

On the Existence, Geometry and p -Ranks of Vertical Fibers of Coverings of Curves

By

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Abstract

Let R be a complete DVR with algebraically closed residue field of characteristic $p > 0$ and X, Y stable curves over R with smooth generic fibers. Let $f : Y \rightarrow X$ be a morphism over R such that the morphism of generic fibers induced by f is a Galois étale covering. A closed point x of X is called a vertical point if $\dim f^{-1}(x) = 1$. In this case, $f^{-1}(x)$ is called the vertical fiber associated to x . We study the existence, the geometry, and the p -ranks of vertical fibers under certain assumptions.

§ 1. Preliminaries

Let R be a complete discrete valuation ring with algebraically closed residue field k , K the quotient field of R , and \overline{K} an algebraic closure of K . We use the notation S to denote the spectrum of R . Write $\eta, \overline{\eta}$, and s for the generic point, the geometric generic point, and the closed point of S corresponding to the natural morphisms $\text{Spec } K \rightarrow S$, $\text{Spec } \overline{K} \rightarrow S$, and $\text{Spec } k \rightarrow S$, respectively. Let X be a stable curve of genus g_X over S . Write $X_\eta, X_{\overline{\eta}}$ and X_s for the generic fiber, the geometric generic fiber and the special fiber, respectively. Moreover, we suppose that X_η is nonsingular.

§ 1.1. Admissible fundamental groups and specialization

Definition 1.1. Let $\phi : Z \rightarrow X_s$ be a morphism of stable curves over s . We shall call ϕ a **Galois admissible covering** over s (or Galois admissible covering for short) if the following conditions hold: (i) there exists a finite group $G \subseteq \text{Aut}_k(Z)$ such that $Z/G = X_s$, and ϕ is equal to the quotient morphism $Z \rightarrow Z/G$; (ii) for each $z \in Z^{\text{sm}}$, ϕ

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is étale at z , where $(-)^{\text{sm}}$ denotes the smooth locus of $(-)$; (iii) for any $z \in Z^{\text{sing}}$, the image $\phi(z)$ is contained in X_s^{sing} , where $(-)^{\text{sing}}$ denotes the singular locus of $(-)$; (iv) let $z \in Z^{\text{sing}}$ and $D_z \subseteq G$ the decomposition group of z ; the local morphism between two nodes (cf. (iii)) induced by ϕ may be described as follows:

$$\begin{aligned} \hat{\mathcal{O}}_{X_s, \phi(z)} &\cong k[[u, v]]/uv \rightarrow \hat{\mathcal{O}}_{Z, z} \cong k[[s, t]]/st \\ u &\mapsto s^n \\ v &\mapsto t^n, \end{aligned}$$

where $(n, \text{char}(k)) = 1$ if $\text{char}(k) = p > 0$; moreover, $\tau(s) = \zeta_{\#D_z} s$ and $\tau(t) = \zeta_{\#D_z}^{-1} t$ for each $\tau \in D_z$, where $\#D_z$ denotes the order of D_z , and $\zeta_{\#D_z}$ is a primitive $\#D_z$ -th root of unit. We shall call ϕ an **admissible covering** if there exists a morphism of stable curves $\phi' : Z' \rightarrow Z$ over s such that the composite morphism $\phi \circ \phi' : Z' \rightarrow X_s$ is a Galois admissible covering over s .

Let Y be the disjoint union of finitely many stable curves over s . We shall call a morphism $\psi : Y \rightarrow X_s$ over s **multi-admissible** if the restriction of ψ to each connected component of Y is an admissible covering.

We use the notation $\text{Cov}^{\text{adm}}(X_s)$ to denote the category which consists of (empty object and) all the multi-admissible coverings of X_s . It is well-known that $\text{Cov}^{\text{adm}}(X_s)$ is a Galois category. Thus, by choosing a base point $x \in X_s$, we obtain a fundamental group $\pi_1^{\text{adm}}(X_s, x)$ which is called the **admissible fundamental group** of X_s . For simplicity, we omit the base point and denote the admissible fundamental group by $\pi_1^{\text{adm}}(X_s)$.

Remark. Note that by the definition of admissible coverings, if $\text{char}(k) = p > 0$, the maximal pro- p quotient of the admissible fundamental group $\pi_1^{\text{adm}}(X_s)$ is isomorphic to the maximal pro- p quotient of the étale fundamental group $\pi_1(X_s)$.

Remark. Let $\overline{\mathcal{M}}_{g,r}$ be the moduli stack of pointed stable curves of type (g, r) over $\text{Spec } \mathbb{Z}$ and $\mathcal{M}_{g,r}$ the open substack of $\overline{\mathcal{M}}_{g,r}$ parametrizing pointed smooth curves. Write $\overline{\mathcal{M}}_{g,r}^{\text{log}}$ for the log stack obtained by equipping $\overline{\mathcal{M}}_{g,r}$ with the natural log structure associated to the divisor with normal crossings $\overline{\mathcal{M}}_{g,r} \setminus \mathcal{M}_{g,r} \subset \overline{\mathcal{M}}_{g,r}$ relative to $\text{Spec } \mathbb{Z}$. We use the notation $\overline{\mathcal{M}}_g$ (resp. $\overline{\mathcal{M}}_g^{\text{log}}$) to denote the stack $\overline{\mathcal{M}}_{g,0}$ (resp. the log stack $\overline{\mathcal{M}}_{g,0}^{\text{log}}$).

Let $s^{\text{log}} \rightarrow \overline{\mathcal{M}}_{g_X}^{\text{log}}$ be a morphism from an fs log point s^{log} (i.e., an fs log scheme whose underlying scheme is s) whose underlying morphism $s \rightarrow \overline{\mathcal{M}}_{g_X}$ is determined by $X_s \rightarrow s$. Thus, we obtain a stable log curve $X_s^{\text{log}} := s^{\text{log}} \times_{\overline{\mathcal{M}}_{g_X}^{\text{log}}} \overline{\mathcal{M}}_{g_X,1}^{\text{log}}$ whose underlying scheme is X_s . Then the admissible fundamental group of X_s is isomorphic to the geometric log étale fundamental group of X_s^{log} .

For more details on admissible coverings, log admissible coverings and the fundamental groups for (pointed) stable curves, see [3], [12].

By applying the theory of deformation of stable log curves, we obtain a **specialization morphism** from the geometric étale fundamental group of the generic fiber to the admissible fundamental group of the special fiber:

$$Sp : \pi_1(X_{\bar{\eta}}) \rightarrow \pi_1^{\text{adm}}(X_s).$$

Sp is always a surjection, but Sp is not an injection in general. Moreover, we have the following theorem.

Theorem 1.2. (i) ([1, Exposé X Corollaire 3.9], [11, Théorème 2.2]) *If $\text{char}(K) = \text{char}(k) = 0$, then Sp is an isomorphism.*

(ii) *If $\text{char}(K) = 0$ and $\text{char}(k) = p > 0$, then Sp is not an isomorphism (cf. the following Remark).*

(iii) *If $\text{char}(K) = \text{char}(k) = p > 0$, then we have the following results:*

(a) ([4, Theorem A and Theorem B], [7, Proposition 2.2.5], [9, Theorem 0.1]) *if $k = \bar{\mathbb{F}}_p$, X_s is smooth over s and X is not a trivial family over S , then Sp is not an isomorphism;*

(b) ([10, Corollary 3.11]) *if X_s is singular, then Sp is not an isomorphism.*

Remark. By the first remark under Definition 1.1, if $\text{char}(k) = p > 0$, we have that the maximal pro- p quotient $\pi_1^p(X_s)$ of $\pi_1(X_s)$ is isomorphic to the maximal pro- p quotient $\pi_1^{p\text{-adm}}(X_s)$ of $\pi_1^{\text{adm}}(X_s)$. Then Theorem 1.2 (ii) follows from the following fact (cf. the third remark of Definition 1.3):

$$\dim_{\mathbb{F}_p}(\mathrm{H}_{\text{ét}}^1(X_{\bar{\eta}}, \mathbb{F}_p)) = 2g_X > g_X \geq \dim_{\mathbb{F}_p}(\mathrm{H}_{\text{ét}}^1(X_s, \mathbb{F}_p)).$$

§ 1.2. Some definitions

In this subsection, we give some definitions. From now on, we assume that $\text{char}(k) = p > 0$.

Definition 1.3. Write $\pi_1^p(X_s)$ for the maximal pro- p quotient of the étale fundamental group $\pi_1(X_s)$ of X_s . It is well-known that $\pi_1^p(X_s)$ is a finitely generated free pro- p group. We define the **p -rank** $\sigma(X_s)$ of X_s as follows:

$$\sigma(X_s) := \text{rank}(\pi_1^p(X_s)) = \dim_{\mathbb{F}_p}(\mathrm{H}_{\text{ét}}^1(X_s, \mathbb{F}_p)).$$

Remark. For a semi-stable curve Z over k , we may also define the p -rank $\sigma(Z)$ of Z as follows:

$$\sigma(Z) := \text{rank}(\pi_1^p(Z)) = \dim_{\mathbb{F}_p}(\mathrm{H}_{\text{ét}}^1(Z, \mathbb{F}_p)).$$

Remark. If X_s is smooth, then the p -rank $\sigma(X_s)$ is equal to the dimension of the p -torsion points of the Jacobian J_{X_s} of X_s as an \mathbb{F}_p -vector space.

Suppose that X_s is a singular curve. Write Γ_{X_s} for the dual graph of X_s and $v(\Gamma_{X_s})$ for the set of vertices of Γ_{X_s} . For $v \in v(\Gamma_{X_s})$, write X_v for the irreducible component of X_s corresponding to v and \widetilde{X}_v for the normalization of X_v . Then the p -rank $\sigma(X_s)$ of X_s is equal to

$$\sum_{v \in v(\Gamma_{X_s})} \sigma(\widetilde{X}_v) + \text{rank}(\mathrm{H}^1(\Gamma_{X_s}, \mathbb{Z})),$$

where $\text{rank}(\mathrm{H}^1(\Gamma_{X_s}, \mathbb{Z}))$ denotes the rank of $\mathrm{H}^1(\Gamma_{X_s}, \mathbb{Z})$ as a finitely generated free \mathbb{Z} -module.

Remark. Note that we have $\sigma(X_s) \leq g_X$.

Definition 1.4. We shall call X_s **ordinary** if $\sigma(X_s) = g_X$.

Definition 1.5. Let $f : Y \rightarrow X$ be a morphism of stable curves over S and G a finite group. We shall call f a **stable covering** (resp. **G -stable covering**) if the morphism of generic fibers f_η is an étale covering (resp. a Galois étale covering with Galois group G).

Remark. Let $f_\eta : Y_\eta \rightarrow X_\eta$ be a morphism of smooth, geometrically connected projective curves over $\text{Spec } K$ and G a finite group. Suppose that f_η is a G -étale covering. Then by applying the stable reduction theorem for curves, after possibly replacing S by a finite extension of S , we can extend f_η to a G -stable covering over S (cf. [2, Theorem 0.2]).

Definition 1.6. Let $f : Y \rightarrow X$ be a stable covering. Suppose that the morphism of special fibers $f_s : Y_s \rightarrow X_s$ is not finite. A closed point $x \in X$ is called a **vertical point** associated to f , or for simplicity, a vertical point when there is no fear of confusion, if $\dim(f^{-1}(x)) = 1$. The inverse image $f^{-1}(x)$ is called the **vertical fiber** associated to x .

§ 2. Questions and results

Let G be a finite group and $f : Y \rightarrow X$ a G -stable covering. By Theorem 1.2, Sp is not an isomorphism in general. It is natural to pose the following question:

Question 2.1. *Is f_s always a finite morphism? When Y_s an ordinary curve? How to compute the p -ranks of vertical fibers?*

Remark. The motivations of Question 2.1 are as follows:

- (1) to understand the reduction of an étale covering of X_η ;
- (2) to understand the structure of the admissible fundamental groups of stable curves over an algebraically closed field of positive characteristic.

§ 2.1. Existence of vertical fibers

Since S_p is not an isomorphism in general by Theorem 1.2, the morphism of special fibers induced by a stable covering is not an admissible covering in general. In this subsection, we consider whether or not there exists a non-finite stable covering of X (i.e., the existence of vertical fibers). Moreover, we consider a sufficient condition for a given G -stable covering over S to restrict an admissible covering of the special fibers.

First, we define the following set which consists of the vertical points:

$$X^{\text{ver}} := \{x \in X_s \mid x \text{ is a vertical point associated to a stable covering of } X\}.$$

Theorem 2.2. *If $\text{char}(K) = 0$, we have the following results:*

- (i) ([10, Theorem 0.2]) *if $k = \overline{\mathbb{F}}_p$, then $X^{\text{ver}} = X^{\text{cl}}$, where X^{cl} denotes the set of closed points of X .*
- (ii) ([13, Theorem 2.5]) *the closure of X^{ver} in X_s is equal to X_s and X_s^{sing} is contained in X^{ver} , where X_s^{sing} denotes the singular locus of X_s .*

Theorem 2.3. *If $\text{char}(K) = p > 0$ and X_s is irreducible, we have the following results:*

- (i) ([13, Theorem 2.7]) *if $k = \overline{\mathbb{F}}_p$, X_s is smooth over s and X is not a trivial family over S , then $X^{\text{ver}} \neq \emptyset$.*
- (ii) ([13, Theorem 2.8]) *if X_s is singular, then $X^{\text{ver}} \neq \emptyset$.*
- (iii) ([14, Theorem 1.3]) *for any finite group G , a G -stable covering $f : Y \rightarrow X$ is finite if and only if f_s is an admissible covering.*

§ 2.2. p -ranks of vertical fibers

In this subsection, we study the p -ranks of vertical fibers of stable coverings. The following theorem was proved by M. Raynaud (cf. [5, Théorème 1]).

Theorem 2.4. *Let G be a p -group, $f : Y \rightarrow X$ a G -stable covering, and $x \in X$ a vertical point associated to f . Suppose that x is a smooth point of X_s . Then the p -rank of each connected component of the vertical fiber $f^{-1}(x)$ is equal to 0. In particular, the dual graph of each connected component of the vertical fiber $f^{-1}(x)$ is a tree.*

Raynaud considered the vertical fibers associated to smooth vertical points. In the following, we consider a similar assertion for the vertical fibers associated to singular vertical points.

Let G be a finite p -group, $f : Y \rightarrow X$ a G -stable covering and x a vertical point associated to f . Suppose that x is a singular point of X_s . Then there are two irreducible components X_{v_1} and X_{v_2} (which may be equal) of X_s such that $x \in X_{v_1} \cap X_{v_2}$. Write Y' for the normalization of X in Y and $\psi : Y' \rightarrow X$ for the resulting normalization morphism. Let y' be a closed point of Y' such that $\psi(y') = x$. In order to compute the p -rank of each connected component of the vertical fibers associated to x , by applying the Zariski-Nagata purity and replacing $f : Y \rightarrow X$ by the quotient morphism $Y \rightarrow Y/I_{y'}$, we may assume that the inertia subgroup $I_{y'} \subseteq G$ of y' is equal to G . Let Y'_{v_1} (resp. Y'_{v_2}) be an irreducible component of Y'_s such that $\psi(Y'_{v_1}) = X_{v_1}$ and $y' \in Y'_{v_1}$ (resp. $\psi(Y'_{v_2}) = X_{v_2}$ and $y' \in Y'_{v_2}$). Write $I_{Y'_{v_1}} \subseteq I_{y'}$ (resp. $I_{Y'_{v_2}} \subseteq I_{y'}$) for the inertia subgroup of Y'_{v_1} (resp. Y'_{v_2}). Write V_x for the vertical fiber $f^{-1}(x)$. Note that since $I_{y'}$ is equal to G , V_x is connected.

The following theorem was proved by M. Saïdi (cf. [8, Theorem]).

Theorem 2.5. *If $I_{y'}$ is isomorphic to a cyclic p -group $\mathbb{Z}/p^r\mathbb{Z}$, then we have $\sigma(V_x) \leq p^r - 1$.*

We generalize Saïdi's result to the case where $I_{y'}$ is a finite abelian p -group as follows:

Theorem 2.6. (1) ([15, Lemma 2.1]) *Write Γ_x for the dual graph of the vertical fiber V_x . If $I_{y'}$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$, we have the following results:*(a) *If $I_{Y'_{v_1}} = \mathbb{Z}/p\mathbb{Z}$ and $I_{Y'_{v_2}}$ is trivial, then $\sigma(V_x) = 0$.* (b) *If $I_{Y'_{v_1}} = I_{Y'_{v_2}} = \mathbb{Z}/p\mathbb{Z}$, then one of the following conditions are satisfied:* (i) $\sigma(V_x) = 0$; (ii) $\sigma(V_x) = p - 1$ and $\text{rank}(\text{H}^1(\Gamma_x, \mathbb{Z})) = p - 1$; (iii) $\sigma(V_x) = p - 1$ and Γ_x is a tree.

(2) ([16, Theorem 1.4]) *If $I_{y'}$ is a finite abelian p -group of order p^r , then there exists a bound of $\sigma(V_x)$ which only depends on p^r .*

Remark. We can construct some examples for Theorem 2.6 (1-a) and (1-b-ii) (cf. [15, Section 4]).

§ 2.3. Ordinarity

In Subsection 2.2, we studied the p -ranks of vertical fibers of stable coverings. We also have some global results concerning the p -ranks of the special fibers of stable coverings. In order to study an étale covering of X_η with bad reduction, Raynaud (cf. [6, Proposition 3]) proved the following theorem:

Theorem 2.7. *Let G be a finite group and $f : Y \rightarrow X$ a G -stable covering. Suppose that X is smooth over S and f_s is not generically étale. Then Y_s is not ordinary.*

By applying Theorem 2.6 (1), we partially generalize Theorem 2.7 to the case where X is not necessarily smooth over S and G is solvable as follows:

Theorem 2.8 ([15, Theorem 3.4]). *Let G be a finite solvable group and $f : Y \rightarrow X$ a G -stable covering. Suppose that the genus of the normalization of each irreducible component of X_s is > 1 , and f_s is not generically étale. Then Y_s is not ordinary.*

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