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One visualization of Shimura's complex multiplication theorem via hypergeometric modular functions

By

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Abstract

This article is a sketch of our research work on the subject given by the above title. In the work "Construction of class fields and zeta functions of algebraic curves" (1967) by Goro Shimura (cf. [SmrA], [SmrB]), it was proved that there exists a modular function (that is called canonical model) that enables to obtain a certain class field (the Shimura class field) of some kind of CM field. In this article we show that for the case of the CM field embedded into the quaternion albebra coming from a co-compact arithmetic triangle group we can determine the canonical model as a hypergeometric modular function in an explicit way. Moreover we give several examples of Hilbert class fields of such kind of CM fields coming from the triangle group $\Delta(3,3,5)$. For our work, we use Shimura's reciplocity law and the existense of the canonical model together with the result by K. Takeuchi (1977) (see [Tku1],[Tku2]). To construct explicit examples we use the modular function for genus 4 pentagonal curves discovered by K. Koike ([Kik]). The first author has written the same subject in the book [Shg] (chapter 8). There he made a detailed explanation of the modular function for $\Delta(3,3,5)$. In contract, here we tried to explain the back ground of our research work. By both of two explanations he expects that the readers will have a nicer perspective of the story. For the full argument with exact proofs refer the paper [N-S].

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§ 1. Around the Hilbert class field, classical theory

1. The Hilbert class field Let K be a number field, and let L be a Galois extension of K which is also a number field. The following conditions are equivalent.

(I) L/K is an unranified abelian extension and it holds $\mathrm{Gal}(L/K) \cong C(K)$, where $C(K)$ is the ideal class group of K.

(II) L/K is a maximal unramified abelian extension over K.

(III) Let $\mathcal{O}_K$ be the ring of integers of K, and let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$ that is unramified in L. $\mathfrak{p}$: principal $\iff \mathfrak{p} = \mathfrak{P}_1 \cdots \mathfrak{P}_g, g = [L : K]$ (complete decomposition) $\iff$ The minimal polynomial of the generating element $\xi \in L$ over K has a solution in $\mathcal{O}_K/\mathfrak{p}$.

Remark 1.1. Such a L exists and is unique up to isomorphism.

Definition 1.2. The above L is called the Hilbert class field (or absolute class field) of K and is denoted by C_K or $C(K)$.

For the meaning of (III), see [Cox] p. 110 – p.115.

2. The elliptic λ function and the j function We consider a non-Euclidean triangle $\nabla(p, q, r)$ in the upper half complex plane $\boldsymbol{H}$ with angles $\frac{\pi}{p}, \frac{\pi}{q}$ and $\frac{\pi}{r}$ respectively, where we suppose $p, q, r \in \boldsymbol{N} \cup \{\infty\}$. We set $\nabla = \nabla(\infty, \infty, \infty)$ with vertices $z_1 = i\infty, z_2 = 0$ and $z_3 = -1$. We make a conformal map $\Phi : \boldsymbol{H}_- \to \nabla$ with $\Phi(0) = i\infty, \Phi(1) = 0$ and $\Phi(\infty) = -1$, as Fig. 1.1.

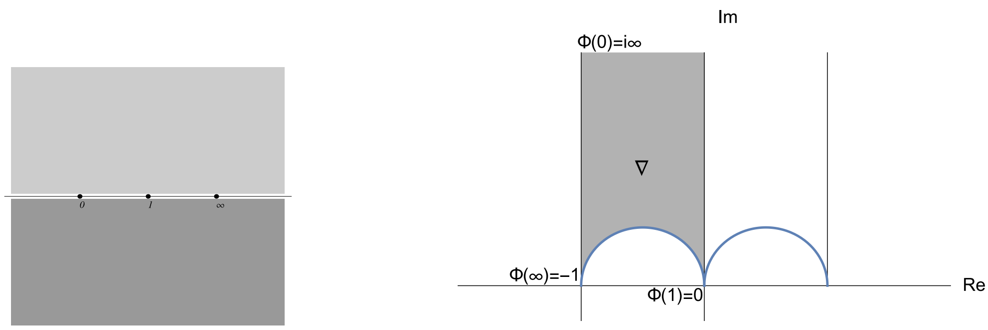

Fig.1.1 : Triangle Map Φ

By making the analytic continuation of this mapping through the interval $(0, 1)$ we get the image of $\nabla' = \boldsymbol{H}$ in Fig.1.2. We can make other continuations to obtain the other images in Fig. 1.2. Finaly they constitutes a tesselation of $\boldsymbol{H}$. As the inverse mapping of this multivalued map, we obtain a modular function $\lambda(\tau)$ that is invariant under the action of $\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \rangle = \Gamma(2)$.

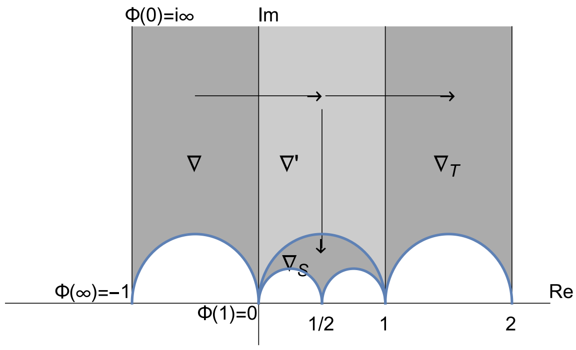

Fig.1.2 : Triangle Map ,analytic continuation

Setting

$$\vartheta_{00} = \vartheta_{00}(\tau) = 1 + 2\sum_{n=1}^{\infty} \tilde{q}^{n^2}, \vartheta_{01} = \vartheta_{01}(\tau) = 1 + 2\sum_{n=1}^{\infty}(-1)^n \tilde{q}^{n^2}, \tilde{q} = e^{\pi i \tau},$$

we have

$$\lambda(\tau) = 1 - \frac{\vartheta_{01}^4}{\vartheta_{00}^4}. \tag{1.1}$$

The elliptic modular function $j(\tau)$ is defined by

$$j(\tau) = 2^8 \frac{(1-\lambda+\lambda^2)^2}{\lambda^2(1-\lambda)^2}. \tag{1.2}$$

3. The classical complex multiplication theorem For precise argument of this subject, see [Kwd], [Hss].

Theorem 1.3 (Hasse, (together with the class field theory by Takagi and Weber))**.** *Let $k = \boldsymbol{Q}(\sqrt{D})$ $(D < 0)$ be an imaginary quadratic field, and let $\mathcal{O}_k = \boldsymbol{Z}+\boldsymbol{Z}\xi$ be its ring of integers, $\xi = \begin{cases} \sqrt{D} \text{ if } D \equiv 2,3 \pmod 4 \\ (1+\sqrt{D})/2 \text{ if } D \equiv 1 \pmod 4 \end{cases}$. Then it holds $C(k) = k(j(\xi))$.*

4. One Example By using (1.1), (1.2) we have the following approximate calculation.

Example 1.4. For the case $k = \boldsymbol{Q}(\sqrt{-6})$, we have the class number $h = 2$. And representatives of ideal classes $J_1 = [1, \sqrt{-6}]$, $J_2 = [2, \sqrt{-6}]$. We have

$$\begin{cases} r_1 = j(\sqrt{-6}) = (4.83190790335133974539736629805\cdots) \times 10^6, \\ r_2 = j(\frac{\sqrt{-6}}{2}) = 3036.09664866025460263370195085\cdots. \end{cases}$$

So, it holds

$$\begin{cases} r_1 + r_2 = (4.8349440000000000000000000000\cdots) \times 10^6, \\ r_1 r_2 = (1.4670139392000000000000000000\cdots) \times 10^{10}, \end{cases}$$

$$D = (r_1 + r_2)^2 - 4r_1r_2 = (2.3318002925568000000000000000\cdots) \times 10^{13}$$
$$= 23318002925568 = 2^{18}3^6 13^2 19^2 \times 2.$$

Because $j\left(\frac{\sqrt{-6}}{2}\right)$ gives the conjugate of $j\left(\sqrt{-6}\right)$ in $C(k)$,

$$K = k(j(\sqrt{-6})) = k(\sqrt{D}) = k(\sqrt{2}) = k(\omega) \quad (\omega^3 = 1)$$

is the Hilbert class field of k.

Note that by putting $G = \begin{pmatrix} 0 & -6 \\ 1 & 0 \end{pmatrix}$ it holds $G^2 + 6 = 0$ (so $\boldsymbol{Q}(G) \cong k$) and $G(\sqrt{-6}) = \sqrt{-6}$ (fixed point). In the Shimura complex multiplication we use the same procedure.

§ 2. The hypergeometric modular function

Let us consider the Gauss hypergeometric differential equation

$$E(a, b, c) : \lambda(1 - \lambda)f'' + (c - (a + b + 1)\lambda)f' - abf = 0 \tag{2.1}$$

with real parameters a, b, c. It has regular singular points at $\lambda = 0, 1, \infty$. The exponents at singularities are given by the Riemann scheme

$$\begin{Bmatrix} 0 & 1 & \infty \\ 0 & 0 & a \\ 1 - c & c - a - b & b \end{Bmatrix}. \tag{2.2}$$

We always assume the condition

$$(*) \quad \begin{cases} |1 - c| + |c - a - b| + |a - b| < 1 \\ p = 1/|1 - c|, q = 1/|c - a - b|, r = 1/|a - b| \in \mathbf{N} \cup \{\infty\}. \end{cases} \tag{2.3}$$

Set $\{\eta_1(\lambda), \eta_2(\lambda)\}$ be a basis of the space of solutions of (2.1). The ratio η_2/η_1 determines a single valued analytic function on the lower complex half plane $\boldsymbol{H}_-$. According to the condition ($*$), by choosing adequate basis the image can be considred to be a hyperbolic triangle $\nabla(p, q, r)$ on the upper half plane $\boldsymbol{H}$ with angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$.

The multivalued analytic map $\mathcal{F}$ obtained as the analytic continuation of the map η_2/η_1 on $\boldsymbol{H}_-$ is called the Schwarz map of (2.1). The image of $\mathcal{F}$ is obtained by the iteration of reflection procedure of $\nabla = \nabla(p,q,r)$. Due to the condition $(*\ )$, the reflection images of ∇ makes a tessellation of $\boldsymbol{H}$. In other words, the monodromy group of (2.1) is given as the totality of the iteration of the reflection procedures of even times. We call this group a triangle group $\Delta = \Delta(p,q,r)$. Note that the fundamental region of this monodromy group Δ is composed of ∇ and its reflection ∇' with respect to one side of ∇ (see Fig.2.1).

The inverse map $\phi(z)$ of $\mathcal{F}$ becomes to be a modular function defined on $\boldsymbol{H}$ with respect to the triangle group $\Delta(p,q,r)$. Let z_1, z_2, z_3 be the verteces of ∇ obtained as $\mathcal{F}(0), \mathcal{F}(1), \mathcal{F}(\infty)$, respectively.

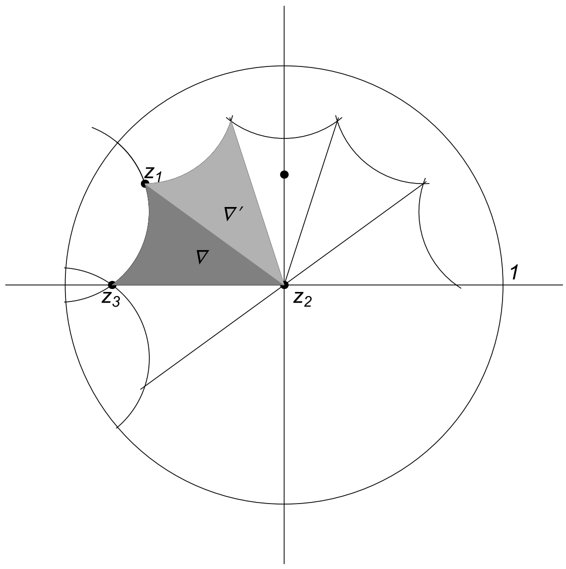

Fig. 2.1 : Triangle $\nabla(5,5,5)$ and its reflections

Definition 2.1. Let (2.1) be a Gauss hypergeometric differential equation with the condition $(*\ )$. We call the inverse $\phi(z)$ of the above Schwarz map **a normalized hypergeometric modular function**. Note that its values at the verteces are fixed in the form : $\phi(z_1) = 0, \phi(z_2) = 1, \phi(z_3) = \infty$.

§ 3. Quaternion algebra over a totally real field

For general introduction of the quaternion algebra, see [VgtL]. Let F be a totally real number field. Let a, b be elements of F satisfying the condition

$$(\mathrm{Cd}) : \begin{cases} a, b \in F : a < 0, b > 0 \\ \\ \text{all their conjugates other than } a, b \text{ are negative.} \end{cases}$$

By them we set a quaternion algebra $\boldsymbol{B} = F + F\alpha + F\beta + F\alpha\beta$ $,\beta\alpha = -\alpha\beta, \alpha^2 = a, \beta^2 = b$. We denote it by $\left(\frac{a,b}{F}\right)$. For $x = x_1 + x_2\alpha + x_3\beta + x_4\alpha\beta \in \boldsymbol{B}$, we define its conjugate $\overline{x} = x_1 - x_2\alpha - x_3\beta - x_4\alpha\beta$.

The reduced trace and the reduced norm of $x \in \boldsymbol{B}$ are defined by $\mathrm{Trd}(x) = x + \overline{x}, \mathrm{Nrd}(x) = x\overline{x}$, respectively.

Setting

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, M_x = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}, M_y = \begin{pmatrix} \sqrt{b} & 0 \\ 0 & -\sqrt{b} \end{pmatrix}, M_z = M_x M_y,$$

we have an isomorphism

$$\boldsymbol{B} \cong F\ M_1 + F\ M_x + F\ M_y + F\ M_z.$$

of noncommutative F algebras. In this way $\boldsymbol{B}$ is embedded in $M_2(\overline{\boldsymbol{Q}})$. We identify $\boldsymbol{B}$ with the subalgebra $F\ M_1 + F\ M_x + F\ M_y + F\ M_z \subset M_2(\boldsymbol{R})$ by this embedding. So we have $\mathrm{Trd}(x) = \mathrm{Tr}(x), \mathrm{Nrd}(x) = \det(x)$.

Set $\boldsymbol{B}^+ = \{x \in \boldsymbol{B} : \det(x) \gg 0\}$, where $x \gg 0$ means x is totally positive as an element of F.

Let $\mathcal{O}_F$ stand for the ring of integers of F. We denote the group of units in $\mathcal{O}_F$ by E_1. And set $E_0 = \{g \in E_1 : \det(g) \gg 0\}$. If it holds $\mathrm{Tr}(\gamma) \in \mathcal{O}_F, \det(\gamma) \in \mathcal{O}_F$ for an element $\gamma \in \boldsymbol{B}$, we say it is an integral element of $\boldsymbol{B}$.

If an $\mathcal{O}_F$ -module of rank 4 is a subring in $\boldsymbol{B}$, we say it is an **order** of $\boldsymbol{B}$. Note that any element of an order is an integral element. Under the condition (Cd) the quaternion algebra $\boldsymbol{B}$ has the unique maximal order up to isomorphism of F-algebras. We fix it and denote it by $\mathcal{O} = \mathcal{O}_{\boldsymbol{B}}$.

Definition 3.1. According to Takeuchi [Tku2] (left) and Shimura [SmrB] (right) we use the following notation:

(3.1) $$\Gamma^{(1)}(\boldsymbol{B}, \mathcal{O}) = \Gamma(\mathcal{O}) = \{\gamma \in \mathcal{O} : \det(\gamma) = 1\},$$

(3.2) $$\Gamma^{+}(\boldsymbol{B}, \mathcal{O}) = \Gamma(\mathcal{O}, 1) = \{\gamma \in \mathcal{O} : \det(\gamma) \in E_0\},$$

(3.3) $$\Gamma^{(*)}(\boldsymbol{B}, \mathcal{O}) = \Gamma^{*}(\mathcal{O}) = \{\gamma \in \boldsymbol{B}^+ : \gamma\mathcal{O} = \mathcal{O}\gamma\}.$$

These groups are called conventionally as the **norm 1 group**, the **unit group** and the **normalizer group**, respectively. Note that the norm 1 group is a subgroup of the unit group of finite index, and the unit group is a subgroup of the normalizer group of finite index. Under the condition (Cd), these groups acts on $\boldsymbol{H}$ as discrete groups.

Definition 3.2. If a co-compact triangle group Δ is commensurable with the unit group of a certain quaternion algebra $\boldsymbol{B} = \left(\frac{a,b}{F}\right)$ up to conjugation in $SL_2(\boldsymbol{R})$, we say Δ is **a (co-compact) arithmetic triangle group**.

K. Takeuchi (see [Tku1], [Tku2]) has determined all arithmetic triangle groups. There are 85 in total, and they are classified into 19 classes of commensurable families (it contains one commensurable class of non-compact type, see Table App.1 in Appendix).

Moreover Takeuchi showed that the quaternion algebra $\boldsymbol{B} = \left(\frac{a,b}{F}\right)$ corresponding to $\Delta = \Delta(e_1, e_2, e_3) \quad (e_1 \le e_2 \le e_3)$ is given by

$$
\begin{aligned}
&a = t_2^2(t_2^2 - 4),\\
&b = t_2^2 t_3^2(t_1^2 + t_2^2 + t_3^2 + t_1 t_2 t_3 - 4),\\
&F = \boldsymbol{Q}(t_1^2, t_2^2, t_3^2, t_1 t_2 t_3), \text{where } t_i = 2\cos\frac{\pi}{e_i}.
\end{aligned}
$$

Table of norm 1 groups, unit groups and normalizer groups according to K. Takeuchi [Tku2] :

class	base field F	discriminant	$\Gamma^{(1)}(\boldsymbol{B},\mathcal{O})$	$\Gamma^{+}(\boldsymbol{B},\mathcal{O})$	$\Gamma^{(*)}(\boldsymbol{B},\mathcal{O})$
(I)	$\boldsymbol{Q}$	(1)	$(2,3,\infty)$	$(2,3,\infty)$	$(2,3,\infty)$
(II)	$\boldsymbol{Q}$	(2)(3)	$(0;2,2,3,3)$	$(0;2,2,3,3)$	$(2,4,6)$
(III)	$\boldsymbol{Q}(\sqrt{2})$	$\mathfrak{p}_2$	$(3,3,4)$	$(3,3,4)$	$(2,3,8)$
(IV)	$\boldsymbol{Q}(\sqrt{3})$	$\mathfrak{p}_2$	$(3,3,6)$	$(2,3,12)$	$(2,3,12)$
(V)	$\boldsymbol{Q}(\sqrt{3})$	$\mathfrak{p}_3$	$(0;2,2,2,6)$	$(2,4,12)$	$(2,4,12)$
(VI)	$\boldsymbol{Q}(\sqrt{5})$	$\mathfrak{p}_2$	$(2,5,5)$	$(2,5,5)$	$(2,4,5)$
(VII)	$\boldsymbol{Q}(\sqrt{5})$	$\mathfrak{p}_3$	$(3,5,5)$	$(3,5,5)$	$(2,5,6)$
(VIII)	$\boldsymbol{Q}(\sqrt{5})$	$\mathfrak{p}_5$	$(3,3,5)$	$(3,3,5)$	$(2,3,10)$
(IX)	$\boldsymbol{Q}(\sqrt{6})$	$\mathfrak{p}_2$	$(0;2,3,3,3)$	$(3,4,6)$	$(3,4,6)$
(X)	$\boldsymbol{Q}(\cos(\frac{\pi}{7}))$	(1)	$(2,3,7)$	$(2,3,7)$	$(2,3,7)$
(XI)	$\boldsymbol{Q}(\cos(\frac{\pi}{9}))$	(1)	$(2,3,9)$	$(2,3,9)$	$(2,3,9)$
(XII)	$\boldsymbol{Q}(\cos(\frac{\pi}{9}))$	$\mathfrak{p}_2\mathfrak{p}_3$	$(0;2,2,9,9)$	$(0;2,2,9,9)$	$(2,4,18)$
(XIII)	$\boldsymbol{Q}(\cos(\frac{\pi}{8}))$	$\mathfrak{p}_2$	$(3,3,8)$	$(3,3,8)$	$(2,3,16)$
(XIV)	$\boldsymbol{Q}(\cos(\frac{\pi}{10}))$	$\mathfrak{p}_2$	$(5,5,10)$	$(2,5,20)$	$(2,5,20)$
(XV)	$\boldsymbol{Q}(\cos(\frac{\pi}{12}))$	$\mathfrak{p}_2$	$(3,3,12)$	$(2,3,24)$	$(2,3,24)$
(XVI)	$\boldsymbol{Q}(\cos(\frac{\pi}{15}))$	$\mathfrak{p}_3$	$(5,5,15)$	$(2,5,30)$	$(2,5,30)$
(XVII)	$\boldsymbol{Q}(\cos(\frac{\pi}{15}))$	$\mathfrak{p}_5$	$(3,3,15)$	$(2,3,30)$	$(2,3,30)$
(XVIII)	$\boldsymbol{Q}(\sqrt{2},\sqrt{5})$	$\mathfrak{p}_2$	$(4,5,5)$	$(4,5,5)$	$(2,5,8)$
(XIX)	$\boldsymbol{Q}(\cos(\frac{\pi}{11}))$	(1)	$(2,3,11)$	$(2,3,11)$	$(2,3,11)$

Table 3.1, List by K. Takeuchi

Remark 3.3. The class number $h(F)$ of F is always equal to 1.

Remark 3.4. We are interested in the unit group $\Gamma^{+}(\boldsymbol{B},\mathcal{O})$. In Table 3.1, there are two cases, Class II and Class XII, where it is not a triangle group but is a quadrangle group. We exclude these cases in the following argument.

§ 4. Shimura's complex multiplication theorem for triangle cases

Let F be a totally real number field, and let M be a CM field over F. Namey, M is a totally imaginary field that is a qudratic extension of F. Shimura made a long research on the complex multiplication theory of CM fields ([SmrA]). We can find his main result in [SmrB]. He showed four main theorems there. Especially we are concerned with the first main theorem for a special case $r = 1$. We can restate this specialized case as the following.

Let F be a totally real number field, and let $\boldsymbol{B} = \left(\frac{a,b}{F}\right)$ be a quaternion algebra satisfying the condition (Cd). In this case $\boldsymbol{B}$ satisfies the condition $r = 1$ with the terminology of Shimura.

[Shimura's Main theorem I for the case $r = 1$] ([SmrB] Theorem 3.2 p.73). Take above mentioned F, M and $\boldsymbol{B}$. Assume an embedding $f : M \hookrightarrow \boldsymbol{B}$ satisfying $f(\mathcal{O}_M) \subset \mathcal{O}_{\boldsymbol{B}}$, where $\mathcal{O}_M$ stands for the ring of integers of M. Then there are a nonsingular compact complex variety V and a modular function $\psi(z)$ on $\boldsymbol{H}$ with respect to $\Gamma(\mathcal{O}, 1)$ satisfying the following condition:

(1) $\psi(z)$ induces a biholomorphic correspondence $\boldsymbol{H}/\Gamma(\mathcal{O}, 1) \cong V$,

(2) V is defined over $C(F)$ (the Hilbert class field of F),

(3) for a regular fixed point (that is explained below) $z_0 \in \boldsymbol{H}$ of M, it holds $M(\psi(z_0)) \cdot C(F) = C(M)$, where $C(M)$ stands for the Hilbert class field of M.

[Regular fixed point $z_0 \in \boldsymbol{H}$ of M]. Recall the embedding $f : M \hookrightarrow \boldsymbol{B}$. We can put $M = F(\alpha), \alpha \in \mathcal{O}_M$. The linear transformation $g = f(\alpha)$ has unique fixed point in $\boldsymbol{H}$. That gives our regular fixed point.

Definition 4.1. The above pair (ψ, V) is called **a canonical model** for $\boldsymbol{H}/\Gamma(\mathcal{O}, 1)$.

Remark. The canonical model is unique up to $\mathrm{Aut}_{C(F)}(V)$. (see [SmrB] Theorem 3.3).

We are going to give a visualization of the above canonical model theorem for arithmetic triangle cases.

Suppose a quaternion algebra $\boldsymbol{B} = \left(\frac{a,b}{F}\right)$ is corresponding to a certain arithmetic triangle group. Moreover, assume that we have $\Delta(e_1, e_2, e_3) = \Gamma^+(\boldsymbol{B}, \mathcal{O})(= \Gamma(\mathcal{O}, 1))$. Suppose the triangle group $\Delta(e_1, e_2, e_3)$ is generated by the triangle $\nabla = \nabla(e_1, e_2, e_3)$. Let z_1, z_2, z_3 be the corresponding verteces of ∇, respectively.

Let us observe the shape of ∇. Due to Takeuchi's Table 1, there are 18 commensurable classes for co-compact arithmetic triangle groups. As for the unit groups of the corresponding quaternion algebras, there are 10 triangles with three different angles as their generating ∇'s and 6 ∇'s with a pair of equal sides that is not a regular triangle. We call the former **of scalene type** unit group and the latter **of isosceles type** unit group.

Proposition 4.2. *Let* $\boldsymbol{B} = \left(\frac{a,b}{F}\right)$ *be quaternion algebra corresponding to an isosceles type unit group* $\Delta(e_1, e_1, e_3)$ $(e_1 = e_2 \neq e_3)$ *in Table 1. The extension* $M_0 = F(\zeta_{e_1})$ *becomes to be a CM field over* F*, where* ζ_ν *means the primitive root of unity of order* ν*. Moreover, it holds* $M_0 = F(i\sqrt{\rho})$ *with the following table of* ρ*.*

Generators of CM field M_0 for isosceles types

class	F	$\Gamma(\mathcal{O},1)$	M_0	ρ
(III)	$\boldsymbol{Q}(\sqrt{2})$	$\Delta(3,3,4)$	$F(\zeta_3)$	3
(VI)	$\boldsymbol{Q}(\sqrt{5})$	$\Delta(5,5,2)$	$F(\zeta_5)$	$\sin(\frac{\pi}{5})$
(VII)	$\boldsymbol{Q}(\sqrt{5})$	$\Delta(5,5,3)$	$F(\zeta_5)$	$\sin(\frac{\pi}{5})$
(VIII)	$\boldsymbol{Q}(\sqrt{5})$	$\Delta(3,3,5)$	$F(\zeta_3)$	3
(XIII)	$\boldsymbol{Q}(\cos(\frac{\pi}{8}))$	$\Delta(3,3,8)$	$F(\zeta_3)$	3
(XVIII)	$\boldsymbol{Q}(\sqrt{2},\sqrt{5})$	$\Delta(5,5,4)$	$F(\zeta_5)$	$\sin(\frac{\pi}{5})$

Table 4.1.

Proof. See [N-S]. □

Theorem 4.3 (Main Theorem)**.** *Let* $\boldsymbol{B}/F$ *a quaternion algebra coming from a co-compact arithmetic triangle group. Set* $\Delta = \Delta(e_1, e_2, e_3) = \Gamma(\mathcal{O}_{\boldsymbol{B}}, 1)$*. Let* $\nabla(z_1, z_2, z_3)$ *be a triangle on* $\boldsymbol{H}$ *with verteces* $z_i (i = 1, 2, 3)$ *which generates the triangle group* $\Delta(e_1, e_2, e_3)$*. Assume the order of* z_i *is equal to* $e_i (i = 1, 2, 3)$*.*

(I) *The case* Δ *is a unit group of scalene type. Let* $\varphi(z)$ *be a hypergeometric modular function with respect to* $\Delta(e_1, e_2, e_3)$ *which is normalized with the condition*

$$\text{(Ncd)}: \qquad \varphi(z_1) = 1, \varphi(z_2) = -1, \varphi(z_3) = \infty.$$

Then it gives the canonical model of $\boldsymbol{H}/\Gamma(\mathcal{O}_{\boldsymbol{B}}, 1)$ *together with the Riemann sphere* $S = \boldsymbol{P}^1$ *that is the image of* $\varphi(z)$*.*

(II) *The case* Δ *is a unit group of isosceles type.*

Let $\varphi(z)$ *be a hypergeometric modular function with the same condition* (Ncd) *in* (I)*. We make another function* $\tilde{\varphi}(z)$ *by using a pure imaginary number* $i\sqrt{\rho}$ *, where* ρ *is the number obtained in the previous proposition:*

$$\tilde{\varphi}(z) = i\sqrt{\rho} \cdot \varphi(z),$$

Then one of $\varphi(z)$ *and* $\tilde{\varphi}(z)$ *becomes to be the canonical model together with the image* $S = \boldsymbol{P}^1$*.*

Proof. See [N-S]. □

Remark 4.4. At this moment we don't have a criterion to determine which of two candidates in (II) becomes to be the canonical model. Later, we shall show the example that $\tilde{\varphi}(z)$ is the canonical model.

§ 5. An explicit form of Shimura's canonical model for $\Delta(3,3,5)$

Koike's modular function for a family of Pentagonal curves The Schwarz map for the Gauss hypergeometric differential equation $E(\frac{2}{5},\frac{3}{5},\frac{6}{5})$ can be identified (by way of the integral representation of the two independent solutions) with the period map for a family of algebraic curves of genus 4:

$$C(\lambda) : y^5 = x^2(x-1)(x-\lambda)$$

with a parameter λ. The monodromy group is the triangle group $\Delta = \Delta(5,5,5)$. By considering the inverse map we obtain the modular function $\lambda(u)$ with respect to Δ that is defined on the period domain.

K. Koike [Kik] showed a representation of $\lambda(u)$ in terms of the Riemann theta constants.

Suppose $0 < \lambda < 1$. We regard $C(\lambda)$ as a 5-sheeted branched cover over the x-plane with cut lines connecting the base point $x_0 \in \boldsymbol{H}_-$ and critical points $x = 0, \lambda, 1, \infty$. Set γ_2, γ_3 be two homology cycles on $C(\lambda)$ indicated in Fig.5.1 below, where (1) means the analytic continuation of the real branch of y on $x > 1$ along an arc that does not intersect the indicated cut lines, and (2) and (3) means the branches of y given by $\rho_5(1)$ and $\rho_5^2(1)$, respectively, where $\rho_5 = e^{2\pi i/5}$.

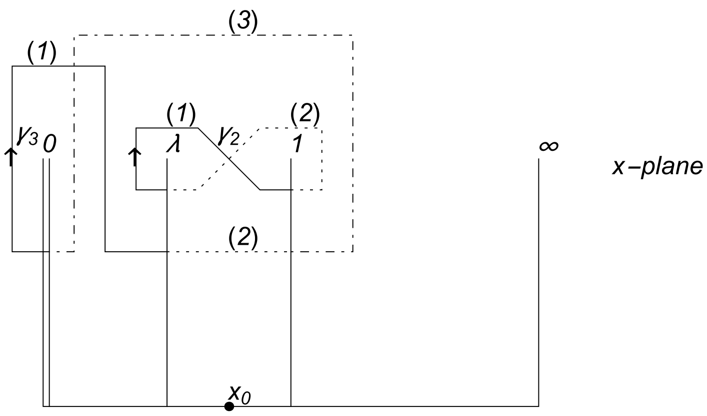

Fig.5.1 : homology cycles on $C(\lambda)$

By making the analytic continuation, the integrals

$$\eta_2(\lambda) = \int_{\gamma_2} \frac{dx}{y^2}, \eta_3(\lambda) = \int_{\gamma_3} \frac{dx}{y^2}$$

give multivalued analytic functions on the λ space. They are independent solutions of $E(\frac{2}{5},\frac{3}{5},\frac{6}{5})$. According to Koike[Kik] , the image of the Schwarz map $\mathcal{F}(\lambda)=\frac{\eta_2(\lambda)}{\eta_3(\lambda)}$ is given by the disc in $\boldsymbol{P}^1$:

$$\mathcal{D}=\{[\eta_2,\eta_3]\in\boldsymbol{P}^1:|\eta_2|^2+\omega|\eta_3|^2<0\},\ (\omega=(1-\sqrt{5})/2). \tag{5.1}$$

We have a modular embedding of $\mathcal{D}$ into the Siegel upper half space $\mathfrak{S}_4=\{\Omega\in GL_4(\boldsymbol{C}):{}^t\Omega=\Omega,\Im\Omega>0\}$ by the following manner:

$$\Omega(u)=\frac{1}{\eta_2^2-\left(1+e^{\frac{2\pi i}{5}}\right)e^{-\frac{4i\pi}{5}}\eta_3^2}$$

$$\left[\begin{pmatrix} \left(-1+e^{-\frac{4i\pi}{5}}\right)\left(\eta_2^2+\eta_3^2\right) & \left(1-e^{\frac{4i\pi}{5}}\right)\eta_2\eta_3 & 0\,0\\ \left(1-e^{\frac{4i\pi}{5}}\right)\eta_2\eta_3 & \left(-1+e^{\frac{4i\pi}{5}}\right)\left(\eta_2^2-e^{-\frac{4i\pi}{5}}\eta_3^2\right) & 0\,0\\ e^{-\frac{4i\pi}{5}}\left(\left(1+e^{\frac{2i\pi}{5}}\right)\eta_2^2+\eta_3^2\right) & \left(1-e^{-\frac{4i\pi}{5}}\right)\eta_2\eta_3 & 0\,0\\ \left(-e^{\frac{2i\pi}{5}}+e^{-\frac{4i\pi}{5}}\right)\eta_2\eta_3 & \left(e^{\frac{2i\pi}{5}}+e^{\frac{4i\pi}{5}}\right)\left(\eta_2^2-e^{-\frac{4i\pi}{5}}\left(1+e^{\frac{4i\pi}{5}}\right)\eta_3^2\right) & 0\,0 \end{pmatrix}\right.$$

$$\left.+\begin{pmatrix} 0\,0\; e^{-\frac{4i\pi}{5}}\left(\left(1+e^{\frac{2i\pi}{5}}\right)\eta_2^2+\eta_3^2\right) & \left(-e^{\frac{2i\pi}{5}}+e^{-\frac{4i\pi}{5}}\right)\eta_2\eta_3\\ 0\,0\quad \left(1-e^{-\frac{4i\pi}{5}}\right)\eta_2\eta_3 & \left(e^{\frac{2i\pi}{5}}+e^{\frac{4i\pi}{5}}\right)\left(\eta_2^2-e^{-\frac{4i\pi}{5}}\left(1+e^{\frac{4i\pi}{5}}\right)\eta_3^2\right)\\ 0\,0\,-e^{\frac{4i\pi}{5}}\left(\eta_2^2-\left(1+e^{\frac{2i\pi}{5}}\right)\eta_3^2\right) & \left(e^{-\frac{2i\pi}{5}}-e^{\frac{2i\pi}{5}}\right)\eta_2\eta_3\\ 0\,0\quad \left(e^{-\frac{2i\pi}{5}}-e^{\frac{2i\pi}{5}}\right)\eta_2\eta_3 & -e^{-\frac{4i\pi}{5}}\left(\eta_2-\left(1+e^{-\frac{2i\pi}{5}}\right)\eta_3^2\right) \end{pmatrix}\right],$$

where $u=\frac{\eta_2}{\eta_3}$. Note that the monodromy group $\Delta(5,5,5)$ is given as a subgroup of $Sp_8(\boldsymbol{Z})$ that preserves $\Omega(\mathcal{D})$.

Set the Riemann theta constant on $\mathfrak{S}_4$ with a characteristic $(a,b)\in(\boldsymbol{Q}^4)^2$:

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\Omega)=\sum_{n\in\boldsymbol{Z}^4}\exp[\pi i\,{}^t(n+a)\Omega(n+a)+2\pi i\,{}^t(n+a)b].$$

We define two following theta characteristics:

$$a_{11}=\begin{bmatrix}a\\b\end{bmatrix}=\frac{1}{10}\begin{bmatrix}1&1&1&1\\-2&-2&-1&-1\end{bmatrix}$$

and

$$a_{19}=\begin{bmatrix}a\\b\end{bmatrix}=\frac{1}{10}\begin{bmatrix}1&9&1&9\\-2&-8&-1&-9\end{bmatrix}.$$

We define two theta functions on $\mathcal{D}$:

$$\theta_{11}(u)=\vartheta[a_{11}](\Omega(u)),\theta_{19}(u)=\vartheta[a_{19}](\Omega(u)).$$

Theorem 5.1 (K. Koike [Kik]). *The function* $\lambda(u) = \left(\frac{\theta_{11}(u)}{\theta_{19}(u)}\right)^5$ *on* $\mathcal{D}$ *gives the inverse of the Schwarz map* $\mathcal{F}(\lambda)$*, and it gives a holomorphic isomorphism:* $\mathcal{D}/\Delta(5,5,5) \xrightarrow{\sim} \boldsymbol{P}^1(\boldsymbol{C})$*. Especially, it holds* $\lambda(\omega e^{-2\pi i/5}) = 0, \lambda(\omega) = \infty, \lambda(0) = 1$*, where* $\omega = \frac{1-\sqrt{5}}{2}$.

Remark. Set $(\zeta_1, \zeta_2, \zeta_3) = (\omega e^{-2\pi i/5}, 0, \omega)$. The triangle $\mathcal{F}(\boldsymbol{H}_-) = \nabla(\zeta_1, \zeta_2, \zeta_3)$ is a generating triangle of $\Delta(5,5,5)$ (see Fig. 5.2).

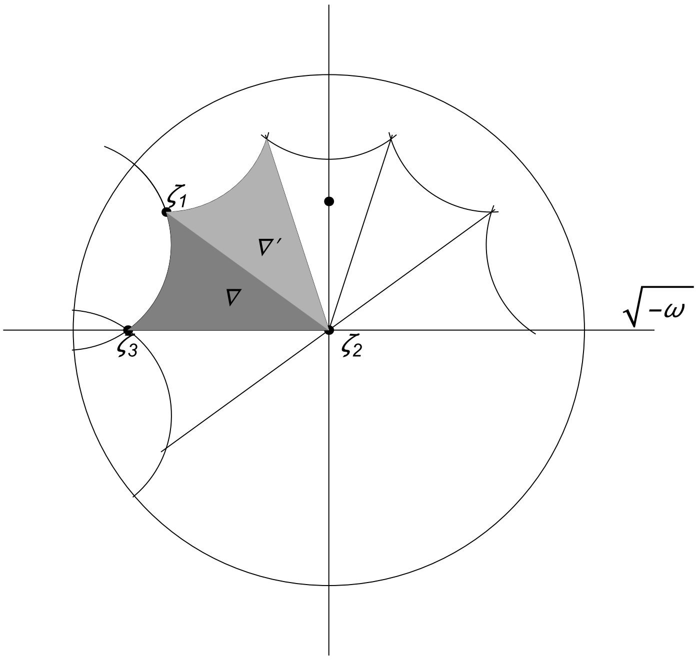

Fig.5.2 : Koike Period disc $\mathcal{D}$

§ 6. Examples of the Hilbert class fields of higer degree

The normalized hypergeometric function for the class VIII. Let us consider the quaternion algebra $\boldsymbol{B} = \left(\frac{a,b}{F}\right)$ arising in the class (VIII) in the table of Takeuchi. In this case, we have the unit group $\Gamma(\mathcal{O},1) = \Delta(3,3,5)$ (see Table 2.1). We may regard $\Delta(5,5,5)$ is a subgroup of $\Delta(3,3,5)$ of index 3. Set u_1 be the varicentric point of $\nabla(\zeta_1,\zeta_2,\zeta_3)$, and set $(u_1,u_2,u_3) = (u_1,\overline{u_1},0)$. The triangle $\nabla(u_1,u_2,u_3)$ becomes to be a generating triangle of $\Delta(3,3,5)$ (see Fig. 6.1)

By using Koike's λ function, set

$$\Phi(u) = \frac{1}{3}\frac{\lambda^3 - 3\lambda + 1}{\lambda(\lambda - 1)}.$$

Then $\varphi(u) = \frac{2}{\sqrt{-3}}\left(\Phi(u) - \frac{1}{2}\right)$ is the normalized hypergeometric modular function for $\Delta(3,3,5)$ in the sense that $(\varphi(u_1), \varphi(u_2), \varphi(u_3)) = (1, -1, \infty)$. So, $\tilde{\varphi}(u) = \Phi(u) - \frac{1}{2}$ gives the modular function of the same symbol in the main theorem 4.3.

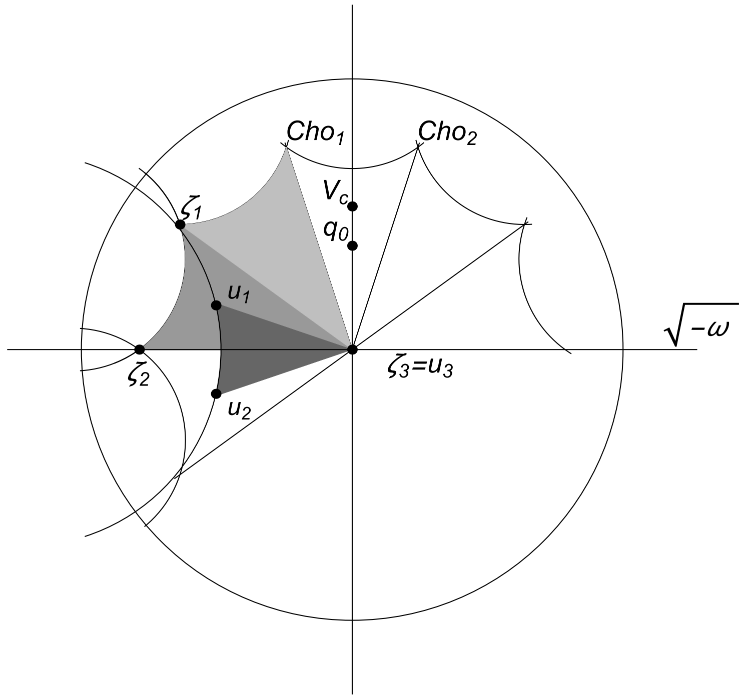

Fig.6.1. : Relation between $\Delta(5,5,5)$ and $\Delta(3,3,5)$

Set

$$\begin{aligned}
BG_1 &= M_1,\\
BG_2 &= M_1 + (\frac{1}{2}\omega)M_y + \frac{1}{2}M_z,\\
BG_3 &= (\frac{1}{2} - \frac{1}{2}\omega)M_1 + (\frac{1}{2}\omega)M_x,\\
BG_4 &= M_1 + \omega M_y,\ (\omega = \frac{1-\sqrt{5}}{2}).
\end{aligned}$$

They give a system of basis of the maximal order $\mathcal{O}_{\boldsymbol{B}}$ as a F-module.

Remark 6.1. Set

$$M_{mc} = \begin{pmatrix} \sqrt{\sqrt{5}\overline{\omega}} & 0 \\ 0 & 1 \end{pmatrix}, M_{mr} = \begin{pmatrix} i & i\sqrt{-\omega} \\ -1 & \sqrt{-\omega} \end{pmatrix}.$$

The composition $M_{hd} = M_{mc}M_{mr}$ induces an isomorphism $\mathcal{D} \xrightarrow{\sim} \boldsymbol{H}$. For a linear transformation h acting on $\mathcal{D}$, we obtain a transformation $\tilde{h} = M_{hd} \circ h \circ M_{hd}^{-1}$ acting on $\boldsymbol{H}$. This shifting procedure induces an identification of the triangle group $\Delta(3,3,5)$

and the unit group $\Gamma(\mathcal{O}, 1)$. So, we identify the modular function on $\boldsymbol{H}$ with that on $\mathcal{D}$ through this correspondence.

Examples

Example 6.1 For a CM field $M = \boldsymbol{Q}(\sqrt{5}, \sqrt{-7})$ over $F = \boldsymbol{Q}(\sqrt{5})$, we have $h(M) = 1$, ($h(M)$ is the class number of M). By taking

$$G_0 = (-3 + 2\omega)BG_1 + 2\omega BG_2 + (4 - 2\omega)BG_3 + (-2\omega)BG_4,$$

it holds $\mathrm{Tr}(G_0) = 0, \det(G_0) = 7$. So $G_0^2 + 7E = 0$. By the correspondence $\sqrt{-7} \mapsto G_0$, we realize an embedding of M into $\boldsymbol{B}$. We have unique fixed point of G_0 in $\mathcal{D}$:

$$u_0 = (-0.205396\cdots) - (0.0667372\cdots)i.$$

By an approximate calculation, we see $\tilde{\varphi}(u_0) = (8.378212485037870203255116553190991358 9\cdots)i$. It holds $\tilde{\varphi}(u_0)^2 = -\frac{2527}{36} = -2^{-2}3^{-2}19^2 \times 7$. Then, $\tilde{\varphi}(u_0) \equiv \sqrt{-7}$ ($\mod \boldsymbol{Q}^*$) $\in M = C(M)$.

It means that $\tilde{\varphi}(u)$ is the modular function that gives the canonical model of $\boldsymbol{H}/\Gamma(\mathcal{O}_{\boldsymbol{B}}, 1)$ for the class VIII, and that the normalized modular function $\varphi(u)$ does not bring the canonical model.

Example 6.2 We find in [HHRWH] that a CM field $M = \boldsymbol{Q}(\sqrt{-(5 + \sqrt{5})})$ has $h(M) = 2$. Setting

$$G_0 = (3 - 3\omega)BG_1 + BG_2 + (-4 + 2\omega)BG_3 + (-1 + \omega)BG_4,$$

we obtain

$$G_0^2 + 5 + \sqrt{5} = 0.$$

So $M = F(G_0)$ in $\boldsymbol{B}$. We have the unique fixed point of G_0 :

$$u_0 = -0.164894 - 0.119803I \in \mathcal{D}.$$

We have

$$\varphi(u_0)^2 = -165.3749999999584 = -3^3 7^2/2^3$$

(Note that $\tilde{\varphi}(u) = \sqrt{-3}\varphi(u)$).

So we obtain the Hilbert class field $C(M) = M(\sqrt{2})$.

Example 6.3 Set $M = \boldsymbol{Q}(\sqrt{-(65 - 26\sqrt{5})})$. Due to [HHRWH] $h(M) = 2$. Take

$$G_0 = (1 - 2\omega)BG1 + 2BG2 + (-8 - 2\omega)BG3 + (-2\omega)BG4.$$

Then

$$G_0^2 + 65 - 26\sqrt{5} = 0.$$

Hence G_0 is a generator of M in $\boldsymbol{B}$. We have a fixed point u_0 of G_0:

$$u_0 = (-0.28840319370820624304292929605443107245953523857814338756287042769 40\cdots)$$
$$+(0.20953718544157995477915015322282420209591214649546391497133897907 53\cdots)i \in \mathcal{D}$$

Hence,

$$\varphi(u_0)^2 = -0.16717727965490681624739779831313980904\cdots.$$

By the expansion into a continued fraction

$$-\varphi(u_0)^2 = [0, 5, 1, 53, 1, 1, 3, 4, 1, 12, 7, 74, 2, 2, 41105985538320721741, \cdots]$$

Hence,

$$\varphi(u_0)^2 = -13 \cdot 29^2 \cdot 79^2 \cdot 2^{-8} \cdot 3^{-13}$$

As a consequence we have the Hilbert class field $C(M) = M(\sqrt{13})$.

Example 6.4 The case $M = \boldsymbol{Q}(\sqrt{5}, \sqrt{-23})$. It holds $h(M) = 3$ (note that $h(\boldsymbol{Q}(\sqrt{-23})) = 3$ also). We may choose two different generators of M in $\mathcal{O}_{\boldsymbol{B}}$:

$$\begin{aligned}
&\frac{1}{2}(M_1 + (-3+\omega)M_x + (1-\omega)M_y - 3\omega M_z) \\
&= -2BG_1 + 3\omega BG_2 + (4-3\omega)BG_3 + (-1-\omega)BG_4 \\
&\frac{1}{2}(M_1 + (1-3\omega)M_x + (1-\omega)M_y - \omega M_z) \\
&= (4-3\omega)BG_1 + \omega BG_2 + (-4+\omega)BG_3 - BG_4.
\end{aligned}$$

Let $\tilde{\varphi}_4, \tilde{\varphi}_{11}$ be the values of $\tilde{\varphi} = \Phi(u) - \frac{1}{2}$ at the regular fixed point respectively. We have approximate values

$$\begin{aligned}
&\tilde{\varphi}_4 = 0.41467884460813106945405483718570037432931049417786518217 \\
&-0.99585830192876518343449922550065726567619225232766421487i, \\
&\tilde{\varphi}_{11} = 13.71950951993067249305997446719063079570023469944061038 7i.
\end{aligned}$$

According to our main theorem any of $\tilde{\varphi}_4, \overline{\tilde{\varphi}_4}, \tilde{\varphi}_{11}$ can be a generator of $C(M)$ over F. Because $[C(M) : M] = 3$, also any of $\tilde{\varphi}_4^2, \overline{\tilde{\varphi}_4}^2, \tilde{\varphi}_{11}^2$ be a generator of $C(M)$. We have approximate values

$$\begin{aligned}
\check{r}_1 &= -(2^8\tilde{\varphi}_4^2 + 2^8\overline{\tilde{\varphi}_4}^2 + 2^8\tilde{\varphi}_{11}^2) = \frac{298002375630573376}{6131066257801}, \\
\check{r}_2 &= 2^8\tilde{\varphi}_{11}^2(2^8\tilde{\varphi}_4^2 + 2^8\overline{\tilde{\varphi}_4}^2) + 2^{16}\tilde{\varphi}_4^2\overline{\tilde{\varphi}_4}^2 = \frac{2794455869937937 2032}{1375668606321}, \\
\check{r}_3 &= -2^{24}\tilde{\varphi}_{11}^2\tilde{\varphi}_4^2\overline{\tilde{\varphi}_4}^2 = \frac{146663661576709210112}{34296447249}.
\end{aligned}$$

Hence, we obtain a cubic equation for $2^8\tilde{\varphi}_4^2, 2^8\overline{\tilde{\varphi}_4}^2, 2^8\tilde{\varphi}_{11}$ with rational coefficients:

$$t^3 + \check{r}_1 t^2 + \check{r}_2 t + \check{r}_3 = 0, \tag{6.1}$$

where $t = 2^8\tilde{\varphi}^2$. The above (6.1) is a defining equation of the Hilbert class field of M. Putting $Y = 11^3 2^8 5^4 \tilde{\varphi}^{-1}$, we have integral defining equation

$$Y^3 + 19268Y^2 + 12444768657Y + 6131066257801 = 0. \tag{6.2}$$

Setting $Y^3 + r_1Y^2 + r_2Y + r_3 = 0$ for it,

$$\begin{aligned}
&r_3 = 19^{10}, \\
&r_2 = 3^2 1382752073, \\
&r_1 = 2^2 4817, \\
&\text{the discriminant dsc} = -19^4 61^2 79^2 89^2 109^2 149^2 229^2 \times 23.
\end{aligned}$$

So we see directly (6.2) defines a Galois extension over $M = F(\sqrt{-23})$ and non-Galois over $\boldsymbol{Q}$.

§ 7. Appendix

In this section, we put several background data for our study. Some of them are not complete list, but it will help to understand the sistuation about what we are doing.

Appendix 1: The Takeuchi list Total list of arithmetic triangle groups by K. Takeuchi ([Tku1]) :

Class	(e_1, e_2, e_3)	Base field F	Discriminant
I	$(2,3,\infty),(2,4,\infty),(2,6,\infty),(2,\infty,\infty),(3,3,\infty),$ $(3,\infty,\infty),(4,4,\infty),(6,6,\infty),(\infty,,\infty,\infty)$	$\boldsymbol{Q}$	(1)
II	$(2,4,6),(2,6,6),(3,4,4),(3,6,6)$	$\boldsymbol{Q}$	(2)(3)
III	$(2,3,8),(2,4,8),(2,6,8),(2,8,8),(3,3,4)$ $(3,8,8),(4,4,4),(4,6,6),(4,8,8)$	$\boldsymbol{Q}(\sqrt{2})$	$\mathfrak{p}_2$
IV	$(2,3,12),(2,6,12),(3,3,6)$ $(3,4,12),(3,12,12),(6,6,6)$	$\boldsymbol{Q}(\sqrt{3})$	$\mathfrak{p}_2$
V	$(2,4,12),(2,12,12),(4,4,6),(6,12,12)$	$\boldsymbol{Q}(\sqrt{3})$	$\mathfrak{p}_3$
VI	$(2,4,5),(2,4,10),(2,5,5)$ $(2,10,10),(4,4,5),(5,10,10)$	$\boldsymbol{Q}(\sqrt{5})$	$\mathfrak{p}_2$
VII	$(2,5,6),(3,5,5)$	$\boldsymbol{Q}(\sqrt{5})$	$\mathfrak{p}_3$
VIII	$(2,3,10),(2,5,10),(3,3,5),(5,5,5)$	$\boldsymbol{Q}(\sqrt{5})$	$\mathfrak{p}_5$
IX	$(3,4,6)$	$\boldsymbol{Q}(\sqrt{6})$	$\mathfrak{p}_2$
X	$(2,3,7),(2,3,14),(2,4,7),(2,7,7)$ $(2,7,14),(3,3,7),(7,7,7)$	$\boldsymbol{Q}(\cos(\pi/7))$	(1)
XI	$(2,3,9),(2,3,18),(2,9,18)$ $(3,3,9),(3,6,18),(9,9,9)$	$\boldsymbol{Q}(\cos(\pi/9))$	(1)
XII	$(2,4,18),(2,18,18),(4,4,9),(9,18,18)$	$\boldsymbol{Q}(\cos(\pi/9))$	$\mathfrak{p}_2\mathfrak{p}_3$
XIII	$(2,3,16),(2,8,16),(3,3,8),(4,16,16),(8,8,8)$	$\boldsymbol{Q}(\cos(\pi/8))$	$\mathfrak{p}_2$
XIV	$(2,5,20),(5,5,10)$	$\boldsymbol{Q}(\cos(\pi/10))$	$\mathfrak{p}_2$
XV	$(2,3,24),(2,12,24),(3,3,12)$ $(3,8,24),(6,24,24),(12,12,12)$	$\boldsymbol{Q}(\cos(\pi/12))$	$\mathfrak{p}_2$
XVI	$(2,5,30),(5,5,15)$	$\boldsymbol{Q}(\cos(\pi/15))$	$\mathfrak{p}_3$
XVII	$(2,3,30),(2,15,30),(3,3,15),(3,10,30),(15,15,15)$	$\boldsymbol{Q}(\cos(\pi/15))$	$\mathfrak{p}_5$
XVIII	$(2,5,8),(4,5,5)$	$\boldsymbol{Q}(\sqrt{2},\sqrt{5})$	$\mathfrak{p}_2$
XIX	$(2,3,11)$	$\boldsymbol{Q}(\cos(\pi/11))$	(1)

Table App.1: Total list of arithmetic triangle group

Remark 7.1. Recently, K. Koike obtained a modular function for the class (X) (see [Kik2]).

Appendix 2. Biquadratic CM fields We perform a calculation of several class numbers for biquadratic fields of type $\boldsymbol{Q}(\sqrt{5},\sqrt{-\ell})$.

Let $k/\boldsymbol{Q}$ be a totally imaginary abelian field , k^+ be its maximal real field.Let h, h^+ be the class numbers of k, k^+, resp. It is known $h^+|h$. $h^- = \frac{h}{h^+}$ is called the relative class number of k. We have a formula for h^-. To state it we need the following terminology:

($\cdot$) $w(k)$: the number of roots of unity in k,

($\cdot$) $W = \langle\zeta\rangle$: the cyclic group of roots of unity in k,

($\cdot$) E, E^+ : the group of units in k, k^+ , resp.

($\cdot$) Let χ be a Dirichlet character for the field k with the conductor f. The generalized Bernoulli number $B_{n,\chi}$ is determined by

$$\sum_{a=1}^{f} \frac{\chi(a)\, t \exp(at)}{\exp(ft)-1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}.$$

$(\cdot)$ The symbol $X(k)$ stands for the collection of all Dilichlet characters for k. A Dirichlet character χ for k is said to be even (odd), if it holds $\chi(-1) = 1$, $\chi(-1) = -1$, respectively. The symbol $X^-(k)$ stands for the collection of odd Dilichlet characters for k.

$(\cdot)$ By $\overline{\chi}$ denote the complex conjugate of a Dilichlet character χ.

$(\cdot)$ $Q(k) = 1$ or 2, and $Q(k) = 2 \iff \varepsilon/\overline{\varepsilon} = \zeta$ for some $\varepsilon \in E$.

Theorem 7.2 ([Kmr] p.19, Thm. 4.11)**.**

$$h^- = w(k)Q(k) \prod_{\chi \in X^-(k)} \left(-\frac{1}{2} B_{1,\overline{\chi}}\right).$$

Let us observe the case $k = \boldsymbol{Q}(\sqrt{5}, \sqrt{-\ell})$. It holds $X(k) = X(k^+) \times X(\boldsymbol{Q}(\sqrt{-\ell}))$. By observing the definition of the above terminology, we can put

$$w(k) = \begin{cases} 2 & (\ell \neq -3) \\ 6 & (\ell = -3), \end{cases}$$
$$Q(M) = 1,$$
$$B_{1,\chi} = \sum_{a=1}^{f} \chi(a) \left(\frac{a}{f} - \frac{1}{2}\right).$$

Moreover, $k = \boldsymbol{Q}(\sqrt{5})$. So it holds $h^+ = 1$ and $h^- = h$.

Remark 7.3. In our case, $\chi^-(k)$ is composed of only two elements. One is the Dirichlet character χ^2 for $\boldsymbol{Q}(\sqrt{-\ell})$ and another χ^3 is the product of χ^2 and the Dirichlet character χ^1 for $\boldsymbol{Q}(\sqrt{5})$. By consulting with the classical theory of quadratic fields, we can find explicit forms of χ^2 and χ^3 (for example [Tkg] p. 295) .

Dirichlet character for $F = \boldsymbol{Q}(\sqrt{d})$. The conductor f_χ of the Dirichlet character χ is $|d|$ with the discriminant d of F.

(i) The case $d \equiv 1 (\mod 4)$. $\chi(a) = \left(\frac{a}{|d|}\right)$.

(ii) The case $d = 4m, m \equiv 3 \ (\mod 4)$. $\chi(a) = (-1)^{(a-1)/2} \left(\frac{a}{|m|}\right) \left|\left(\frac{a}{2}\right)\right|$.

(iii) The case $d = 8m', m' \equiv 1 \ (\mod 4)$. $\chi(a) = (-1)^{(a^2-1)/8} \left(\frac{a}{|m'|}\right) \left|\left(\frac{a}{2}\right)\right|$.

(iv) The case $d = 8m', m' \equiv 3 \pmod 4$. $\chi(a) = (-1)^{(a^2-1)/8+(n-1)/2}\left(\frac{a}{|m'|}\right)\left|\left(\frac{a}{2}\right)\right|$. The fractional bracket means the Jacobi symbol. $\left|\left(\frac{a}{2}\right)\right|$ represents the operation to kill the even a.

Example 7.4. The class number of $k = \boldsymbol{Q}(\sqrt{5}, \sqrt{-6})$, ($k^- = \boldsymbol{Q}(\sqrt{-6})$). Its discriminant is $5^2 \cdot (-24)^2$. The odd Dirichlet character for $\boldsymbol{Q}(\sqrt{-6})$ is given by

$$\chi^2(a) = (-1)^{(a^2-1)/8}\left(\frac{3}{a}\right)\left|\left(\frac{2}{a}\right)\right|.$$

And we have $\chi^3 = \left(\frac{5}{a}\right)\chi^2$. We know their conductors are 24, 120, respectively. So we have $h = h^- = 2\left(-\frac{1}{2}B_{1,\chi^2}\right)\cdot\left(-\frac{1}{2}B_{1,\chi^3}\right) = 4$. Other examples are listed below.

$-\ell$	discriminant(k^-)	Embed(yes,no)	$h(k^-)$	h	$\left(-\frac{1}{2}B_{1,\chi^2}\right)$	$\left(-\frac{1}{2}B_{1,\chi^3}\right)$
-2	-8	y	1	1	$\frac{1}{2}$	1
-3	-3	y	1	1	$\frac{1}{6}$	1
-6	-24	n	2	4	1	2
-7	-7	y	1	1	$\frac{1}{2}$	1
-11	-11	n	1	2	$\frac{1}{2}$	2
-13	-52	y	2	8	1	4
-14	-56	n	4	8	2	2
-17	-68	y	4	8	2	2
-19	-19	n	1	4	$\frac{1}{2}$	4
-21	-84	n	4	16	2	4
-22	-88		2	12	1	6
-23	-23	y	3	3	$\frac{3}{2}$	1
-26	-104		6	12	3	2
-29	-116		6	24	3	4
-31	-31	n	3	6	$\frac{3}{2}$	2
-33	-132		4	16	2	4
-34	-136		4	24	2	6
-37	-148		2	16	1	8
-38	-152		6	12	3	2
-39	-39		4	8	2	2
-41	-164		8	32	4	4
-42	-168		4	16	2	4
-43	-43	y	1	7	$\frac{1}{2}$	7
-46	-184		4	40	2	10
-47	-47	y	5	5	$\frac{5}{2}$	1

Table App.2: Several biquadratic extensions over $\boldsymbol{Q}$

This table tells us that it is not easy to find out such kind of CM fields with lower class number which has an embedding into our quaternion algebra $\boldsymbol{B}$.

Appendix 3. Examples of CM fields of degree 4 those are cyclic over $\boldsymbol{Q}$
According to [HHRWH], [H-P] we have the following:

If a CM field M over $k = \boldsymbol{Q}(\sqrt{D})$ is cyclic over $\boldsymbol{Q}$, then it can be described in the form $M = \boldsymbol{Q}(\sqrt{A(D+B\sqrt{D})})$, where A is a square free odd integer, $B, C \in \boldsymbol{N}$, $(A, D) = 1$ and $D = B^2 + C^2$. Among them with relatively small conductor are given by the following. Especially there are only three CM fields over $F = \boldsymbol{Q}(\sqrt{5})$ with class number 2. Those are appearing in the list:

Cyclic CM field M of degree 4 with small conductor:

number	conductor	discriminant$D$	-A	B	C	$h(M)$
1	5	5	1	2	1	1
6	40	5	1	1	2	2
10	60	5	3	2	1	4
12	65	5	13	2	1	2
18	85	5	17	2	1	2
23	105	5	21	2	1	4
27	120	5	3	1	2	4
30	140	5	7	2	1	4
37	165	5	33	2	1	8
41	185	5	37	2	1	10

Table App.3: Cyclic Extensions of degree 4 over $\boldsymbol{Q}$ with small conductors

Appendix 4. Gauss HGDE on $x + y = 1$ for F_1 Sometimes we obtain an arithmetic triangle group by restricting the Appell hypergeometric differential equation $E_1(a, b, b', c)$ to the hyperplane $x + y = 1$. For all 2- variables cases in the Terada and Deligne-Mostow table (see [Trd], [D-M]), we have the following. Where we use the notation:

$N - 3 =$ the number of variables,
$\frac{\mu_1}{d} = b, \frac{\mu_2}{d} = b', \frac{\mu_3}{d} = c - b - b', \frac{\mu_4}{d} = a + 1 - c, \frac{\mu_5}{d} = 1 - a,$
NA means non arithmetic monodromy group (no indication means to be arithmetic),

∞ means non compact type monodromy group (no indication means to be of compact type),

$\Delta(p, q, r)$ indicates the trangle monodromy group of the hypergeometric differential equation appering on $x + y = 1$, no indication means it does not happen.

In many cases of them, we have an interpretation as families of $K3$ surfaces with a fixed cyclic automorphism. For it , for example see [A-S-T] and [Knd].

no.	N	d	μ_i	Arith	comp	(a,b,b',c)	$\Delta(p,q,r)$
1	5	3	$2,1,1,1,1$		∞	$(\frac{1}{3},\frac{1}{3},\frac{1}{3},1)$	$(3,6,6)$
2	5	4	$2,2,2,1,1$		∞	$(\frac{1}{2},\frac{1}{4},\frac{1}{4},1)$	$(0,4,4)$ nc
3	5	4	$3,2,1,1,1$		∞		
4	5	5	$2,2,2,2,2$			$(\frac{3}{5},\frac{2}{5},\frac{2}{5},\frac{6}{5})$	$(5,10,10)$
5	5	6	$3,3,2,2,2$		∞		
6	5	6	$3,3,3,2,1$		∞		
7	5	6	$4,3,2,2,1$		∞		
8	5	6	$5,2,2,2,1$				
9	5	8	$4,3,3,3,3$			$(\frac{1}{2},\frac{3}{8},\frac{3}{8},\frac{9}{8})$	$(4,8,8)$
10	5	8	$5,5,2,2,2$				
11	5	8	$6,3,3,3,1$				
12	5	9	$4,4,4,4,2$				
13	5	10	$7,4,4,4,1$				
14	5	12	$5,5,5,5,4$				
15	5	12	$6,5,5,4,4$	NA		$(\frac{1}{2},\frac{1}{3},\frac{1}{3},\frac{13}{12})$	$(4,6,12)$ na
16	5	12	$6,5,5,5,3$				
17	5	12	$7,5,4,4,4$	NA	∞		
18	5	12	$7,6,5,3,3$	NA	∞		
19	5	12	$7,7,4,4,2$				
20	5	12	$8,5,5,3,3$			$(\frac{1}{3},\frac{1}{4},\frac{1}{4},\frac{11}{12})$	$(3,4,12)$
21	5	12	$8,5,5,5,1$				
22	5	12	$8,7,3,3,3$	NA			
23	5	12	$10,5,3,3,3$				
24	5	15	$8,6,6,6,4$	NA			
25	5	18	$11,8,8,8,1$				
26	5	20	$14,11,5,5,5$	NA			
27	5	24	$14,9,9,9,7$	NA			

Table App. 4: List of hypergeometric differential equations on $x+y=1$

Example of calculation [Case 4]. For $F_1(\frac{3}{5},\frac{2}{5},\frac{2}{5},\frac{6}{5})(x,y)$ we perform the transformation $x+y=v, x-y=u$. After that we put $v=1$. Then we have the Fuchsian diffeential equation of order 3:

$$p_0 f''' + p_1 f'' + p_2 f' + p_3 =$$
$$125(-1+u)^2u^2(1+u)^2 f''' + 50(-1+u)u(1+u)(-2+15u^2)f'' + 20(-5-21u^2+42u^4)f' + 96u^3 f$$
$$= 0.$$

We have the decomposition :

$$p_0 f''' + p_1 f'' + p_2 f' + p_3 =$$
$$[-5(-u+u^3)\partial - 5 - 3u^2][-25(-1+u)u(1+u)\partial^2 - 20(-1+3u^2)\partial - 12u]f.$$

By putting $t=u^2$, we obtain

$$[-25(-1+u)u(1+u)\partial^2 - 20(-1+3u^2)\partial - 12u]f = E(\frac{3}{10},\frac{4}{10},\frac{9}{10};t).$$

By the correspondence between $E(a,b,c)$ and $\Delta(p,q,r)$ with

$$p=\frac{1}{|1-c|},\ q=\frac{1}{|c-a-b|},\ r=\frac{1}{|a-b|},$$

we obtain $\Delta(5,10,10)$ for it.

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