resp. $S_{2,\nu}$) is a central extension of $\mathrm{Ker}(\mathrm{tr}\colon k^n \to k)$ (resp. $\mathrm{Ker}(\mathrm{tr}\colon k_n \to k)$) by $k \subset \mathrm{Aut}(Z_\nu)$. Moreover, we have $S_{1,\nu} \cap S_{2,\nu} = k \subset \mathrm{Aut}(Z_\nu)$.*

(3) *Assume that $\nu$ is even and coprime to $n$. Then the centers of $S_{1,\nu}$ and $S_{2,\nu}$ are both $k \subset \mathrm{Aut}(Z_\nu)$. Moreover, for any non-trivial character $\psi$ of $k$, there exists a unique irreducible representation, up to isomorphism, of $S_{i,\nu}$ $(i = 1, 2)$ with the central character $\psi$.*

In fact, in the situation of (2) one can explicitly describe the group structure of $S_{i,\nu}$ $(i = 1, 2)$. The assertion (3) is proved by studying the structure of these groups and applying the elementary representation theory of a central extension of an abelian group (often called *a Heisenberg group*).

**Proposition 3.3.** *Assume that $n$ and $\nu$ are coprime. Put $H_{c,\nu}^{n-1} = H_c^{n-1}(Z_\nu, \overline{\mathbb{Q}}_\ell)$. Then we have the following decomposition*

$$H_{c,\nu}^{n-1} \simeq \bigoplus_{\psi \in k^\vee \setminus \{1\}} \rho_{1,\nu,\psi} \boxtimes \rho_{2,\nu,\psi} \tag{3.1}$$

*as representations of $S_{1,\nu} \times S_{2,\nu}$, where, for $i = 1, 2$, $\rho_{i,\nu,\psi}$ is the unique irreducible representation of $S_{i,\nu}$ with the central character $\psi$. Moreover, for any non-trivial character $\psi$, the traces on each $\psi$-isotypic component*[5] $H^{n-1}_{c,\nu,\psi} \simeq \rho_{1,\nu,\psi} \boxtimes \rho_{2,\nu,\psi}$ *of various automorphisms of $Z_\nu$ are computed as follows*:

$$\begin{aligned} \operatorname{tr}\left(\gamma^j \mid H^{n-1}_{c,\nu,\psi}\right) &= (-1)^{n-1} \qquad (\textit{for } j \textit{ coprime to } n), \\ \operatorname{tr}\left(\iota \mid H^{n-1}_{c,\nu,\psi}\right) &= (-1)^{n-1} \qquad (\textit{if } \nu \textit{ is odd}), \\ \operatorname{tr}\left(\mathrm{Frob}_q \mid H^{n-1}_{c,\nu,\psi}\right) &= (-1)^{n-1} \sum_y \psi(P_\nu(y_1, \ldots, y_n)), \end{aligned} \tag{3.2}$$

*where the summation is taken over $y = (y_i) \in k^n$ such that $y_1 + \cdots + y_n = 0$.*

*Remark.*

(1) If $\nu$ is odd, then $\rho_{1,\nu,\psi}$ and $\rho_{2,\nu,\psi}$ are nothing other than $\psi$ itself. Thus, the decomposition of the cohomology group (3.1) amounts to a statement on the multiplicity in this case.

(2) The key to the computation of the traces of $\gamma^j$ and $\iota$ on $H^{n-1}_{c,\nu,\psi}$ is the fixed point formula [DL76, Theorem 3.2] of Deligne-Lusztig. By the formula, we are reduced to study the trace of $k \subset \mathrm{Aut}(Z_\nu)$ on the cohomology of the fixed point varieties $Z_\nu^{\gamma^j}$ and $Z_\nu^\iota$, which are both merely discrete sets of points indexed by $k$, where $j$ is coprime to $n$ in the former case. While the traces for general $j$ may be more difficult to compute, as it turned out, they are not necessary in the step (iii). (Also, even when $\nu$ is even, the automorphism $\iota$ can be defined and the trace can be computed in the same way. However, it is not necessary in the step (iii).)

(3) In fact, the sum on the right hand side of (3.2) can be expressed more explicitly and it is used to prove the realization of the local Langlands and Jacquet-Langlands correspondences in the step (iii).

(4) One can also prove that if $n$ and $\nu$ are not coprime, then the cohomology group vanishes in degree $n - 1$.

**Step (iii)** Denote by $\overline{\mathrm{Stab}}_\nu$ the image of $\mathrm{Stab}_\nu$ under the canonical homomorphism $G^0 \to G \to D^\times \times W_K$. Then, for an irreducible smooth representation $\pi$ of $\mathrm{GL}_n(K)$, the Frobenius reciprocity induces a natural isomorphism

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{GL}_n(K)}\left(\text{c-Ind}^G_{\mathrm{Stab}_\nu} H^{n-1}_c\left(\overline{\mathscr{Z}}_\nu, \overline{\mathbb{Q}}_\ell\right)((n-1)/2), \pi\right) \\ &\quad\simeq \mathrm{Ind}^{D^\times \times W_K}_{\overline{\mathrm{Stab}}_\nu} \mathrm{Hom}_{\mathrm{Stab}_\nu \cap \mathrm{GL}_n(K)}\left(H^{n-1}_c\left(\overline{\mathscr{Z}}_\nu, \overline{\mathbb{Q}}_\ell\right)((n-1)/2), \pi\right) \end{aligned} \tag{3.3}$$

[5] Note that the action of $k \subset \mathrm{Aut}(Z_\nu)$ on $Z_\nu$ clearly commutes with those of $\gamma, \iota, \mathrm{Frob}_q$.

of representations of $D^\times \times W_K$.

We are to examine the condition for the space of homomorphisms in the right hand side of (3.3) to be non-zero and prove that, if non-zero, the induced representation yields the desired representation of $D^\times \times W_K$. This step is largely representation-theoretic. The main ingredients are the following.

- *The theory of types* (see [BK93] and [Br98]; the author found the exposition in [BH05b, BH11] helpful), which allows one to read off properties of an irreducible smooth representation of $\mathrm{GL}_n(K)$ or $D^\times$ from the containment of a certain irreducible representation of an open subgroup, and also to express an irreducible supercuspidal representation as a compact induction of a finite-dimensional representation of an open subgroup.
- *The essentially tame local Langlands and Jacquet-Langlands correspondences* (see [BH05a, BH05b, BH10, BH11]; in our case [BH05b, BH11] are the most relevant), which explicitly describe the local Langlands correspondence and the local Jacquet-Langlands correspondence for essentially tame representations, in terms of the expression of representations as compact inductions provided by the theory of types.

The condition for $\pi$ to occur in $H_c^{n-1}\left(\overline{\mathscr{X}}_\nu, \overline{\mathbb{Q}}_\ell\right)((n-1)/2)$ can be examined by applying (a simple instance of) the theory of types: $\rho_{1,\nu,\psi}$ in Proposition 3.3 are closely related to irreducible representations appearing in the theory.

The proof that, if non-zero, the right hand side of (3.3) is isomorphic to $\mathrm{LJ}(\pi) \boxtimes \mathrm{LL}(\pi)$ is more subtle and difficult to explain. We simply remark that after some arguments one only needs to study the action of a certain subgroup of $\mathrm{Stab}_\nu$ of finite index and this reduces to studying the actions of $S_{1,\nu}$, $S_{2,\nu}$, $\gamma^j$ (with $j$ coprime to $n$), $\iota$ (if $\nu$ is odd), $\mathrm{Frob}_q$ as in Proposition 3.3, which is relatively easy.

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