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Remarks on the probabilistic well-posedness for quadratic nonlinear Schrödinger equations

By

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Abstract

We consider the Cauchy problem for the quadratic nonlinear Schrodinger equation without gauge invariance: $i\partial_t u + \Delta u = |u|^2$. First, we show the probabilistic well-posedness in $H^s(\mathbb{R}^d)$ for $d \geq 5$ and $\frac{d-3}{d-2}s_c < s < s_c$, where $s_c := \frac{d}{2} - 2$ is the scaling critical regularity. Second, as in the paper of Bényi et al., by performing a fixed point argument around the higher order expansion, we improve the regularity threshold for almost sure local well-posedness, i.e., $\frac{d-3}{d-2}s_c$ is replaced by $\frac{d-4}{d-3}s_c$.

§ 1. Introduction

We consider the following Cauchy problem for the nonlinear Schrödinger equation:

$$i\partial_t u + \Delta u = |u|^2, \quad u(0,x) = \phi(x) \tag{1.1}$$

where $u = u(t,x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ is an unknown function, ϕ is a given initial datum.

The equation (1.1) is invariant under the following scaling transformation:

$$u(t,x) \mapsto u^\lambda(t,x) := \lambda^2 u(\lambda^2 t, \lambda x)$$

for $\lambda > 0$. Hence, the scaling critical Sobolev regularity is $s_c := \frac{d}{2} - 2$.

Well-posedness for (1.1) has been extensively studied. In particular, Tsutsumi [18] and Cazenave and Weissler [5] showed that (1.1) is locally well-posed in $H^s(\mathbb{R}^d)$ for $s \geq \max(s_c, 0)$. Moreover, local well-posedness in $H^{-\frac{1}{4}+\varepsilon}(\mathbb{R}^d)$ was proved by Kenig et al. [10], Colliander et al. [6], and Tao [16] for $d = 1$, 2, and 3, respectively. In one

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dimension, Kishimoto and Tsugawa [12] proved that (1.1) is locally well-posed in $H^s(\mathbb{R})$ if and only if $s \geq -\frac{1}{4}$. For $d = 2, 3$, the exponent $-\frac{1}{4}$ is sharp up to epsilons. Namely, Iwabuchi and Uriya [9] and Kishimoto [11] proved that (1.1) is ill-posed in $H^s(\mathbb{R}^d)$ if $d = 2, 3$ and $s < -\frac{1}{4}$ (see also [15]).

Recently, Ikeda and Inui [8] showed nonexistence of solutions of (1.1) with initial data in $H^s(\mathbb{R}^d)$ satisfying a certain condition when $d \geq 3$ and $s < s_c$. Oh and Pocovnicu with the author [14] proved probabilistic well-posedness of (1.1) for $d = 6$, which is a corollary of the well-posedness result for the energy-critical nonlinear Schrödinger equation. To handle non-algebraic nonlinearities, they avoid thorough case-by-case analysis. In this paper, by using the case-by-case analysis, we consider the probabilistic well-posedness for (1.1) with low regularity data. As in [14], thanks to randomizing the initial data, we can avoid these initial data in [8] for which no solution exists.

Following the papers [19, 13, 1, 2, 7, 4], we define the randomization. Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ satisfy

$$\operatorname{supp}\psi \subset [-1,1]^d \qquad \text{and} \qquad \sum_{n\in\mathbb{Z}^d} \psi(\xi - n) = 1 \quad \text{for any } \xi \in \mathbb{R}^d.$$

Then, given a function ϕ on $\mathbb{R}^d$, we have

$$\phi = \sum_{n\in\mathbb{Z}^d} \psi(D-n)\phi.$$

This replaces the role of the Fourier series expansion on compact domains. We then define the Wiener randomization of ϕ by

$$\phi^\omega := \sum_{n\in\mathbb{Z}^d} g_n(\omega)\psi(D-n)\phi, \tag{1.2}$$

where $\{g_n\}$ is a sequence of independent mean zero complex-valued random variables on a probability space $(\Omega, \mathcal{F}, P)$. In the following, we assume that the real and imaginary parts of g_n are independent and endowed with probability distributions $\mu_n^{(1)}$ and $\mu_n^{(2)}$, satisfying the following exponential moment bound:

$$\int_{\mathbb{R}} e^{\kappa x} d\mu_n^{(j)}(x) \leq e^{c\kappa^2}$$

for all $\kappa \in \mathbb{R}$, $n \in \mathbb{Z}^d$, $j = 1, 2$.

The randomization has no smoothing in terms of differentiability. However, it improves the integrability (see for example Lemma 2.3 in [1]).

The following is our main result.

Theorem 1.1. *Let $d \geq 5$ and let $\frac{d-3}{d-2}s_c < s < s_c$. Given $\phi \in H^s(\mathbb{R}^d)$, let ϕ^ω be its Wiener randomization defined in* (1.2). *Then, the Cauchy problem* (1.1) *with*

$u(0) = \phi^\omega$ is almost surely locally well-posed in $H^s(\mathbb{R}^d)$. Moreover, the solution u lies in the class:

$$S(t)\phi^\omega + C([-T,T]; H^{s_c+}(\mathbb{R}^d)) \subset C([-T,T]; H^s(\mathbb{R}^d))$$

for $T = T(\phi,\omega) > 0$ almost surely, where $S(t) := e^{it\Delta}$.

Let $z(t) = z^\omega(t) := S(t)\phi^\omega$ denote the random linear solution with ϕ^ω as initial data. If u is a solution to (1.1), then the residual term $v := u - z$ satisfies the following perturbed nonlinear Schrödinger equation:

$$\begin{cases} i\partial_t v + \Delta v = |v + z^\omega|^2 \\ v|_{t=0} = 0. \end{cases} \tag{1.3}$$

We use the contraction mapping theorem to find a solution to (1.3) (or the corresponding integral equation).

By performing a fixed point argument around the second order expansion, Bényi et al. [3] improved the regularity threshold for almost sure local well-posedness from their previous work [2]. Following their approach, we obtain the improved well-posedness result. Set $z_1(t) := z(t)$ and

$$z_2(t,x) := -i\int_0^t S(t-t')|z_1(t',x)|^2 dt'.$$

Theorem 1.2. *Let $d \geq 5$ and let $\frac{(d-3)^2}{d^2-5d+7}s_c < s < s_c$. Given $\phi \in H^s(\mathbb{R}^d)$, let ϕ^ω be its Wiener randomization defined in* (1.2)*. Then, the Cauchy problem* (1.1) *with $u(0) = \phi^\omega$ is almost surely locally well-posed in $H^s(\mathbb{R}^d)$. Moreover, the solution u lies in the class:*

$$z_1 + z_2 + C([-T,T]; H^{s_c+}(\mathbb{R}^d)) \subset C([-T,T]; H^s(\mathbb{R}^d))$$

for $T = T(\phi,\omega) > 0$ almost surely.

As in [3], we can also consider the (unbalanced) higher order expansion. Set $\zeta_1 := z_1$, $\zeta_2 := z_2$, and

$$\zeta_k(t) := -2i\int_0^t S(t-t')\Re(\zeta_1\overline{\zeta_{k-1}})(t')dt'$$

for $k \geq 3$. Then, we have

$$\zeta_2(t) \in H^{\frac{d-2}{d-3}s-}(\mathbb{R}^d), \quad \zeta_3(t) \in H^{\frac{d^2-5d+7}{(d-3)^2}s-}(\mathbb{R}^d)$$

for $0 < s < s_c$. In general, Lemma 2.5 below shows that $\zeta_k \in C([-T,T]; H^{\alpha_k s-}(\mathbb{R}^d))$ for $0 < s < s_c$, where

$$\alpha_k := \sum_{l=0}^{k-1}(d-3)^{-l} = \frac{d-3}{d-4}\left(1-(d-3)^{-k}\right).$$

Then, we have $\alpha_2 = \frac{d-2}{d-3}$, $\alpha_3 = \frac{d^2-5d+7}{(d-3)^2}$, and $\alpha_\infty = \frac{d-3}{d-4}$.

Theorem 1.3. *Let $d \geq 6$ and let $\frac{d-4}{d-3}s_c < s < s_c$. Given $\phi \in H^s(\mathbb{R}^d)$, let ϕ^ω be its Wiener randomization defined in* (1.2). *Then, the Cauchy problem* (1.1) *with $u(0) = \phi^\omega$ is almost surely locally well-posed in $H^s(\mathbb{R}^d)$. Moreover, by letting $k \in \mathbb{N}$ such that $\frac{s_c}{\alpha_{k+1}} < s \leq \frac{s_c}{\alpha_k}$, the solution u lies in the class:*

$$\zeta_1 + \zeta_2 + \cdots + \zeta_k + C([-T,T]; H^{s_c+}(\mathbb{R}^d)) \subset C([-T,T]; H^s(\mathbb{R}^d))$$

for $T = T(\phi, \omega) > 0$ almost surely.

The upper bound $s \leq \frac{s_c}{\alpha_k}$ implies that the expansion is meaningful. Indeed, because $\zeta_k(t) \in H^{s_c+}(\mathbb{R}^d)$ for $s > \frac{s_c}{\alpha_k}$, the solution u lies in the class

$$\zeta_1 + \zeta_2 + \cdots + \zeta_{k-1} + C([-T,T]; H^{s_c+}(\mathbb{R}^d)) \subset C([-T,T]; H^s(\mathbb{R}^d))$$

provided that $\frac{s_c}{\alpha_k} < s < s_c$.

Because the regularity of $|\zeta_2|^2$ is not large enough when $d = 5$ (see Remark 3 below), we need to modify the expansion for $d = 5$. Set $\eta_k := \zeta_k$ for $k = 1, 2, 3$ and

$$\eta_4 := \zeta_4 - i\int_0^t S(t-t')|\zeta_2(t')|^2 dt', \quad \eta_k := -2i\int_0^t S(t-t')\Re(\eta_1\overline{\eta}_{k-1})(t')dt'$$

for $k \geq 5$. Since the regularity of $\zeta_1\overline{\zeta_3}$ is worse than that of $|\zeta_2|^2$, η_k has the same regularity as that of ζ_k, i.e., $\eta_k \in C([-T,T]; H^{\alpha_k s-}(\mathbb{R}^5))$.

Theorem 1.4. *Let $\frac{s_c}{2} < s < s_c$. Given $\phi \in H^s(\mathbb{R}^5)$, let ϕ^ω be its Wiener randomization defined in* (1.2). *Then, the Cauchy problem* (1.1) *with $u(0) = \phi^\omega$ is almost surely locally well-posed in $H^s(\mathbb{R}^5)$. Moreover, by letting $k \in \mathbb{N}$ such that $\frac{s_c}{\alpha_{k+1}} < s \leq \frac{s_c}{\alpha_k}$, the solution u lies in the class:*

$$\eta_1 + \eta_2 + \cdots + \eta_k + C([-T,T]; H^{s_c+}(\mathbb{R}^5)) \subset C([-T,T]; H^s(\mathbb{R}^5))$$

for $T = T(\phi, \omega) > 0$ almost surely.

§ 1.1. Notation

We summarize the notation used throughout this paper. We set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We denote the space of all rapidly decaying functions on $\mathbb{R}^d$ by $\mathcal{S}(\mathbb{R}^d)$.

In estimates, we use C to denote a positive constant that can change from line to line. If C is absolute or depends only on parameters that are considered fixed, then we often write $X \lesssim Y$, which means $X \leq CY$. We define $X \sim Y$ to mean $C^{-1}Y \leq X \leq CY$.

Let θ be a smooth even function with $0 \le \theta \le 1$ and $\theta(\xi) = \begin{cases} 1, & \text{if } |\xi| \le 1, \\ 0, & \text{if } |\xi| \ge 2. \end{cases}$ For any $R > 0$, we set

$$\theta_{\le R}(\xi) := \theta\left(\frac{\xi}{R}\right), \quad \theta_R(\xi) := \theta_{\le R}(\xi) - \theta_{\le \frac{R}{2}}(\xi).$$

For any $N \in 2^{\mathbb{N}_0}$, we define

$$P_N f := \mathcal{F}^{-1}[\theta_N \widehat{f}].$$

We use $a+$ and $a-$ to denote quantities $a + \varepsilon$ and $a - \varepsilon$, respectively, when $\varepsilon > 0$ is arbitrarily small and implicit constants are allowed to depend on ε. We also use $\infty-$ to denote $\frac{1}{0+}$.

§ 2. Proof of Theorem 1.1

First, we collect Strichartz-type estimates. We call (q, r) admissible if $q, r \in [2, \infty]$, $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$, and $(q, r, d) \neq (2, \infty, 2)$. Then, the following Strichartz estimate holds.

Theorem 2.1. *Let (q, r) be admissible. Then, we have*

$$\|S(t)\phi\|_{L_t^q L_x^r} \lesssim \|\phi\|_{L_x^2}.$$

For the nonlinear estimates, we use the Fourier restriction norm space.

Definition 2.2. Let $s, b \in \mathbb{R}$. The space $X^{s,b}$ is defined to be the closure of the Schwartz functions $\mathcal{S}(\mathbb{R} \times \mathbb{R}^d)$ under the norm

$$\|u\|_{X^{s,b}} := \|\langle\xi\rangle^s \langle\tau + |\xi|^2\rangle^b \mathcal{F}_{t,x}[u](\tau, \xi)\|_{L^2_{\tau,\xi}}.$$

Note that $X^{s,b} \hookrightarrow C(\mathbb{R}; H^s(\mathbb{R}))$ for $b > \frac{1}{2}$ holds (see, for example, Tao [17, Corollary 2.10]). Moreover, the transfer principle yields the following (see, for example, Tao [17, Lemma 2.9]): For any admissible pair (q, r) and $b > \frac{1}{2}$, we have

$$\|u\|_{L_t^q L_x^r} \lesssim \|u\|_{X^{0,b}}.$$

Remark 1. Since $\|u\|_{L^2_{t,x}} = \|u\|_{X^{0,0}}$, for any admissible pair (q, r) with $q > 2$, an interpolation shows that

$$\|u\|_{L_t^{q-} L_x^{r-}} \lesssim \|u\|_{X^{0, \frac{1}{2}-}}.$$

When $q = 2$, we have

$$\|u\|_{L_t^2 L_x^{\frac{2d}{d-2}-}} \lesssim \|u\|_{X^{0, \frac{1}{2}-}}.$$

The following lemma shows an improvement of the Strichartz estimates upon the randomization of initial data (See, for example, [1]).

Lemma 2.3. *Given ϕ on $\mathbb{R}^d$, let ϕ^ω be its Wiener randomization defined in* (1.2). *Then, given finite $q, r \geq 2$, there exist $C, c > 0$ such that*

$$P\Big(\|S(t)\phi^\omega\|_{L_t^q L_x^r([-T,T]\times\mathbb{R}^d)} > \lambda\Big) \leq C\exp\left(-c\frac{\lambda^2}{T^{\frac{2}{q}}\|\phi\|_{L^2}^2}\right) \tag{2.1}$$

for all $T > 0$ and $\lambda > 0$.

Second, we recall the following bilinear estimate by Tao [16].

Theorem 2.4. *Let N_1, N_2, and N_3 be dyadic numbers. Then, we have*

$$\|P_{N_3}(P_{N_1}f\overline{P_{N_2}g})\|_{X^{0,-\frac{1}{2}+}} \lesssim N_{\max}^{-\frac{1}{2}+}N_{\min}^{\frac{d-3}{2}}\|P_{N_1}f\|_{X^{0,\frac{1}{2}+}}\|P_{N_2}g\|_{X^{0,\frac{1}{2}+}},$$

where $N_{\max} := \max(N_1, N_2, N_3)$ and $N_{\min} := \min(N_1, N_2, N_3)$.

The resonance function which corresponds our nonlinearity

$$h(\xi_1, \xi_2) := |\xi_1 + \xi_2|^2 - |\xi_1|^2 + |\xi_2|^2$$

can vanish when $\xi_1 + \xi_2$ and ξ_2 are orthogonal. Indeed,

$$|h(\xi_1, \xi_2)| = 2|(\xi_1 + \xi_2)\cdot\xi_2| \sim |\xi_1 + \xi_2||\xi_2|\left|\frac{\pi}{2} - \angle(\xi_1 + \xi_2, \xi_2)\right|,$$

where $\angle(\xi_1, \xi_2)$ denotes the angle between ξ_1 and ξ_2. Hence, by using the Fourier restriction norm, we have gained $N_{\min}^{-1}$ on the right hand side of the estimate in Theorem 2.4.

Third, we observe the following bilinear estimate.

Lemma 2.5. *Let $d \geq 5$. Assume that*

$$0 < s \leq \rho, \quad \sigma < \min\left(s + \frac{\rho}{d-3}, s + \frac{1}{2}\right).$$

Then, we have

$$\|z\overline{F}\|_{X^{\sigma,-\frac{1}{2}+}} + \|F\overline{z}\|_{X^{\sigma,-\frac{1}{2}+}} \lesssim \|z\|_{A^s}\|F\|_{X^{\rho,\frac{1}{2}+}},$$

where

$$\|z\|_{A^s} := \|z\|_{X^{s,\frac{1}{2}+}} + \|\langle\nabla\rangle^s z\|_{L_{t,x}^{\frac{d+2}{2}+}}.$$

Proof. We only consider the estimate for $z\overline{F}$, because $F\overline{z}$ is similarly handled. It suffices to show that

$$\|P_{N_3}(P_{N_1}z\overline{P_{N_2}F})\|_{X^{\sigma,-\frac{1}{2}+}} \lesssim N_{\max}^{0-}\|z\|_{A^s}\|F\|_{X^{\rho,\frac{1}{2}+}} \tag{2.2}$$

for $N_1, N_2, N_3 \in 2^{\mathbb{N}_0}$ and $\max(N_1, N_2) \gtrsim N_3$.

First, we consider the case $N_1 \sim N_2 \gtrsim N_3$. The duality, Hölder's inequality, and Remark 1 imply that

$$\begin{aligned}
\|P_{N_3}(P_{N_1}z\overline{P_{N_2}F})\|_{X^{\sigma,-\frac{1}{2}+}} &\lesssim N_3^{\sigma} \sup_{\|w\|_{X^{0,\frac{1}{2}-}}=1} \left| \int_{\mathbb{R}\times\mathbb{R}^d} P_{N_1}z\overline{P_{N_2}FP_{N_3}w}dtdx \right| \\
&\lesssim N_3^{\sigma} \sup_{\|w\|_{X^{0,\frac{1}{2}-}}=1} \|P_{N_1}z\|_{L_{t,x}^{\frac{d+2}{2}+}}\|P_{N_2}F\|_{L_{t,x}^{\frac{2(d+2)}{d}}}\|P_{N_3}w\|_{L_{t,x}^{\frac{2(d+2)}{d}-}} \\
&\lesssim N_1^{-s-\rho+}N_3^{\sigma}\|\langle\nabla\rangle^s z\|_{L_{t,x}^{\frac{d+2}{2}+}}\|F\|_{X^{\rho,\frac{1}{2}+}}.
\end{aligned}$$

Hence, (2.2) follows from $\sigma < s + \rho$.

Second, we consider the case $N_1 \sim N_3 \geq N_2$. Then, we divide the proof into two cases: $N_2 \leq N_1^{\frac{1}{d-3}}$ or $N_2 \geq N_1^{\frac{1}{d-3}}$.

When $N_2 \leq N_1^{\frac{1}{d-3}}$, Theorem 2.4 yields that

$$\begin{aligned}
\|P_{N_3}(P_{N_1}z\overline{P_{N_2}F})\|_{X^{\sigma,-\frac{1}{2}+}} &\lesssim N_1^{\sigma-\frac{1}{2}+}N_2^{\frac{d-3}{2}}\|P_{N_1}z\|_{X^{0,\frac{1}{2}+}}\|P_{N_2}F\|_{X^{0,\frac{1}{2}+}} \\
&\lesssim N_1^{\sigma-s-\frac{1}{2}+}N_2^{-\rho+\frac{d-3}{2}+}\|z\|_{X^{s,\frac{1}{2}+}}\|F\|_{X^{\rho,\frac{1}{2}+}} \\
&\lesssim N_1^{0-}N_2^{(d-3)\sigma-(d-3)s-\rho+}\|z\|_{X^{s,\frac{1}{2}+}}\|F\|_{X^{\rho,\frac{1}{2}+}} \\
&\lesssim N_1^{0-}\|z\|_{X^{s,\frac{1}{2}+}}\|F\|_{X^{\rho,\frac{1}{2}+}}
\end{aligned}$$

provided that $\sigma < s + \frac{1}{2}$ and $\sigma < s + \frac{\rho}{d-3}$.

When $N_2 \geq N_1^{\frac{1}{d-3}}$, the duality, Hölder's inequality, and Remark 1 imply that for $\rho > 0$

$$\begin{aligned}
\|P_{N_3}(P_{N_1}z\overline{P_{N_2}F})\|_{X^{\sigma,-\frac{1}{2}+}} &\lesssim N_3^{\sigma} \sup_{\|w\|_{X^{0,\frac{1}{2}-}}=1} \left| \int_{\mathbb{R}\times\mathbb{R}^d} P_{N_1}z\overline{P_{N_2}FP_{N_3}w}dtdx \right| \\
&\lesssim N_3^{\sigma} \sup_{\|w\|_{X^{0,\frac{1}{2}-}}=1} \|P_{N_1}z\|_{L_{t,x}^{\frac{d+2}{2}+}}\|P_{N_2}F\|_{L_{t,x}^{\frac{2(d+2)}{d}}}\|P_{N_3}w\|_{L_{t,x}^{\frac{2(d+2)}{d}-}} \\
&\lesssim N_1^{\sigma-s+}N_2^{-\rho}\|\langle\nabla\rangle^s z\|_{L_{t,x}^{\frac{d+2}{2}+}}\|F\|_{X^{\rho,\frac{1}{2}+}} \\
&\lesssim N_1^{\sigma-s-\frac{\rho}{d-3}+}\|\langle\nabla\rangle^s z\|_{L_{t,x}^{\frac{d+2}{2}+}}\|F\|_{X^{\rho,\frac{1}{2}+}}.
\end{aligned}$$

Hence, (2.2) follows from $\sigma < s + \frac{\rho}{d-3}$.

Third, we consider the case $N_1 \le N_2 \sim N_3$. The same calculation as above yields that (2.2) is valid if $s > 0$ and $\sigma < \min\left(\rho + \frac{s}{d-3}, \rho + \frac{1}{2}\right)$, which is better than the condition above. □

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1.

From a standard argument, it suffices to show that

$$\|v\overline{v}\|_{X^{s_c+,-\frac{1}{2}+}} + \|v\overline{z}\|_{X^{s_c+,-\frac{1}{2}+}} + \|z\overline{v}\|_{X^{s_c+,-\frac{1}{2}+}} + \|z\overline{z}\|_{X^{s_c+,-\frac{1}{2}+}} \lesssim \|v\|^2_{X^{s_c+,\frac{1}{2}+}} + \|z\|^2_{A^s}.$$

Here, we note that Lemma 2.3 yields that $\|\theta_{\le T}(t)z\|_{A^s}$ is bounded almost surely.

The estimate for $v\overline{v}$ is essentially the same as that for the deterministic setting. For reader's convenience, we give the proof. The Littlewood-Paley decomposition and Theorem 2.4 yield that

$$\begin{aligned}
\|v\overline{v}\|_{X^{s_c+,-\frac{1}{2}+}} &\lesssim \sum_{\substack{N_1,N_2,N_3 \in 2^{\mathbb{N}_0} \\ \max(N_1,N_2) \gtrsim N_3}} \|P_{N_3}(P_{N_1}v\overline{P_{N_2}v})\|_{X^{s_c+,-\frac{1}{2}+}} \\
&\lesssim \sum_{\substack{N_1,N_2,N_3 \in 2^{\mathbb{N}_0} \\ N_1 \sim N_2 \gtrsim N_3}} N_1^{-\frac{1}{2}+} N_3^{2s_c+\frac{1}{2}+} \|P_{N_1}v\|_{X^{0,\frac{1}{2}+}} \|P_{N_2}v\|_{X^{0,\frac{1}{2}+}} \\
&\quad + \sum_{\substack{N_1,N_2,N_3 \in 2^{\mathbb{N}_0} \\ N_1 \le N_2 \sim N_3}} N_1^{s_c+\frac{1}{2}} N_2^{s_c-\frac{1}{2}+} \|P_{N_1}v\|_{X^{0,\frac{1}{2}+}} \|P_{N_2}v\|_{X^{0,\frac{1}{2}+}} \\
&\lesssim \sum_{\substack{N_1,N_2,N_3 \in 2^{\mathbb{N}_0} \\ N_1 \sim N_2 \gtrsim N_3}} N_1^{-2s_c-\frac{1}{2}-} N_3^{2s_c+\frac{1}{2}+} \|P_{N_1}v\|_{X^{s_c+,\frac{1}{2}+}} \|P_{N_2}v\|_{X^{s_c+,\frac{1}{2}+}} \\
&\quad + \sum_{\substack{N_1,N_2,N_3 \in 2^{\mathbb{N}_0} \\ N_1 \le N_2 \sim N_3}} N_1^{\frac{1}{2}-} N_2^{-\frac{1}{2}+} \|v\|_{X^{s_c+,\frac{1}{2}+}} \|v\|_{X^{s_c+,\frac{1}{2}+}} \\
&\lesssim \|v\|^2_{X^{s_c+,\frac{1}{2}+}}.
\end{aligned}$$

We apply Lemma 2.5 with $F = v$ and $\sigma = \rho = s_c$ to estimate $v\overline{z}$ and $z\overline{v}$. Then, the condition $\frac{d-4}{d-3}s_c < s < s_c$ appears.

The estimate for $z\overline{z}$ is reduced to Lemma 2.5 with $F = z$, $\sigma = s_c$, and $\rho = s$. Then, we need $s > \frac{d-3}{d-2}s_c$. □

§ 3. Proof of Theorem 1.2

For simplicity, we use the abbreviation $\|f\|_{L^q_T L^r_x} := \|f\|_{L^q_t([-T,T];L^r_x(\mathbb{R}^d))}$.

As in [3], we can estimate the $L^q_t L^r_x$-norm of z_2.

Lemma 3.1. *Let $d \geq 4$. For any finite $q, r \geq 2$, we have*

$$\|P_N z_2\|_{L^q_T L^r_x} \lesssim \begin{cases} T^{d(\frac{1}{r}-\frac{1}{2})+1}\|z_1\|^2_{L^{2q}_T L^{2r'}_x} & \text{when } 2 \leq r < \frac{2d}{d-2}, \\ T^{0+}N^{d(\frac{1}{2}-\frac{1}{r})-1+}\|z_1\|^2_{L^{2q}_T L_x^{\frac{4d}{d+2}+}} & \text{when } r \geq \frac{2d}{d-2} \end{cases}$$

for any $T > 0$ and $N \in 2^{\mathbb{N}_0}$.

Proof. We first consider the case $r < \frac{2d}{d-2}$. We use the dispersive estimate to obtain

$$\begin{aligned} \|P_N z_2\|_{L^q_T L^r_x} &\leq \left\| \int_0^t \|P_N S(t-t')|z_1(t')|^2\|_{L^r_x} dt' \right\|_{L^q_T} \\ &\lesssim \left\| \int_0^t |t-t'|^{-d(\frac{1}{2}-\frac{1}{r})} \|z_1(t')\|^2_{L^{2r'}_x} dt' \right\|_{L^q_T} \\ &\lesssim T^{d(\frac{1}{r}-\frac{1}{2})+1}\|z_1\|^2_{L^{2q}_T L^{2r'}_x} \end{aligned}$$

When $r \geq \frac{2d}{d-2}$, we apply Sobolev's embedding $W^{d(\frac{1}{2}-\frac{1}{r})-1+,\frac{2d}{d-2}-}(\mathbb{R}^d) \hookrightarrow L^r(\mathbb{R}^d)$ as the following:

$$\begin{aligned} \|P_N z_2\|_{L^q_T L^r_x} &\leq \left\| \int_0^t \|P_N S(t-t')|z_1(t')|^2\|_{L^r_x} dt' \right\|_{L^q_T} \\ &\leq N^{d(\frac{1}{2}-\frac{1}{r})-1+} \left\| \int_0^t \|P_N S(t-t')|z_1(t')|^2\|_{L_x^{\frac{2d}{d-2}-}} dt' \right\|_{L^q_T} \\ &\lesssim N^{d(\frac{1}{2}-\frac{1}{r})-1+} \left\| \int_0^t |t-t'|^{-1+} \|z_1(t')\|^2_{L_x^{\frac{4d}{d+2}+}} dt' \right\|_{L^q_T} \\ &\lesssim T^{0+}N^{d(\frac{1}{2}-\frac{1}{r})-1+}\|z_1\|^2_{L^{2q}_T L_x^{\frac{4d}{d+2}+}}, \end{aligned}$$

which concludes the proof. □

We define

$$\|F\|_{B^s} := \|F\|_{X^{\frac{d-2}{d-3}s-,\frac{1}{2}+}} + \|\langle\nabla\rangle^s F\|_{L_t^{\infty-} L_x^{\frac{2d}{d-2}-}}.$$

Lemmas 2.5 and 3.1 show that $\|\theta_{\leq T}(t) z_2\|_{B^s}$ is bounded almost surely for $0 < s < s_c$.

Lemma 3.2. *Let $d \geq 6$. Assume that*

$$s > s_c - \frac{2d-7}{3d-7}.$$

Then, we have

$$\|z_2\overline{v}\|_{X^{s_c+,-\frac{1}{2}+}} + \|v\overline{z_2}\|_{X^{s_c+,-\frac{1}{2}+}} \lesssim \|z_2\|_{B^s}\|v\|_{X^{s_c+,\frac{1}{2}+}}.$$

Proof. We only consider the estimate for $z_2\overline{v}$, because $v\overline{z_2}$ is similarly handled. It suffices to show that

$$\|P_{N_3}(P_{N_1}z_2\overline{P_{N_2}v})\|_{X^{s_c+,-\frac{1}{2}+}} \lesssim N_{\max}^{0-}\|z_2\|_{B^s}\|v\|_{X^{s_c+,\frac{1}{2}+}} \tag{3.1}$$

for $N_1, N_2, N_3 \in 2^{\mathbb{N}_0}$ and $\max(N_1, N_2) \gtrsim N_3$.

First, we consider the case $N_1 \sim N_2 \gtrsim N_3$. Since $\left(2, \frac{2d}{d-2}\right)$ is admissible, we use Hölder's inequality and Sobolev's embedding $W^{\frac{d}{2}-3,\frac{2d}{d-2}}(\mathbb{R}^d) \hookrightarrow L^{\frac{d}{2}}(\mathbb{R}^d)$, and Remark 1 to obtain

$$\begin{aligned}
&\|P_{N_3}(P_{N_1}z_2\overline{P_{N_2}v})\|_{X^{s_c+,-\frac{1}{2}+}} \\
&\sim N_3^{s_c+} \sup_{\|w\|_{X^{0,\frac{1}{2}-}}=1} \left|\int_{\mathbb{R}\times\mathbb{R}^d} P_{N_1}z_2\overline{P_{N_2}vP_{N_3}w}\,dtdx\right| \\
&\lesssim N_3^{s_c+} \sup_{\|w\|_{X^{0,\frac{1}{2}-}}=1} \|P_{N_1}z_2\|_{L_t^{\infty-}L_x^{\frac{d}{2}+}}\|P_{N_2}v\|_{L_t^{2+}L_x^{\frac{2d}{d-2}}}\|P_{N_3}w\|_{L_t^2L_x^{\frac{2d}{d-2}-}} \\
&\lesssim N_1^{-s-1+}N_3^{s_c+}\|\langle\nabla\rangle^s z_2\|_{L_t^{\infty-}L_x^{\frac{2d}{d-2}-}}\|v\|_{X^{s_c+,\frac{1}{2}+}}.
\end{aligned}$$

Thus, (3.1) is valud if $s > s_c - 1$.

Second, we consider the case $N_1 \leq N_2 \sim N_3$. Theorem 2.4 shows

$$\begin{aligned}
\|P_{N_3}(P_{N_1}z_2\overline{P_{N_2}v})\|_{X^{s_c+,-\frac{1}{2}+}} &\lesssim N_1^{\frac{d-3}{2}}N_2^{s_c-\frac{1}{2}+}\|P_{N_1}z_2\|_{X^{0,\frac{1}{2}+}}\|P_{N_2}v\|_{X^{0,\frac{1}{2}+}} \\
&\lesssim N_1^{-\frac{d-2}{d-3}s+\frac{d-3}{2}}N_2^{-\frac{1}{2}+}\|P_{N_1}z_2\|_{X^{\frac{d-2}{d-3}s-,\frac{1}{2}+}}\|P_{N_2}v\|_{X^{s_c+,\frac{1}{2}+}}.
\end{aligned}$$

On the other hand, since $\left(2, \frac{2d}{d-2}\right)$ is admissible, we use Hölder's inequality and Sobolev's embedding $W^{\frac{d}{2}-3,\frac{2d}{d-2}}(\mathbb{R}^d) \hookrightarrow L^{\frac{d}{2}}(\mathbb{R}^d)$ to obtain

$$\begin{aligned}
&\|P_{N_3}(P_{N_1}z_2\overline{P_{N_2}v})\|_{X^{s_c+,-\frac{1}{2}+}} \\
&\sim N_3^{s_c+} \sup_{\|w\|_{X^{0,\frac{1}{2}-}}=1} \left|\int_{\mathbb{R}\times\mathbb{R}^d} P_{N_1}z_2\overline{P_{N_2}vP_{N_3}w}\,dtdx\right| \\
&\lesssim N_2^{s_c+} \sup_{\|w\|_{X^{0,\frac{1}{2}-}}=1} \|P_{N_1}z_2\|_{L_t^{\infty-}L_x^{\frac{d}{2}+}}\|P_{N_2}v\|_{L_t^{2+}L_x^{\frac{2d}{d-2}}}\|P_{N_3}w\|_{L_t^2L_x^{\frac{2d}{d-2}-}} \\
&\lesssim N_1^{-s+s_c-1+}N_2^{0+}\|\langle\nabla\rangle^s z_2\|_{L_t^{\infty-}L_x^{\frac{2d}{d-2}-}}\|v\|_{X^{s_c+,\frac{1}{2}+}}.
\end{aligned}$$

We apply the first estimate only to the (0+)-power of the factor in $\|P_{N_3}(P_{N_1}z_2\overline{P_{N_2}v})\|_{X^{s_c,-\frac{1}{2}+}}$.

Then, we have

$$\begin{aligned}
&\|P_{N_3}(P_{N_1}z_2\overline{P_{N_2}v})\|_{X^{s_c+,-\frac{1}{2}+}} \\
&\leq \|P_{N_3}(P_{N_1}z_2\overline{P_{N_2}v})\|^{0+}_{X^{s_c+,-\frac{1}{2}+}}\|P_{N_3}(P_{N_1}z_2\overline{P_{N_2}v})\|^{1-}_{X^{s_c+,-\frac{1}{2}+}} \\
&\lesssim N_1^{-s+s_c-1+}N_2^{0-}\left(\|z_2\|_{X^{\frac{d-2}{d-3}s-,\frac{1}{2}+}}+\|\langle\nabla\rangle^s z_2\|_{L_t^{\infty-}L_x^{\frac{2d}{d-2}-}}\right)\|v\|_{X^{s_c+,\frac{1}{2}+}}.
\end{aligned}$$

Hence, (3.1) follows from this bound and $s > s_c - 1$.

Third, we consider the case $N_2 \leq N_1 \sim N_3$. We divide the proof into two: $N_2 \leq N_1^{\frac{2d-7}{3d-7}}$ or $N_2 \geq N_1^{\frac{2d-7}{3d-7}}$.

Subcase 1. $N_2 \leq N_1^{\frac{2d-7}{3d-7}}$: We apply Theorem 2.4 to obtain

$$\begin{aligned}
\|P_{N_3}(P_{N_1}z_2\overline{P_{N_2}v})\|_{X^{s_c+,-\frac{1}{2}+}} &\lesssim N_1^{s_c-\frac{1}{2}+}N_2^{\frac{d-3}{2}}\|P_{N_1}z_2\|_{X^{0,\frac{1}{2}+}}\|P_{N_2}v\|_{X^{0,\frac{1}{2}+}} \\
&\lesssim N_1^{-\frac{d-2}{d-3}s+\frac{d-5}{2}+}N_2^{\frac{1}{2}}\|z_2\|_{X^{\frac{d-2}{d-3}s-,\frac{1}{2}+}}\|v\|_{X^{s_c+,\frac{1}{2}+}} \\
&\lesssim N_1^{-\frac{d-2}{d-3}s+\frac{(d-2)(3d-14)}{2(3d-7)}+}\|z_2\|_{X^{\frac{d-2}{d-3}s-,\frac{1}{2}+}}\|v\|_{X^{s_c+,\frac{1}{2}+}},
\end{aligned}$$

which shows (3.1) provided that $s > \frac{(d-3)(3d-14)}{2(3d-7)} = s_c - \frac{2d-7}{3d-7}$.

Subcase 2. $N_2 \geq N_1^{\frac{2d-7}{3d-7}}$: Hölder's inequality yields that for $\rho > 0$

$$\begin{aligned}
&\|P_{N_3}(P_{N_1}z_2\overline{P_{N_2}v})\|_{X^{s_c+,-\frac{1}{2}+}} \\
&\sim N_3^{s_c+}\sup_{\|w\|_{X^{0,\frac{1}{2}-}}=1}\left|\int_{\mathbb{R}\times\mathbb{R}^d}P_{N_1}z_2\overline{P_{N_2}vP_{N_3}w}dtdx\right| \\
&\lesssim N_1^{s_c+}\sup_{\|w\|_{X^{0,\frac{1}{2}-}}=1}\|P_{N_1}z_2\|_{L_t^{\infty-}L_x^{\frac{2d}{d-2}-}}\|P_{N_2}v\|_{L_t^{2+}L_x^{\frac{d}{2}+}}\|w\|_{L_t^2L_x^{\frac{2d}{d-2}-}}.
\end{aligned}$$

For $d \geq 6$, since $\left(2, \frac{2d}{d-2}\right)$ is admissible, we use Sobolev's embedding $W^{\frac{d}{2}-3,\frac{2d}{d-2}}(\mathbb{R}^d) \hookrightarrow L^{\frac{d}{2}}(\mathbb{R}^d)$ to show that

$$\|P_{N_2}v\|_{L_t^{2+}L_x^{\frac{d}{2}+}} \lesssim N_2^{\frac{d}{2}-3}\|P_{N_2}v\|_{L_t^{2+}L_x^{\frac{2d}{d-2}+}} \lesssim N_2^{-1+}\|v\|_{X^{s_c+,\frac{1}{2}+}}.$$

We therefore have

$$\begin{aligned}
\|P_{N_3}(P_{N_1}z_2\overline{P_{N_2}v})\|_{X^{s_c+,-\frac{1}{2}+}} &\lesssim N_1^{-s+s_c+}N_2^{-1+}\|\langle\nabla\rangle^s z_2\|_{L_t^{\infty-}L_x^{\frac{2d}{d-2}}}\|v\|_{X^{s_c+,\frac{1}{2}+}} \\
&\lesssim N_1^{-s+s_c-\frac{2d-7}{3d-7}+}\|\langle\nabla\rangle^s z_2\|_{L_t^{\infty-}L_x^{\frac{2d}{d-2}}}\|v\|_{X^{s_c+,\frac{1}{2}+}}.
\end{aligned}$$

Hence, (3.1) holds provided that $s > s_c - \frac{2d-7}{3d-7}$, which concludes the proof. □

Lemma 3.3. *Let $d \geq 6$. If $s > \max\left\{\frac{(d-3)(3d-13)}{6(d-2)}, \frac{(d-3)(3d-14)}{2(3d-8)}\right\}$, we have*

$$\|z_2\overline{z_2}\|_{X^{s_c+,-\frac{1}{2}+}} \lesssim \|z_2\|_{B^s}^2.$$

Proof. From the Littlewood-Paley decomposition, it suffices to show that

$$\|P_{N_3}(P_{N_1}z_2\overline{P_{N_2}z_2})\|_{X^{s_c+,-\frac{1}{2}+}} \lesssim N_2^{0-}\|z_2\|_{B^s}^2 \tag{3.2}$$

for $N_1, N_2, N_3 \in 2^{\mathbb{N}_0}$ with $N_1 \leq N_2$.

When $N_1 \sim N_2 \gtrsim N_3$, we use the duality argument, the fact that $\left(2, \frac{2d}{d-2}\right)$ is admissible, Sobolev's embedding $W^{\frac{d}{2}-3,\frac{2d}{d-2}}(\mathbb{R}^d) \hookrightarrow L^{\frac{d}{2}}(\mathbb{R}^d)$, and Remark 1 to obtain

$$\begin{aligned}
&\|P_{N_3}(P_{N_1}z_2\overline{P_{N_2}z_2})\|_{X^{s_c+,-\frac{1}{2}+}} \\
&\leq N_3^{s_c+}\sup_{\|w\|_{X^{0,\frac{1}{2}-}}=1}\left|\int_{\mathbb{R}\times\mathbb{R}^d}P_{N_1}z_2\overline{P_{N_2}z_2P_{N_3}w}dtdx\right| \\
&\lesssim N_3^{s_c+}\sup_{\|w\|_{X^{0,\frac{1}{2}-}}=1}\|P_{N_1}z_2\|_{L_t^{2+}L_x^{\frac{d}{2}+}}\|P_{N_2}z_2\|_{L_t^{\infty-}L_x^{\frac{2d}{d-2}-}}\|P_{N_3}w\|_{L_t^2L_x^{\frac{2d}{d-2}-}} \\
&\lesssim N_1^{-\frac{2d-5}{d-3}s+s_c-1+}N_3^{s_c+}\|z_2\|_{X^{\frac{d-2}{d-3}s-,\frac{1}{2}+}}\|\langle\nabla\rangle^sP_{N_2}z_2\|_{L_t^{\infty-}L_x^{\frac{2d}{d-2}-}}.
\end{aligned}$$

Thus, (3.2) follows from $s > \frac{(d-3)(d-5)}{2d-5}$, which is better than $s > \frac{(d-3)(3d-13)}{6(d-2)}$ for $d \geq 6$.

Next, we focus on the case $N_1 \leq N_2 \sim N_3$. We divide the proof into two cases: $N_1 \leq N_2^{\frac{1}{2}}$ or $N_1 \geq N_2^{\frac{1}{2}}$. First, we consider the case $N_1 \leq N_2^{\frac{1}{2}}$. Theorem 2.4 yields that

$$\begin{aligned}
\|P_{N_3}(P_{N_1}z_2\overline{P_{N_2}z_2})\|_{X^{s_c+,-\frac{1}{2}+}} &\lesssim N_1^{\frac{d-3}{2}}N_2^{s_c-\frac{1}{2}+}\|P_{N_1}z_2\|_{X^{0,\frac{1}{2}+}}\|P_{N_2}z_2\|_{X^{0,\frac{1}{2}+}} \\
&\lesssim N_1^{-\frac{d-2}{d-3}s+\frac{d-3}{2}}N_2^{-\frac{d-2}{d-3}s+s_c-\frac{1}{2}+}\|z_2\|_{X^{\frac{d-2}{d-3}s-,\frac{1}{2}+}}^2 \\
&\lesssim N_1^{-3\frac{d-2}{d-3}s+\frac{3d-13}{2}+}N_2^{0-}\|z_2\|_{X^{\frac{d-2}{d-3}s-,\frac{1}{2}+}}^2
\end{aligned}$$

provided that $s > \frac{(d-3)(d-5)}{2(d-2)}$. Hence, (3.2) follows from $s > \frac{(d-3)(3d-13)}{6(d-2)}$.

Second, we focus on the case $N_1 \geq N_2^{\frac{1}{2}}$. By the duality argument, we have

$$\|P_{N_3}(P_{N_1}z_2\overline{P_{N_2}z_2})\|_{X^{s_c+,-\frac{1}{2}+}} \leq N_3^{s_c+}\sup_{\|w\|_{X^{0,\frac{1}{2}-}}=1}\left|\int_{\mathbb{R}\times\mathbb{R}^d}P_{N_1}z_2\overline{P_{N_2}z_2P_{N_3}w}dtdx\right|.$$

We use the fact that $\left(2, \frac{2d}{d-2}\right)$ is admissible and Sobolev's embedding $W^{\frac{d}{2}-3,\frac{2d}{d-2}}(\mathbb{R}^d) \hookrightarrow$

$L^{\frac{d}{2}}(\mathbb{R}^d)$ to show that

$$
\begin{aligned}
&\left|\int_{\mathbb{R}\times\mathbb{R}^d} P_{N_1}z_2\overline{P_{N_2}z_2P_{N_3}w}dtdx\right| \\
&\lesssim \|P_{N_1}z_2\|_{L_t^{2+}L_x^{\frac{d}{2}+}}\|P_{N_2}z_2\|_{L_t^{\infty-}L_x^{\frac{2d}{d-2}-}}\|P_{N_3}w\|_{L_t^2L_x^{\frac{2d}{d-2}-}} \\
&\lesssim N_1^{-\frac{d-2}{d-3}s+\frac{d}{2}-3+}N_2^{-s+}\|z_2\|_{X^{\frac{d-2}{d-3}s-,\frac{1}{2}+}}\|\langle\nabla\rangle^s P_{N_2}z_2\|_{L_t^{\infty-}L_x^{\frac{2d}{d-2}-}}\|w\|_{X^{0,\frac{1}{2}-}}.
\end{aligned}
$$

Hence, we obtain that

$$
\|P_{N_3}(P_{N_1}z_2\overline{P_{N_2}z_2})\|_{X^{s_c+,-\frac{1}{2}+}} \lesssim N_1^{-\frac{d-2}{d-3}s+\frac{d}{2}-3+}N_2^{-s+s_c+}\|z_2\|_{B^s}^2.
$$

Then, (3.2) holds provided that $s > \frac{(d-3)(d-6)}{2(d-2)}$ and $s > \frac{(d-3)(3d-14)}{2(3d-8)}$. □

Remark 2. Although the lower bound of the regularity in the assumption in Lemma 3.3 is not sharp, it is enough to prove our results. Indeed, the value is smaller than $\frac{d-4}{d-3}s_c\left(<\frac{(d-3)^2}{d^2-5d+7}s_c\right)$.

For $d=5$, we need the following bit more general estimate.

Lemma 3.4. *Let $s\in(0,\frac{1}{2})$. Assume that real numbers a, σ, and ρ satisfy*

$$
s<a<s+\frac{1}{2}, \quad a\le\rho, \quad \sigma<\min\left\{s+\left(a-s+\frac{1}{2}\right)\rho, a+\frac{1}{2}\right\}.
$$

Then, we have

$$
\|Z\overline{F}\|_{X^{\sigma,-\frac{1}{2}+}}+\|F\overline{Z}\|_{X^{\sigma,-\frac{1}{2}+}} \lesssim \left(\|Z\|_{X^{a,\frac{1}{2}+}}+\|\langle\nabla\rangle^s Z\|_{L_t^4L_x^{\frac{10}{3}-}}\right)\|F\|_{X^{\rho,\frac{1}{2}+}}.
$$

Proof. As in the proof of Lemma 2.5, it suffices to show that

$$
\|P_{N_3}(P_{N_1}Z\overline{P_{N_2}F})\|_{X^{\sigma,-\frac{1}{2}+}} \lesssim N_{\max}^{0-}\left(\|Z\|_{X^{a,\frac{1}{2}+}}+\|\langle\nabla\rangle^s Z\|_{L_t^4L_x^{\frac{10}{3}-}}\right)\|F\|_{X^{\rho,\frac{1}{2}+}} \tag{3.3}
$$

for $N_1,N_2,N_3\in 2^{\mathbb{N}_0}$ and $\max(N_1,N_2)\gtrsim N_3$.

First, we consider the case $N_1\sim N_2\gtrsim N_3$. Since $\left(2,\frac{10}{3}\right)$ and $\left(4,\frac{5}{2}\right)$ are admissible, the duality argument, Hölder's inequality, and Remark 1 imply that

$$
\begin{aligned}
\|P_{N_3}(P_{N_1}Z\overline{P_{N_2}F})\|_{X^{\sigma,-\frac{1}{2}+}} &\lesssim N_3^\sigma \sup_{\|w\|_{X^{0,\frac{1}{2}-}}=1}\left|\int_{\mathbb{R}\times\mathbb{R}^5}P_{N_1}Z\overline{P_{N_2}FP_{N_3}w}dtdx\right| \\
&\lesssim N_3^\sigma \sup_{\|w\|_{X^{0,\frac{1}{2}-}}=1}\|P_{N_1}Z\|_{L_t^4L_x^{\frac{10}{3}-}}\|P_{N_2}F\|_{L_t^4L_x^{\frac{5}{2}+}}\|P_{N_3}w\|_{L_t^2L_x^{\frac{10}{3}-}} \\
&\lesssim N_1^{-s-\rho+}N_3^\sigma\|\langle\nabla\rangle^s Z\|_{L_t^4L_x^{\frac{10}{3}-}}\|F\|_{X^{\rho,\frac{1}{2}+}}.
\end{aligned}
$$

Hence, (3.3) follows from $\sigma < s + \rho$.

Second, we consider the case $N_1 \sim N_3 \geq N_2$. We also divide the proof into two cases: $N_2 \leq N_1^{a-s+\frac{1}{2}}$ or $N_2 \geq N_1^{a-s+\frac{1}{2}}$.

When $N_2 \leq N_1^{a-s+\frac{1}{2}}$, Theorem 2.4 yields that

$$\begin{aligned}
\|P_{N_3}(P_{N_1}Z\overline{P_{N_2}F})\|_{X^{\sigma,-\frac{1}{2}+}} &\lesssim N_1^{\sigma-\frac{1}{2}+}N_2\|P_{N_1}Z\|_{X^{0,\frac{1}{2}+}}\|P_{N_2}F\|_{X^{0,\frac{1}{2}+}} \\
&\lesssim N_1^{\sigma-a-\frac{1}{2}+}N_2^{-\rho+1}\|Z\|_{X^{a,\frac{1}{2}+}}\|F\|_{X^{\rho,\frac{1}{2}+}} \\
&\lesssim N_1^{0-}N_2^{\frac{2}{2a-2s+1}\sigma-\frac{2}{2a-2s+1}s-\rho+}\|Z\|_{X^{a,\frac{1}{2}+}}\|F\|_{X^{\rho,\frac{1}{2}+}} \\
&\lesssim N_1^{0-}\|Z\|_{X^{s,\frac{1}{2}+}}\|F\|_{X^{\rho,\frac{1}{2}+}}
\end{aligned}$$

provided that $\sigma < a + \frac{1}{2}$ and $\sigma < s + \left(a - s + \frac{1}{2}\right)\rho$.

When $N_2 \geq N_1^{a-s+\frac{1}{2}}$, we note that $\left(2, \frac{10}{3}\right)$ and $\left(4, \frac{5}{2}\right)$ are admissible. Then, the duality and Hölder's inequality imply that for $\rho > 0$

$$\begin{aligned}
\|P_{N_3}(P_{N_1}Z\overline{P_{N_2}F})\|_{X^{\sigma,-\frac{1}{2}+}} &\lesssim N_3^{\sigma} \sup_{\|w\|_{X^{0,\frac{1}{2}-}}=1}\left|\int_{\mathbb{R}\times\mathbb{R}^5} P_{N_1}Z\overline{P_{N_2}FP_{N_3}w}dtdx\right| \\
&\lesssim N_3^{\sigma} \sup_{\|w\|_{X^{0,\frac{1}{2}-}}=1}\|P_{N_1}Z\|_{L_t^4L_x^{\frac{10}{3}-}}\|P_{N_2}F\|_{L_t^4L_x^{\frac{5}{2}+}}\|P_{N_3}w\|_{L_t^2L_x^{\frac{10}{3}-}} \\
&\lesssim N_1^{\sigma-s+}N_2^{-\rho+}\|\langle\nabla\rangle^s Z\|_{L_t^4L_x^{\frac{10}{3}-}}\|F\|_{X^{\rho,\frac{1}{2}+}} \\
&\lesssim N_1^{\sigma-s-(a-s+\frac{1}{2})\rho+}\|\langle\nabla\rangle^s Z\|_{L_t^4L_x^{\frac{10}{3}-}}\|F\|_{X^{\rho,\frac{1}{2}+}}.
\end{aligned}$$

Hence, (3.3) follows from $\sigma < s + \left(a - s + \frac{1}{2}\right)\rho$.

Third, we consider the case $N_1 \leq N_2 \sim N_3$. We also divide the proof into two cases: $N_1 \leq N_2^{\frac{1}{2(-a+s+1)}}$ or $N_1 \geq N_2^{\frac{1}{2(-a+s+1)}}$.

When $N_1 \leq N_2^{\frac{1}{2(-a+s+1)}}$, Theorem 2.4 yields that

$$\begin{aligned}
\|P_{N_3}(P_{N_1}Z\overline{P_{N_2}F})\|_{X^{\sigma,-\frac{1}{2}+}} &\lesssim N_1N_2^{\sigma-\frac{1}{2}+}\|P_{N_1}Z\|_{X^{0,\frac{1}{2}+}}\|P_{N_2}F\|_{X^{0,\frac{1}{2}+}} \\
&\lesssim N_1^{-a+1}N_2^{\sigma-\rho-\frac{1}{2}+}\|Z\|_{X^{a,\frac{1}{2}+}}\|F\|_{X^{\rho,\frac{1}{2}+}} \\
&\lesssim N_1^{2(-a+s+1)\sigma-s+2(a-s-1)\rho}N_2^{0-}\|Z\|_{X^{a,\frac{1}{2}+}}\|F\|_{X^{\rho,\frac{1}{2}+}} \\
&\lesssim N_2^{0-}\|Z\|_{X^{s,\frac{1}{2}+}}\|F\|_{X^{\rho,\frac{1}{2}+}}
\end{aligned}$$

provided that $\sigma < \rho + \frac{s}{2(-a+s+1)}$. This condition follows from $s < a \leq \min(\rho, 1)$ and $\sigma < s + \left(a - s + \frac{1}{2}\right)\rho$.

When $N_1 \geq N_2^{\frac{1}{2(-a+s+1)}}$, we note that $\left(2, \frac{10}{3}\right)$ and $\left(4, \frac{5}{2}\right)$ are admissible. Then, the duality and Hölder's inequality imply that for $s>0$

$$\begin{aligned}\|P_{N_3}(P_{N_1}Z\overline{P_{N_2}F})\|_{X^{\sigma,-\frac{1}{2}+}} &\lesssim N_3^{\sigma}\sup_{\|w\|_{X^{0,\frac{1}{2}-}}=1}\left|\int_{\mathbb{R}\times\mathbb{R}^5} P_{N_1}Z\overline{P_{N_2}FP_{N_3}w}dtdx\right|\\ &\lesssim N_3^{\sigma}\sup_{\|w\|_{X^{0,\frac{1}{2}-}}=1}\|P_{N_1}Z\|_{L_t^4L_x^{\frac{10}{3}-}}\|P_{N_2}F\|_{L_t^4L_x^{\frac{5}{2}+}}\|P_{N_3}w\|_{L_t^2L_x^{\frac{10}{3}-}}\\ &\lesssim N_1^{-s+}N_2^{\sigma-\rho+}\|\langle\nabla\rangle^sZ\|_{L_t^4L_x^{\frac{10}{3}-}}\|F\|_{X^{\rho,\frac{1}{2}+}}\\ &\lesssim N_2^{\sigma-\rho-\frac{s}{2(-a+s+1)}}\|\langle\nabla\rangle^sZ\|_{L_t^4L_x^{\frac{10}{3}-}}\|F\|_{X^{\rho,\frac{1}{2}+}}.\end{aligned}$$

Hence, (3.3) follows from $\sigma<\rho+\frac{s}{2(-a+s+1)}$. □

Remark 3. In particular, Lemma 3.4 with $a=\rho=\frac{3}{2}s$ shows that

$$|z_2|^2\in X^{\frac{3s+7}{4}s-,-\frac{1}{2}+}$$

for $s<\frac{1}{2}$. Here, $\frac{3s+7}{4}s>s_c=\frac{1}{2}$ is equivalent to $s>\frac{-7+\sqrt{73}}{6}\approx 0.257$, which is better than $s>\left.\frac{(d-3)^2}{d^2-5d+7}s_c\right|_{d=5}=\frac{2}{7}\approx 0.286$.

Moreover, because $0.254\approx\frac{16}{63}=\left.\frac{s_c}{\alpha_6}\right|_{d=5}<\frac{-7+\sqrt{73}}{6}<\left.\frac{s_c}{\alpha_5}\right|_{d=5}=\frac{8}{31}\approx 0.258$, we need to extract the term $|z_2|^2$ in the higher expansion.

Proof of Theorem 1.2. When $s>\frac{d-3}{d-2}s_c$, the proof is reduced to that of Theorem 1.1, because $\theta_{\le T}(t)z_2\in X^{s_c+,\frac{1}{2}+}$. Hence, we consider the case $s\le\frac{d-3}{d-2}s_c$.

From a standard argument, it suffices to show that

$$\||v+z_1+z_2|^2-|z_1|^2\|_{X^{s_c+,-\frac{1}{2}+}}\lesssim\|v\|^2_{X^{s_c+,\frac{1}{2}+}}+\|z_1\|^2_{A^s}+\|z_2\|^2_{B^s}.$$

Because

$$|v+z_1+z_2|^2-|z_1|^2=|v|^2+|z_2|^2+2\Re\left(v\overline{z_1}+v\overline{z_2}+z_1\overline{z_2}\right),$$

the estimates for $|v|^2$ and $v\overline{z_1}$ are already observed in the proof of Theorem 1.1.

By Lemma 2.5 with $F=z_2$, $\sigma=s_c+$, and $\rho=\frac{d-2}{d-3}s$, we can estimate $z_1\overline{z_2}$ if $s>\frac{(d-3)^2}{d^2-5d+7}s_c$.

The estimates for $v\overline{z_2}$ and $|z_2|^2$ are a consequence of Lemmas 3.2, 3.3, and 3.4, whose conditions are better than $s>\frac{(d-3)^2}{d^2-5d+7}s_c$. □

§ 4. higher order expansion

§ 4.1. higher dimensional cases

Lemma 4.1. *Let $d\ge 4$ and let $s\ge 0$. For any $k\ge 2$ and any finite $q,r>2$, there exists $\delta_k=\delta_k(q,r)>0$ such that*

$$\|P_N\zeta_k\|_{L_T^qL_x^r}\lesssim\max(N^{d(\frac{1}{2}-\frac{1}{r})-1+},1)\|z_1\|^{k(1-\delta_k)}_{L_T^{kq}L_x^2}\|z_1\|^{k\delta_k}_{L_T^{kq}L_x^d}\tag{4.1}$$

for any $T \in (0,1)$ *and* $N \in 2^{\mathbb{N}_0}$.

Proof. We prove (4.1) by induction. We first consider the case $r < \frac{2d}{d-2}$. Lemma 3.1 shows that (4.1) with $k=2$ is valid. Suppose that (4.1) holds for $k-1$. Then, the dispersive estimate yields that

$$\begin{aligned}\|P_N\zeta_k\|_{L_T^qL_x^r} &\le \left\| \int_0^t \|P_N S(t-t')(\zeta_1\overline{\zeta_{k-1}})(t')\|_{L_x^r} dt' \right\|_{L_T^q} \\ &\lesssim \left\| \int_0^t |t-t'|^{-d(\frac{1}{2}-\frac{1}{r})} \|(\zeta_1\overline{\zeta_{k-1}})(t')\|_{L_x^{r'}} dt' \right\|_{L_T^q} \\ &\lesssim T^{d(\frac{1}{r}-\frac{1}{2})+1} \|\zeta_1\|_{L_T^{2q}L_x^{\frac{2dr}{(d+2)r-2d}+}} \|\zeta_{k-1}\|_{L_T^{2q}L_x^{\frac{2d}{d-2}-}}.\end{aligned}$$

Because $d \ge 4$ and $2 < r < \frac{2d}{d-2}$ imply that $\frac{d}{2} < \frac{2dr}{(d+2)r-2d} < d$. We use the interpolation $L^{\frac{2dr}{(d+2)r-2d}}(\mathbb{R}^d) = [L^2(\mathbb{R}^d), L^d(\mathbb{R}^d)]_{\frac{2(d-r)}{(d-2)r}}$ and the induction hypothesis to obtain

$$\begin{aligned}&\|P_N\zeta_k\|_{L_T^qL_x^r} \\ &\lesssim \|z_1\|_{L_T^{kq}L_x^2}^{1-\frac{2(d-r)}{(d-2)r}+} \|z_1\|_{L_T^{kq}L_x^d}^{\frac{2(d-r)}{(d-2)r}-} \cdot \|z_1\|_{L_T^{(k-1)q}L_x^2}^{(k-1)(1-\delta_{k-1}(2q,\frac{2d}{d-2}-))} \|z_1\|_{L_T^{(k-1)q}L_x^d}^{(k-1)\delta_{k-1}(2q,\frac{2d}{d-2}-)} \\ &\lesssim \|z_1\|_{L_T^{kq}L_x^2}^{k(1-\delta_k(q,r))} \|z_1\|_{L_T^{kq}L_x^d}^{k\delta_k(q,r)},\end{aligned}$$

where $\delta_k(q,r) := \frac{2(d-r)}{(d-2)rk} + \left(1-\frac{1}{k}\right)\delta_{k-1}(2q, \frac{2d}{d-2}-)-$.

When $r \ge \frac{2d}{d-2}$, (4.1) follows from Sobolev's embedding $W^{\frac{d}{2}-\frac{d}{r}-1+,\frac{2d}{d-2}-}(\mathbb{R}^d) \hookrightarrow L^r(\mathbb{R}^d)$ and a similar argument as above (see also the proof of Lemma 3.1). □

Proof of Theorem 1.3. By Theorem 1.2, we only consider the case $k \ge 3$. As in the proof of Theorem 1.2, it suffices to estimate the norm

$$\left\| \left| v + \sum_{l=1}^{k} \zeta_l \right|^2 - |\zeta_1|^2 - 2\Re \sum_{l=2}^{k-1} \zeta_1 \overline{\zeta_l} \right\|_{X^{s_c+,-\frac{1}{2}+}}.$$

More precisely, we need to consider the following cases:

$$\text{(A) } \Re(\zeta_1\overline{\zeta_k}) \quad \text{(B) } \Re(\zeta_1\overline{v}) \quad \text{(C) } \Re(\zeta_{j_1}\overline{\zeta_{j_2}}) \quad \text{(D) } \Re(v\zeta_{j_2}) \quad \text{(E) } v\overline{v}$$

where j_1, j_2 can take any value in $\{2, 3, \dots, k\}$.

From Lemma 2.5 with $F = \zeta_k$, $\sigma = s_c+$, and $\rho = \alpha_k s-$, we can treat Case (A) provided that $s > \frac{s_c}{\alpha_{k+1}}$. We note that the worst interaction appears in this case.

We can use the same calculation as in the proof of Theorem 1.1 for the cases (B) and (E).

Lemmas 2.5 and 4.1 show that ζ_j for $k = 3, \dots, k$ enjoys (at least) the same regularity property as ζ_2 both in terms of differentiability and space-time integrability. Therefore, we can simply apply Lemmas 3.3 and 3.2 for cases (C) and (D), respectively. $\square$

§ 4.2. five dimensional case

Because $\frac{5}{2} < \frac{2dr}{(d+2)r-2d} < \frac{10}{3}$ for $d = 5$ and $r \in (\frac{5}{2}, \frac{10}{3})$, the same argument as in the proof of Lemma 4.1 yields the following.

Lemma 4.2. *For any $k \geq 2$ and any $r \in (\frac{5}{2}, \infty)$, $q \in (2, \infty)$, there exists $\widetilde{\delta}_k = \widetilde{\delta}_k(q, r) > 0$ such that*

$$\|P_N \eta_k\|_{L^q_T L^r_x} \lesssim \max(N^{5(\frac{1}{2}-\frac{1}{r})-1+}, 1)\|z_1\|^{k(1-\widetilde{\delta}_k)}_{L^{kq}_T L^2_x} \|z_1\|^{k\widetilde{\delta}_k}_{L^{kq}_T L^d_x}$$

for any $T \in (0, 1)$ and $N \in 2^{\mathbb{N}_0}$.

Proof of Corollary 1.4. Let $k \geq 4$. As in the proof of Theorem 1.3, it suffices to estimate the norm

$$\left\| \left| v + \sum_{l=1}^{k} \eta_l \right|^2 - |\eta_1|^2 - |\eta_2|^2 - 2\Re \sum_{l=2}^{k-1} \eta_1 \overline{\eta_l} \right\|_{X^{s_c+, -\frac{1}{2}+}}.$$

More precisely, we need to consider the following cases:

(A) $\Re(\eta_1 \overline{\eta_k})$ (B) $\Re(\eta_1 \overline{v})$ (C) $\Re(\eta_{j_1} \overline{\eta_{j_2'}})$ (D) $\Re(v \eta_{j_2})$ (E) $v\overline{v}$

where j_1, j_2 can take any value in $\{2, 3, \dots, k\}$ and j_2' can take any value in $\{3, \dots, k\}$.

Lemma 2.5 and Remark 3 say that $\theta_{\leq T}(t)\eta_4 \in X^{\alpha_4 s-, \frac{1}{2}+} + X^{\frac{3s+7}{4}s-, \frac{1}{2}+} = X^{\alpha_4 s-, \frac{1}{2}+}$ for $s > \frac{1}{6}$. Accordingly, $\theta_{\leq T}(t)\eta_k \in X^{\alpha_k s-, \frac{1}{2}+}$ for $\frac{s_c}{2} < s < s_c$. Then, as in the proof of Theorem 1.3, we can treat Case (A) provided that $s > \frac{s_c}{\alpha_{k+1}}$. We note that the worst interaction appears in this case.

We can use the same calculation as in the proof of Theorem 1.1 for the cases (B) and (E).

Lemma 4.2 and the fact that $\theta_{\leq T}(t)\eta_k \in X^{\alpha_k s-, \frac{1}{2}+}$ show that η_{j+l} for $j = 2, 3$ and $l = 1, \dots, k-j$ enjoys (at least) the same regularity property as η_j both in terms of differentiability and space-time integrability. Hence, Cases (C) and (D) are reduced to consider $\Re(\eta_2 \overline{\eta_3})$ and $\Re(v \overline{\eta_2})$, respectively. From $\theta_{\leq T}(t)\eta_2 \in X^{\frac{3}{2}s-, \frac{1}{2}+}$, we can simply apply Lemmas 3.4 for Case (D). Since $\theta_{\leq T}(t)\eta_2 \in X^{\frac{3}{2}s-, \frac{1}{2}+}$ and $\theta_{\leq T}(t)\eta_3 \in X^{\frac{7}{4}s-, \frac{1}{2}+}$, Lemma 3.4 with $a = \frac{3}{2}s-$ and $\rho = \frac{7}{4}s-$ shows that $\eta_2 \overline{\eta_3} \in X^{\frac{7s+15}{8}s-, -\frac{1}{2}+}$ for $0 < s < \frac{1}{2}$. We thus have $\eta_2 \overline{\eta_3} \in X^{\frac{1}{2}+, -\frac{1}{2}+}$ provided that $s > \frac{-15+\sqrt{337}}{14} \approx 0.2398$. Hence, Case (C) can be treated for $s > \frac{1}{4}$. $\square$

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