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Smoothing estimates for velocity averages with radial data

By

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Abstract

This article provides some new results on smoothing estimates for velocity averages of solutions to the kinetic transport equation for initial data which are radially symmetric in the spatial variable. This builds on recent work of Bennett, Gutiérrez, Lee and the first author.

§ 1. Introduction

The kinetic transport equation $(\partial_t + v \cdot \nabla_x)F = 0$ with initial data $F(x, v, 0) = f(x, v)$ has solution $F(x, v, t) = f(x - tv, v)$, where $x \in \mathbb{R}^d$ with $d \geq 2$, $t \in \mathbb{R}$ and the velocity variable v belongs to a prescribed domain in $\mathbb{R}^d$. In this article, we will consider the case where the velocity domain is either the unit sphere $\mathbb{S}^{d-1}$ or the unit ball $\mathbb{B}^d$, and the corresponding velocity averages given by

$$\rho f(x,t) = \int_{\mathbb{S}^{d-1}} f(x - tv, v) \, \mathrm{d}\sigma(v)$$

and

$$\varrho f(x,t) = \int_{\mathbb{B}^d} f(x - tv, v) \, \mathrm{d}v,$$

respectively. Here, σ denotes the induced Lebesgue measure on the unit sphere. For simplicity of the exposition, we initially focus on discussing the spherical case.

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It is natural to seek estimates which quantify the regularising nature of the averaging operator ρ. For $d \geq 3$, a half-derivative gain of regularity can be expressed through the estimates[1]

$$\|D_+^{1/2}\rho f\|_{L^2_{x,t}} \lesssim \|f\|_{L^2_{x,v}} \tag{1.1}$$

for initial data in $L^2_{x,v}$. Here, we are using the notation[2] D_+^β to denote the fractional derivative operator of order β given by

$$\widehat{D_+^\beta g}(\xi,\tau) = (|\xi| + |\tau|)^\beta \widehat{g}(\xi,\tau),$$

where $\widehat{\ }$ denotes the Fourier transform. In three spatial dimensions, (1.1) was established by Bournaveas–Perthame in [8] and in higher dimensions by Bournaveas–Gutiérrez in [7].

Increasing the gain of regularity in (1.1) beyond 1/2 is not achievable; indeed, the estimate can only be true with order 1/2, as may be seen by elementary homogeneity considerations. Nevertheless, it is reasonable to hope for some form of higher regularity gain by appropriately reformulating the structure of the estimates. Furthermore, since the estimate (1.1) fails when $d = 2$, it is also reasonable to seek a framework in which a similar kind of regularity gain is permissible even in two spatial dimensions. It turns out that a natural viewpoint which addresses both of these points is to consider smoothing estimates of the form

$$\|D_+^{\beta_+} D_-^{\beta_-} \rho f\|_{L^2_{x,t}} \lesssim \|f\|_{L^2_{x,v}} \tag{1.2}$$

where D_-^β, given by

$$\widehat{D_-^\beta g}(\xi,\tau) = ||\xi| - |\tau||^\beta \widehat{g}(\xi,\tau),$$

is the so-called "hyperbolic derivative" operator of order β. Although homogeneity considerations mean that (1.2) can only hold in the case where we have a total of one half derivative, that is,

$$\beta_+ + \beta_- = \frac{1}{2},$$

this framework at least allows the possibility to get additional "classical" regularity by raising β_+ at the expense of some hyperbolic derivatives. This is indeed the case and the following theorem generalises (1.1).

Theorem 1.1 ([7], [8]). *Let $d \geq 2$ and assume $\beta_+ + \beta_- = \frac{1}{2}$. Then (1.2) holds if and only if $\beta_+ \leq \frac{d-1}{4}$.*

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[1] By $A \lesssim B$ and $B \gtrsim A$ we mean $A \leq CB$ for some constant C which may depend on d and any exponents which are used to define the relevant function space in use, and when both $A \lesssim B$ and $B \gtrsim A$ hold, we write $A \sim B$. Also, $A \simeq B$ means $A = CB$ for some $C \sim 1$.

[2] The reason for the subscript + will soon become apparent.

In two spatial dimensions, Theorem 1.1 was established by Bournaveas–Perthame [8] and the idea to consider hyperbolic derivatives in this context originates in this paper. For higher dimensions, Theorem 1.1 was established by Bournaveas–Gutiérrez [7]. This result clarifies that when $d = 2$ one is permitted to take up to 1/4 classical derivatives and the inclusion of the hyperbolic derivatives makes it possible to realise a total 1/2-derivative gain. We also mention that the threshold condition $\beta_+ \leq \frac{d-1}{4}$ is of course equivalent to $\beta_- \geq \frac{3-d}{4}$ and thus, in the critical case where equality holds, we will encounter negative indices for $d \geq 4$. This point is pertinent in the sense that $||\xi| - |\tau||^{\beta_-}$ will become the more delicate part of the multiplier to control.

There are many natural ways in which one would like to extend the estimates in Theorem 1.1. As in [3], in this article we focus on the way in which the allowable range of the smoothing parameters β_+ and β_- varies if we consider mixed space-time norms $L^q_t L^r_x$ (in which case we capture the regularising effect of the averaging operators through a combination of smoothness and integrability, reminiscent of null form estimates for the wave equation; see, for example, [2], [5], [9], [10], [12], [13], [15], [16]) and/or we restrict our attention to initial data which are radially symmetric in the spatial variable.

We begin by outlining an approach from [3] in the next section which provides a basis for establishing estimates of the type we seek. This will simultaneously provide an overview of some of the main aspects of [3] and allow us to streamline our presentation of some new results in this direction for radially symmetric data; these new results will be stated and proved in Sections 3 and 4.

§ 2. A schema for proving smoothing estimates

Again, we focus on averaging operator ρ taking averages over the sphere; it will be clear from the argument below how to make the necessary (minor) modifications when one considers the operator ϱ taking averages over the ball.

We begin by noting that the Fourier transform of the velocity average is easily seen to be given by

$$\widehat{\rho f}(\xi, \tau) \simeq \int_{\mathbb{S}^{d-1}} \delta(v \cdot \xi + \tau) \widehat{f}(\xi, v) \, \mathrm{d}\sigma(v), \tag{2.1}$$

where δ denotes the one-dimensional Dirac delta distribution supported at the origin. Clearly, $\widehat{\rho f}$ is supported in the conical region $\mathfrak{C} := \{(\xi, \tau) \in \mathbb{R}^{d+1} : |\tau| \leq |\xi|\}$. At this point, we also observe the explicit formula

$$\int_{\mathbb{S}^{d-1}} \delta(v \cdot \xi + \tau) \, \mathrm{d}\sigma(v) \simeq \frac{\chi_{\mathfrak{C}}(\xi, \tau)}{|\xi|} \left(1 - \frac{\tau^2}{|\xi|^2}\right)^{\frac{d-3}{2}} \tag{2.2}$$

which we shall make use of later on, and at the same time provides an opportunity to highlight why it is natural to capture the regularising effect of ρ using D_+ and D_-.

We may use (2.2) to justify the alternative representation of the Fourier transform of ρf given by

$$\widehat{\rho f}(\xi,\tau) \simeq \frac{\chi_{\mathfrak{C}}(\xi,\tau)}{(|\xi|^2-\tau^2)^{1/2}} \int_{\Sigma_{\xi,\tau}} \widehat{f}(\xi,v)\,\mathrm{d}\sigma_{\xi,\tau}(v), \tag{2.3}$$

where

$$\Sigma_{\xi,\tau} = \{v \in \mathbb{S}^{d-1} : v\cdot\xi+\tau = 0\}$$

is a certain slice of $\mathbb{S}^{d-1}$, which we equip with its induced Lebesgue measure $\sigma_{\xi,\tau}$. This representation will be used to give a proof of Theorem 1.1 which is sufficiently robust to allow various extensions, such as replacing the $L^2_{t,x}$ norm on the left-hand side by a general mixed-norm $L^q_tL^r_x$ for $q,r \geq 2$. This argument originates in [3] and although it will miss out on the critical case $\beta_- = \frac{3-d}{4}$, this is a small price to pay for its robustness (later on, we shall make some remarks about the critical case); we now give a somewhat sketchy overview of this part of [3].

First, we consider initial data f whose Fourier transform (in ξ) is supported in a fixed annulus $\{\xi \in \mathbb{R}^d : |\xi| \sim 1\}$. Concentrating on controlling $D_-^{\beta_-}$, we write

$$D_-^{\beta_-}\rho f = \sum_{k\in\mathbb{Z}} 2^{-k\beta_-}C_k\rho f. \tag{2.4}$$

Here, the Fourier multiplier operator C_k is given by

$$\widehat{C_kg}(\xi,\tau) = \psi(2^k(|\xi|-|\tau|))\phi(|\xi|)\widehat{g}(\xi,\tau),$$

where $\phi,\psi \in C_c^\infty(\mathbb{R})$ are supported in the interval $[1/2,2]$ and ψ is chosen to satisfy

$$s^{\beta_-} = \sum_{k\in\mathbb{Z}} 2^{-k\beta_-}\psi(2^ks)$$

for each s, giving a slight modification of a standard partition of unity. Thus, ϕ is localising the frequency variable ξ to a fixed annulus and ψ is giving rise to a dyadic decomposition of the multiplier away from the edge of the cone. Since $\widehat{\rho f}$ is supported in $\mathfrak{C}$ it follows that the relevant k in the summation in the decomposition (2.4) is bounded below by a fixed number, and the difficulty arises in the contribution for k large; thus, in what follows, k is considered to be sufficiently large.

Our first goal is

$$\|D_-^{\beta_-}\rho f\|_{L^2_{x,t}} \lesssim \|f\|_{L^2_{x,v}} \tag{2.5}$$

for all $\beta_- > \frac{3-d}{4}$, and using (2.4), it is clear that it suffices to prove

$$\|C_k\rho f\|_{L^2_{t,x}} \lesssim 2^{\frac{3-d}{4}k}\|f\|_{L^2_{x,v}}. \tag{2.6}$$

This key estimate is sharp with respect to the exponent $\frac{3-d}{4}$ and here we see the critical exponent (with respect to β_-) explicitly appear. Using that the multiplier for C_k is supported where $||\xi|-|\tau|| \sim 2^{-k}$ and $|\xi| \sim 1$, we may use (2.3) to see that, for $(\xi,\tau) \in \mathfrak{C}$, then

$$|\widehat{C_k\rho f}(\xi,\tau)|^2 \lesssim 2^k \psi(2^k(|\xi|-|\tau|))^2 \phi(|\xi|)^2 \left| \int_{\Sigma_{\xi,\tau}} \widehat{f}(\xi,v)\, \mathrm{d}\sigma_{\xi,\tau}(v) \right|^2.$$

Since $\Sigma_{\xi,\tau}$ is a $(d-2)$-dimensional sphere with radius $(1-\frac{\tau^2}{|\xi|^2})^{1/2} \sim 2^{-k/2}$, it follows from the Cauchy–Schwarz inequality and another application of (2.3) that

$$|\widehat{C_k\rho f}(\xi,\tau)|^2 \lesssim 2^{\frac{3-d}{2}k} \int_{\mathbb{S}^{d-1}} \delta(v\cdot\xi+\tau)|\widehat{f}(\xi,v)|^2\, \mathrm{d}\sigma(v).$$

By integrating this first with respect to τ and then ξ, we get (2.6) and hence (2.5).

Once we have (2.5) for $\beta_- > \frac{3-d}{4}$, then standard arguments can be used to prove the estimates (1.2) in Theorem 1.1 for the same range of β_-. Indeed, proceeding initially with frequency localised initial data as above, then the operator D_+ becomes rather harmless since its multiplier is effectively a smooth and compactly supported function. The desired estimates (1.2) follow for frequency localised initial data and then orthogonality considerations allow us to extend this to general data in $L^2_{x.v}$.

As we have already alluded to, the above argument is sufficiently robust to allow one to prove mixed-norm extensions of (1.2) of the form

$$\|D_+^{\beta_+} D_-^{\beta_-} \rho f\|_{L^q_t L^r_x} \lesssim \|f\|_{L^2_{x,v}}. \tag{2.7}$$

Scaling considerations inform us that

$$\beta_+ + \beta_- = \frac{d}{r} + \frac{1}{q} - \frac{d}{2}$$

is a necessary condition for such an estimate. To prove (2.7) we may proceed initially in the same way as above when $(q,r) = (2,2)$; we first consider data which are frequency localised and use the decomposition (2.4). In order to prove the key estimate

$$\|C_k\rho f\|_{L^q_t L^r_x} \lesssim 2^{\eta(q,r)k} \|f\|_{L^2_{x,v}}, \tag{2.8}$$

for an appropriate exponent $\eta(q,r)$, it suffices to prove that the multiplier operator C_k satisfies a bound of the form

$$\|C_k g\|_{L^q_t L^r_x} \lesssim 2^{\alpha(q,r)k} \|g\|_{L^2_{x,t}}, \tag{2.9}$$

for an appropriate exponent $\alpha(q,r)$. Indeed, $C_k = \widetilde{C}_k C_k$ where $\widetilde{C}_k$ is defined in the same way as C_k in terms of cutoff functions $\widetilde{\phi}, \widetilde{\psi} \in C_c^\infty(\mathbb{R})$ which are equal to one on the support of ϕ and ψ, respectively, and therefore (2.9) and (2.6) imply

$$\|C_k\rho f\|_{L^q_t L^r_x} \lesssim 2^{\alpha(q,r)k} \|C_k\rho f\|_{L^2_{x,t}} \lesssim 2^{(\alpha(q,r)+\frac{3-d}{4})k} \|f\|_{L^2_{x,v}},$$

that is, (2.8) with $\eta(q,r) = \alpha(q,r) + \frac{3-d}{4}$.

The estimate (2.9) was proved in [3] with the sharp exponent

$$\alpha(q,r) = \begin{cases} -\frac{1}{2}, & \frac{1}{q} \leq \frac{d-1}{2}(\frac{1}{2} - \frac{1}{r}) \\ \frac{1}{q} - \frac{1}{2} + \frac{d-1}{2}\left(\frac{1}{r} - \frac{1}{2}\right), & \frac{1}{q} > \frac{d-1}{2}(\frac{1}{2} - \frac{1}{r}) \end{cases} \tag{2.10}$$

for $q, r \geq 2$. For $d \geq 4$, this was shown in [3] by making use of the Strichartz estimates for frequency-localised solutions of the wave equation and interpolation with the trivial estimate corresponding to $\alpha(2,2) = 0$ (for $d = 2, 3$ a little more work is necessary).

Once we have (2.8), one may readily deduce estimates of the form (2.7) in much the same way as outlined above for the case $(q,r) = (2,2)$, where classical Littlewood–Paley theory is used to pass from the frequency-localised case to general initial data. Based on this, the following theorem was obtained in [3].

Theorem 2.1 ([3]). *Let $d \geq 2$, $q, r \in [2, \infty)$ and assume $\beta_+ + \beta_- = \frac{d}{r} + \frac{1}{q} - \frac{d}{2}$. In the case $\frac{1}{q} \leq \frac{d-1}{2}(\frac{1}{2} - \frac{1}{r})$, then (2.7) holds if and only if $\beta_- > \frac{1-d}{4}$. In the case $\frac{1}{q} > \frac{d-1}{2}(\frac{1}{2} - \frac{1}{r})$, then (2.7) holds if $\beta_- > \frac{1}{q} + \frac{d-1}{2}(\frac{1}{r} - 1)$ and fails if $\beta_- < \frac{1}{q} + \frac{d-1}{2}(\frac{1}{r} - 1)$.*

This almost sharp result of course leaves open the critical case $\beta_- = \frac{1}{q} + \frac{d-1}{2}(\frac{1}{r} - 1)$ when $\frac{1}{q} > \frac{d-1}{2}(\frac{1}{2} - \frac{1}{r})$. By use of a more delicate argument, it was shown in [4] that the estimates (2.7) are true in this critical case; interestingly, and perhaps surprisingly, this shows that the dichotomy $\frac{1}{q} \leq \frac{d-1}{2}(\frac{1}{2} - \frac{1}{r})$ and $\frac{1}{q} > \frac{d-1}{2}(\frac{1}{2} - \frac{1}{r})$ is a genuine feature of the problem.

Before we move on to the case of radial data, we end this discussion with a couple of further remarks on the above schema by showcasing some of the results in [3] for initial data in wider classes. By making use of the recently established decoupling (Wolff-type) estimates for the cone by Bourgain and Demeter [6], one may obtain estimates of the form (2.8) for frequency localised data in $L^p_{x,v}$ for $q \geq p \geq 2$, $r = q$, and an appropriate exponent η. In much the same way as above, via use of classical Littlewood–Paley theory, this led to $\dot{B}^s_{p,2} \to L^q_{x,t}$ smoothing estimates, where $\dot{B}^s_{p,2}$ is a homogeneous Besov space. Here we are restricting our attention to the diagonal case $q = r$; indeed, a complete characterisation of *mixed-norm* decoupling estimates is currently unknown and this appears to be an interesting and challenging open problem.

A somewhat different approach to Theorems 1.1 and 2.1 based on duality was also observed in [3]. In the simplest case where $(q,r) = (2,2)$, direct calculations using Plancherel's theorem, the scaling condition $\beta_+ + \beta_- = \frac{1}{2}$, and (2.2) reveal that

$$\begin{aligned} \|(D_+^{\beta_+} D_-^{\beta_-} \rho)^* g\|^2_{L^2_{\xi,v}} &= \int_{\mathfrak{C}} \left(1 + \frac{|\tau|}{|\xi|}\right)^{\beta_+ - \beta_-} \left(1 - \frac{\tau^2}{|\xi|^2}\right)^{2\beta_- + \frac{d-3}{2}} |\widehat{g}(\xi,\tau)|^2 \,\mathrm{d}\xi \mathrm{d}\tau \\ &\sim \int_{\mathfrak{C}} \left(1 - \frac{\tau^2}{|\xi|^2}\right)^{2\beta_- + \frac{d-3}{2}} |\widehat{g}(\xi,\tau)|^2 \,\mathrm{d}\xi \mathrm{d}\tau. \end{aligned}$$

From this expression, it is clear that the dual estimate to (1.2) holds if and only if $2\beta_- + \frac{d-3}{2}$ is nonnegative; that is, $\beta_- \geq \frac{3-d}{4}$. In the mixed-norm case in Theorem 2.1, a similar TT^* argument allows one to reduce (2.7) to estimates on the Fourier multiplier operator with multiplier $(1 - \frac{\tau^2}{|\xi|^2})_+^\alpha \phi(|\xi|)$, where, as above, ϕ is a bump function supported away from the origin. This is a mild variant of the famous cone multiplier operator and, by an appropriate dyadic decomposition, we may establish the desired estimate in Theorem 2.1 in this manner via the estimate (2.9) arising above.

§ 3. Radial data: velocities on the sphere

It was observed in [3] that the allowable range of exponents for the smoothing estimates (1.2) improve if we consider initial data which are radially symmetric in the spatial variable and possess angular regularity in the velocity variable. More precisely, it was shown in [3, Theorem 7.2] that the estimate

$$\|D_+^{\beta_+} D_-^{\beta_-} \rho f\|_{L^2_{x,t}} \lesssim \|(1-\Delta)^{\nu/2} f\|_{L^2_{x,v}} \tag{3.1}$$

holds for all $f \in L^2_\circ$ if $\beta_+ + \beta_- = \frac{1}{2}$, $\nu \in [\frac{2-d}{2}, 0]$ and $\beta_- > -\nu + \frac{2-d}{2}$. Here, $L^2_\circ$ denotes the subspace of $L^2_{x,v}$ of initial data which are radially symmetric in the spatial variable and Δ denotes the Laplace–Beltrami operator on $\mathbb{S}^{d-1}$ (which acts on the velocity variable). In this section, we revisit and refine this result, leading to the following *complete* characterisation of the exponents (β_-, β_+, ν) for which (3.1) holds.

Let $\mathfrak{R}_d$ denote the subset of $\mathbb{R}^2$ given by

$$\mathfrak{R}_d = \{(\nu, \beta_-) \in \mathbb{R}^2 : \nu \geq \tfrac{2-d}{2}, \beta_- \geq \tfrac{2-d}{2} - \tfrac{\nu}{2}, \beta_- > \tfrac{2-d}{2}\} \setminus \{(\tfrac{2-d}{2}, \tfrac{2-d}{4})\},$$

illustrated in Figure 1.

Theorem 3.1. *Let $d \geq 2$ and $\beta_+ + \beta_- = \frac{1}{2}$. Then* (3.1) *holds for all $f \in L^2_\circ$ if and only if $(\nu, \beta_-) \in \mathfrak{R}_d$.*

Proof. We begin with the sufficiency of the condition $(\nu, \beta_-) \in \mathfrak{R}_d$ and, for the most part, we follow the approach in [3, Theorem 7.2]. For $f \in L^2_\circ$, we also have $\widehat{f} \in L^2_\circ$ and thus we may write $\widehat{f}(\xi, v) = F_0(|\xi|, v)$ for an appropriate function F_0 defined on $[0,\infty) \times \mathbb{S}^{d-1}$. Expanding in terms of spherical harmonics, we have

$$F_0(r, v) = \sum_{\ell=0}^{\infty} Y_\ell^r(v) \tag{3.2}$$

and using orthogonality and the well-known fact that each Y_ℓ^r is an eigenfunction of $-\Delta$ with eigenvalue $\ell(\ell+d-2)$, we obtain

$$\|(1-\Delta)^{\nu/2} f\|^2_{L^2_{x,v}} \simeq \sum_{\ell=0}^{\infty} (1 + \ell(\ell+d-2))^\nu \int_0^\infty \|Y_\ell^r\|_2^2 \, r^{d-1} \, \mathrm{d}r.$$

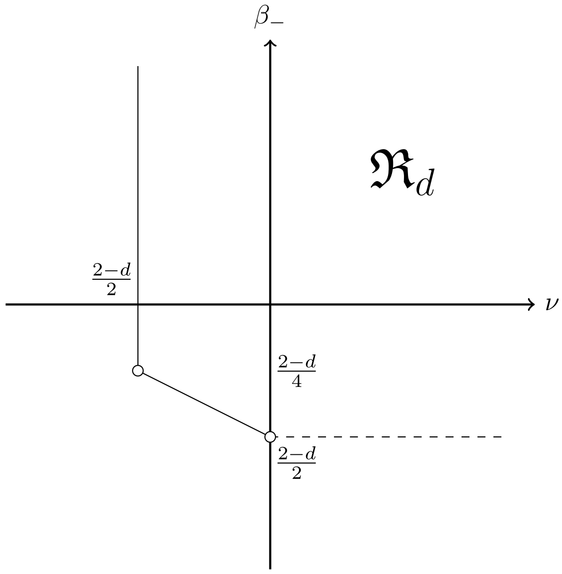

Figure 1. The region $\mathfrak{R}_d$

A similar expression holds for the left-hand side of (3.1). To see this, for each $(r, \theta, \tau) \in (0, \infty) \times \mathbb{S}^{d-1} \times \mathbb{R}$, we use (2.1) to write

$$\widehat{\rho f}(r\theta, \tau) \simeq \frac{1}{r} \sum_{\ell=0}^{\infty} \int_{\mathbb{S}^{d-1}} Y_\ell^r(v) \delta(\tfrac{\tau}{r} + \theta \cdot v) \, \mathrm{d}\sigma(v)$$

and then the Funk–Hecke theorem (see, for example, [1, Theorem 2.2]) to obtain

$$\widehat{\rho f}(r\theta, \tau) \simeq \frac{1}{r} \chi_{|\tau| \leq r} \left(1 - \frac{\tau^2}{r^2}\right)^{\frac{d-3}{2}} \sum_{\ell=0}^{\infty} Y_\ell^r(\theta) p_{d,\ell}(-\tfrac{\tau}{r}). \tag{3.3}$$

Here, $p_{d,\ell}$ is the Legendre polynomial of degree ℓ in d dimensions, given, for example, by the Rodrigues representation formula

$$(1 - t^2)^{\frac{d-3}{2}} p_{d,\ell}(t) = (-1)^\ell \frac{\Gamma(\frac{d-1}{2})}{2^\ell \Gamma(\ell + \frac{d-1}{2})} \frac{\mathrm{d}^\ell}{\mathrm{d}t^\ell} (1 - t^2)^{\ell + \frac{d-3}{2}}.$$

We include this formula merely as a matter of completeness; the remainder of the proof rests on the sharp integral estimates below in (3.4) which involve a certain weighted L^2 norm of $p_{d,\ell}$.

By orthogonality, the assumption $\beta_+ + \beta_- = \frac{1}{2}$, and elementary changes of variables, we have from (3.3) that

$$\|D_+^{\beta_+}D_-^{\beta_-}\rho f\|^2_{L^2_{x,t}} \sim \sum_{\ell=0}^{\infty}\int_0^1 (1-\lambda)^{d-3+2\beta_-}|p_{d,\ell}(\lambda)|^2\,\mathrm{d}\lambda \int_0^\infty \|Y_\ell^r\|^2_{L^2}\,r^{d-1}\,\mathrm{d}r$$

and thus it suffices to show that

$$\int_0^1 (1-\lambda)^{d-3+2\beta_-}|p_{d,\ell}(\lambda)|^2\,\mathrm{d}\lambda \lesssim \ell^{2\nu}$$

for $\ell \geq 1$ if $(\nu,\beta_-) \in \mathfrak{R}_d$. Such an estimate is an immediate consequence of the fact that, for $\beta_- > \frac{2-d}{2}$, we have

$$\int_0^1 (1-\lambda)^{2\beta_-+d-3}|p_{d,\ell}(\lambda)|^2\,\mathrm{d}\lambda \sim \begin{cases} \ell^{4-2d-4\beta_-}, & \beta_- < \frac{2-d}{4}, \\ \ell^{2-d}\log k, & \beta_- = \frac{2-d}{4}, \\ \ell^{2-d}, & \beta_- > \frac{2-d}{4}. \end{cases} \tag{3.4}$$

These integral estimates on $p_{d,\ell}$ may be found in the classical textbook of Szegö [14, p. 391] (see also [11, Lemma 4.1] where such estimates arose recently in estimating the size of the spherical harmonic projection operators) and are based on the pointwise estimates

$$p_{d,\ell}(\cos\theta) = \begin{cases} \theta^{\frac{2-d}{2}}O(\ell^{\frac{2-d}{2}}), & \frac{c}{\ell} \leq \theta \leq \frac{\pi}{2}, \\ O(1), & 0 \leq \theta \leq \frac{c}{\ell}, \end{cases} \tag{3.5}$$

as $\ell \to \infty$ (see Szegö [14, Theorem 7.32.2]).

For the necessity part, we test (3.1) on initial data satisfying

$$\widehat{f}(\xi, v) = \phi(|\xi|)Y_\ell(v)$$

where ϕ is a bump function and Y_k is a spherical harmonic of degree $\ell \geq 0$ normalised so that $\|Y_\ell\|_2 = 1$. Then, following the calculations above, we may deduce that

$$\|(1-\Delta)^{\nu/2}f\|^2_{L^2_{x,v}} \sim (1+\ell)^{2\nu}$$

and

$$\|D_+^{\beta_+}D_-^{\beta_-}\rho f\|^2_{L^2_{x,t}} \sim \int_0^1 (1-\lambda)^{d-3+2\beta_-}|p_{d,\ell}(\lambda)|^2\,\mathrm{d}\lambda.$$

Taking $\ell = 0$, we obtain the necessary condition $\beta_- > \frac{2-d}{2}$. By taking $\ell \to \infty$ and using (3.4), it follows that $\nu \geq \frac{2-d}{2}$ is necessary when $\beta_- > \frac{2-d}{4}$, $\nu > \frac{2-d}{2}$ is necessary when $\beta_- = \frac{2-d}{4}$, and $\nu \geq 2-d-2\beta_-$ is necessary when $\beta_- < \frac{2-d}{4}$. Putting these facts together implies that $(\nu,\beta_-) \in \mathfrak{R}_d$ is necessary, and this completes the proof of Theorem 3.1. □

For the mixed-norm case (without angular regularity on the initial data), we may follow the schema in the previous section along with (3.3) in order to obtain the sharp L^2 bound on each $C_k\rho f$. In particular, for each $k \gtrsim 1$, we have

$$\|C_k\rho f\|^2_{L^2_{x,t}} \simeq \sum_{\ell=0}^{\infty}\int_0^\infty\int_{-1}^1 \phi(r)^2\psi(2^k r(1-\lambda))^2(1-\lambda^2)^{d-3}|p_{d,\ell}(\lambda)|^2\|Y_\ell^r\|_2^2 r^{d-3}\,\mathrm{d}\lambda\mathrm{d}r$$

and since the integration is taken over $r \sim 1$ and $1-\lambda \sim 2^{-k}$, it follows quickly from the uniform bound $|p_{d,\ell}(\lambda)| \leq 1$ that

$$\|C_k\rho f\|_{L^2_{x,t}} \lesssim 2^{\frac{2-d}{2}k}\|f\|_{L^2_{x,v}}.$$

This should be compared with the estimate (2.6) and explains why the allowable range $\beta_- > \frac{2-d}{2}$ appears in Theorem 3.1 for radial data (when $\nu = 0$), extending the range from $\beta_- \geq \frac{3-d}{4}$ for general data in L^2.

Using (2.9) to pass to mixed-norm estimates for $C_k\rho f$, we obtain the following result from [3].

Theorem 3.2 ([3])**.** *Let $d \geq 2$, $q, r \in [2,\infty)$ and assume $\beta_+ + \beta_- = \frac{d}{r} + \frac{1}{q} - \frac{d}{2}$. In the case $\frac{1}{q} \leq \frac{d-1}{2}(\frac{1}{2} - \frac{1}{r})$, (2.7) holds for $f \in L^2_\circ$ if $\beta_- > \frac{1-d}{2}$. In the case $\frac{1}{q} > \frac{d-1}{2}(\frac{1}{2} - \frac{1}{r})$, (2.7) holds for $f \in L^2_\circ$ if $\beta_- > \frac{1}{q} + \frac{d-1}{2}(\frac{1}{r} - \frac{3}{2})$.*

Unlike the other statements presented so far in this note, we are currently unable to say whether the range of β_- in Theorem 3.2 is optimal and we leave this as an interesting open problem.

§ 4. Radial data: velocities in the ball

For the unit ball

$$\mathbb{B}^d = \{v \in \mathbb{R}^d : |v| \leq 1\},$$

we recall that the associated velocity average ϱ is given by

$$\varrho f(x,t) = \int_{\mathbb{B}^d} f(x - tv, v)\,\mathrm{d}v.$$

With the averaging now taken over a larger dimensional space, naturally we expect additional regularising properties of ϱ compared with ρ. Indeed, in this case we have the expression

$$\int_{\mathbb{B}^d} \delta(v\cdot\xi + \tau)\,\mathrm{d}v \simeq \frac{\chi_{\mathfrak{C}}(\xi,\tau)}{|\xi|}\left(1 - \frac{\tau^2}{|\xi|^2}\right)^{\frac{d-1}{2}} \tag{4.1}$$

which is a less singular version of (2.2). This identity leads to the following analogue of Theorem 1.1 for the estimate

$$\|D_+^{\beta_+} D_-^{\beta_-} \varrho f\|_{L^2_{x,t}} \lesssim \|f\|_{L^2_{x,v}}, \tag{4.2}$$

for which we again have the necessary condition $\beta_+ + \beta_- = \frac{1}{2}$.

Theorem 4.1 ([3]). *Let $d \geq 2$ and assume $\beta_+ + \beta_- = \frac{1}{2}$. Then (4.2) holds if and only if $\beta_+ \leq \frac{d+1}{4}$ (or, $\beta_- \geq \frac{1-d}{4}$).*

Following similar ideas to those used in the previous section, we show the following improvement for radially symmetric initial data.

Theorem 4.2. *Let $d \geq 3$ and $\beta_+ + \beta_- = \frac{1}{2}$. Then (4.2) holds for all $f \in L^2_\circ$ if and only if $\beta_- > \frac{1-d}{2}$.*

Proof. For the sufficiency claim, for a given $f \in L^2_\circ$ we write $\widehat{f}(\xi, v) = F_0(|\xi|, v)$ for an appropriate function F_0 defined on $[0, \infty) \times \mathbb{B}^d$, and expand as

$$F_0(r, s\omega) = \sum_{\ell=0}^{\infty} \sum_{m=0}^{n_\ell} f^r_{\ell,m}(s) Y_{\ell,m}(\omega) \tag{4.3}$$

for $(r, s, \omega) \in (0, \infty) \times (0, 1] \times \mathbb{S}^{d-1}$. Here, $\{Y_{\ell,m}\}_{m=0}^{n_\ell}$ is a fixed orthonormal basis for the space of spherical harmonics of degree ℓ and, in the sequel, we write $\sum_{\ell=0}^{\infty} \sum_{m=0}^{n_\ell}$ as $\sum_{\ell,m}$ for brevity. In terms of this expansion, we have

$$\|f\|^2_{L^2_{x,v}} \simeq \sum_{\ell,m} \int_0^\infty \int_0^1 |f^r_{\ell,m}(s)|^2 (rs)^{d-1} \, \mathrm{d}s\mathrm{d}r.$$

Next, we use the Funk–Hecke theorem to write

$$\widehat{\varrho f}(r\theta, \tau) \simeq \frac{1}{r} \chi_{|\tau| \leq r} \sum_{\ell,m} Y_{\ell,m}(\theta) \int_{|\tau|/r}^1 f^r_{\ell,m}(s) p_{d,\ell}(-\tfrac{\tau}{rs}) \left(1 - \frac{\tau^2}{(rs)^2}\right)^{\frac{d-3}{2}} s^{d-2} \, \mathrm{d}s \tag{4.4}$$

from which orthonormality of the $Y_{\ell,m}$ yields

$$\begin{aligned} &\|D_+^{\beta_+} D_-^{\beta_-} \varrho f\|^2_{L^2_{x,t}} \\ &\quad \sim \sum_{\ell,m} \int_0^\infty \int_0^1 (1-\lambda)^{2\beta_-} \left(\int_\lambda^1 f^r_{\ell,m}(s) p_{d,\ell}(\tfrac{\lambda}{s}) \left(1 - \frac{\lambda^2}{s^2}\right)^{\frac{d-3}{2}} s^{d-2} \, \mathrm{d}s \right)^2 r^{d-1} \, \mathrm{d}\lambda \mathrm{d}r. \end{aligned}$$

By the Cauchy–Schwarz inequality in the s-integral and from the uniform bound $|p_{d,\ell}(\frac{\lambda}{s})| \leq 1$, it suffices to show

$$\int_0^1 (1-\lambda)^{2\beta_-} \int_\lambda^1 \left(1 - \frac{\lambda^2}{s^2}\right)^{d-3} s^{d-3} \, \mathrm{d}s \, \mathrm{d}\lambda \lesssim 1.$$

Elementary considerations show that this holds for $\beta_- > \frac{1-d}{2}$.

For the necessity claim, we fix $\varepsilon > 0$ and consider initial data such that

$$\widehat{f}(\xi, v) = \phi(|\xi|)(1 - |v|^2)^{-1/2+\varepsilon}\chi_{B(0,1)}(v),$$

where ϕ is a bump function, in which case $\|f\|_{L^2_{x,v}} < \infty$. Also, straightforward calculations yield

$$\begin{aligned}\widehat{\varrho f}(r\theta, \tau) &\simeq \chi_{|\tau| \le r}\frac{\phi(r)}{r}\int_{|\tau|/r}^1 (1-s^2)^{-1/2+\varepsilon}\left(1 - \frac{\tau^2}{(rs)^2}\right)^{\frac{d-3}{2}} s^{d-2}\,\mathrm{d}s \\ &\simeq \chi_{|\tau| \le r}\frac{\phi(r)}{r}\left(1 - \frac{\tau^2}{r^2}\right)^{\frac{d-2}{2}+\varepsilon}\end{aligned}$$

and therefore, by changing variables, we obtain

$$\|D_+^{\beta_+}D_-^{\beta_-}\varrho f\|^2_{L^2_{x,t}} \simeq \int_0^1 (1-\lambda)^{2\beta_- + d - 2 + 2\varepsilon}\,\mathrm{d}\lambda.$$

This implies $\beta_- > \frac{1-d}{2} - \varepsilon$ and, since $\varepsilon > 0$ was arbitrary, we obtain the necessary condition $\beta_- > \frac{1-d}{2}$. □

We conclude by extending the estimates in Theorem 4.2 to the mixed-norm case

$$\|D_+^{\beta_+}D_-^{\beta_-}\varrho f\|_{L^q_tL^r_x} \lesssim \|f\|_{L^2_{x,v}}, \tag{4.5}$$

giving an analogue of Theorem 3.2 for the ball.

Theorem 4.3. *Let $d \ge 3$, $q, r \in [2, \infty)$ and assume $\beta_+ + \beta_- = \frac{d}{r} + \frac{1}{q} - \frac{d}{2}$. In the case $\frac{1}{q} \le \frac{d-1}{2}(\frac{1}{2} - \frac{1}{r})$, (4.5) holds for $f \in L^2_\circ$ if $\beta_- > -\frac{d}{2}$. In the case $\frac{1}{q} > \frac{d-1}{2}(\frac{1}{2} - \frac{1}{r})$, (4.5) holds for $f \in L^2_\circ$ if $\beta_- > \frac{1}{q} - \frac{1}{2} + \frac{d-1}{2}(\frac{1}{r} - \frac{3}{2})$.*

Proof. We begin by estimating each $\|C_k\varrho f\|_{L^2_{x,t}}$. Using (4.4) and orthonormality of the $Y_{\ell,m}$, we have

$$\begin{aligned}\|C_k\varrho f\|^2_{L^2_{x,t}} \simeq \sum_{\ell,m}\int_0^\infty\int_{-1}^1 &\phi(r)^2\psi(2^kr(1-\lambda))^2 \\ &\times\left|\int_{|\lambda|}^1 f^r_{\ell,m}(s)p_{d,\ell}(\tfrac{\lambda}{s})\left(1 - \frac{\lambda^2}{s^2}\right)^{\frac{d-3}{2}} s^{d-2}\,\mathrm{d}s\right|^2 r^{d-3}\,\mathrm{d}\lambda\mathrm{d}r.\end{aligned}$$

Since we have

$$\begin{aligned}\int_{1-\lambda\sim 2^{-k}}\int_\lambda^1\left(1 - \frac{\lambda^2}{s^2}\right)^{d-3} s^{d-3}\,\mathrm{d}s\mathrm{d}\lambda &= \int_{1-\lambda\sim 2^{-k}}\lambda^{d-2}\int_\lambda^1 (1-u^2)^{d-3}\frac{1}{u^{d-1}}\,\mathrm{d}u\mathrm{d}\lambda \\ &\sim 2^{(1-d)k},\end{aligned}$$

it follows, from the Cauchy–Schwarz inequality and the uniform bound $|p_{d,\ell}(\frac{\lambda}{s})| \leq 1$, that

$$\|C_k \varrho f\|_{L^2_{x,t}} \lesssim 2^{\frac{1-d}{2}k} \|f\|_{L^2_{x,v}}.$$

As in our earlier sketch of the proof of Theorem 3.2, we now use (2.9) to obtain the bound

$$\|C_k \varrho f\|_{L^q_t L^r_x} \lesssim 2^{(\alpha(q,r)+\frac{1-d}{2})k} \|f\|_{L^2_{x,v}},$$

where $\alpha(q,r)$ is given by (2.10), and from this we obtain the claimed estimates in Theorem 4.3. $\square$

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