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Characterization of generalized Besov–Morrey spaces and Triebel–Lizorkin–Morrey spaces by differences

By

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Abstract

In this paper, we consider the characterization of generalized Besov–Morrey spaces and generalized Triebel–Lizorkin–Morrey spaces via ball means by differences and via differences. Since Besov spaces were originally defined by using differences. In this point, it is natural to consider the characterization of Besov and Triebel–Lizorkin type spaces via differences. To obtain the characterization results, we apply the boundedness of Peetre maximal function and the characterization of these spaces by local means.

Contents

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§ 1. Introduction

It is well known that there are the two approaches to define the Besov spaces $B^s_{q,r}(\mathbb{R}^n)$. One uses differences (original approach) and the other uses the Fourier transform. The two definitions are equivalent under the appropriate conditions on q, r and s. The (generalized) Besov–Morrey spaces and the (generalized) Triebel–Lizorkin–Morrey spaces are generalizations of Besov spaces and Triebel–Lizorkin spaces respectively, and these spaces are studied many researchers ([5, 7, 8, 9, 11, 12, 14, 15]). These spaces have been initially defined by the Fourier transform approach. In this paper we consider the characterization of these spaces via ball means by differences and via differences. Characterizations of Besov type spaces and Triebel–Lizorkin type spaces via ball means by differences and via differences are studied by many researchers. Dispa [2] studied the characterization by differences for Besov spaces on Lipschitz domains. Drihem [3] studied the characterization by differences for Besov-type spaces and Triebel–Lizorkin type spaces. Kempka and Vybíral [6] studied the characterization via ball means by differences for two-miocrolocal Besov spaces with variable exponents. Liang, Yang, Yuan, Sawano and Ullrich [8] studied the characterization via differences for generalized Besov type and Triebel–Lizorkin type spaces.

§ 2. Function spaces

To define the Morrey spaces, we use cubes. By a "cube" we mean a compact cube whose edges are parallel to the coordinate axes. If a cube has the center x and the radius r, then we denote it by $Q(x,r)$, that is, $Q(x,r) \equiv \left\{ y \in \mathbb{R}^n : \max_{1 \le i \le n} |x_i - y_i| \le r \right\}$. Below we denote by $|E|$ the Lebesgue measure of a measurable set E. From the definition of $Q(x,r)$ we see that $|Q(x,r)| = (2r)^n$. We additionally write $Q(r) \equiv Q(\mathrm{o}, r)$, where o denotes the origin. Conversely, given a cube Q, we denote by $c(Q)$ *the center of* Q and by $\ell(Q)$ the *side-length of* Q. Then we obtain $\ell(Q) = |Q|^{1/n}$. Let $\mathcal{Q}$ denote the set of all cubes.

Definition 2.1 (Generalized Morrey spaces [10]). Let $0 < q < \infty$. Denote by $\mathcal{G}_q$ the set of all nondecreasing functions $\varphi : (0, \infty) \to (0, \infty)$ such that

$$\varphi(t_1){t_1}^{-n/q} \geq \varphi(t_2){t_2}^{-n/q} \quad (0 < t_1 \leq t_2 < \infty). \tag{2.1}$$

Let $\varphi \in \mathcal{G}_q$. Then define $\|f\|_{\mathcal{M}^\varphi_q} \equiv \sup_{Q \in \mathcal{Q}} \varphi(\ell(Q)) \left(\frac{1}{|Q|} \int_Q |f(y)|^q \, dy \right)^{\frac{1}{q}}$ for a measurable function f. The space $\mathcal{M}^\varphi_q(\mathbb{R}^n)$ is the set of all measurable functions f satisfying that the quasi-norm $\|f\|_{\mathcal{M}^\varphi_q}$ is finite.

Remark 1. Let $q \in (0, \infty)$ and $\varphi \in \mathcal{G}_q$. Then it is easy to see that

$$\|f\|_{\mathcal{M}^\varphi_q} \sim \sup_{x \in \mathbb{R}^n, r>0} \varphi(r) \left(\frac{1}{B(x,r)} \int_{B(x,r)} |f(y)|^q dy \right)^{\frac{1}{q}},$$

where $B(x, r) = \{y \in \mathbb{R}^n : |x - y| \leq r\}$.

To deal with generalized Besov–Morrey spaces and generalized Triebel–Lizorkin–Morrey spaces, we use the following notation in the present paper:

1. Let $A, B \geq 0$. Then $A \lesssim B$ means that there exists a constant $C > 0$ such that $A \leq CB$, where C is independent of the parameters of importance. We write $A \sim B$ whenever $A \lesssim B$ and $B \lesssim A$ hold. If we want to stress that the implicit constants in these symbols depend on important parameters, then we add them as subscripts. For example, $A \lesssim_p B$ means that there exists a constant $C > 0$ depending only on p such that $A \leq CB$.

2. Let $a \in \mathbb{R}^n$ and $r, t > 0$. Then we define $B(a, r) \equiv \{x \in \mathbb{R}^n : |x - a| \leq r\}$, $\mathbb{B} \equiv B(\mathrm{o}, 1) = \{x \in \mathbb{R}^n : |x| \leq 1\}$ and $t\mathbb{B} \equiv \{tx : x \in \mathbb{B}\}$, where $tx \equiv (tx_1, tx_2, \ldots, tx_n)$ for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

3. Given an integrable function f we define the Fourier transform and its inverse by

$$\begin{cases} \mathcal{F}f(\xi) \equiv \dfrac{1}{\sqrt{(2\pi)^n}} \displaystyle\int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} \, dx & (\xi \in \mathbb{R}^n) \\ \mathcal{F}^{-1}f(x) \equiv \dfrac{1}{\sqrt{(2\pi)^n}} \displaystyle\int_{\mathbb{R}^n} f(\xi) e^{ix\cdot\xi} \, d\xi & (x \in \mathbb{R}^n) \end{cases}.$$

   In a standard way, we extend the definition of $\mathcal{F}$ and $\mathcal{F}^{-1}$ to the space of all tempered distributions $\mathcal{S}'(\mathbb{R}^n)$.

4. For $\theta \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$ we write $\theta(D)f \equiv \mathcal{F}^{-1}[\theta \mathcal{F} f]$, or equivalently we define $\theta(D)f(x) \equiv \dfrac{1}{\sqrt{(2\pi)^n}} \langle f, \mathcal{F}^{-1}\theta(x - \cdot)\rangle$.

5. Denote by $\mathrm{BUC}(\mathbb{R}^n)$ the Banach space consisting of all bounded uniformly continuous functions. We additionally define $||f||_{\mathrm{BUC}} \equiv ||f||_\infty$ for $f \in \mathrm{BUC}(\mathbb{R}^n)$.

6. For any $\{f_j\}_{j=0}^\infty \subset \mathcal{M}_q^\varphi(\mathbb{R}^n)$ and $r \in (0,\infty)$, we often use the following notation:

$$\|\{f_j\}_{j=0}^\infty\|_{\ell^r(\mathcal{M}_q^\varphi)} \equiv \left(\sum_{j=0}^\infty \|f_j\|_{\mathcal{M}_q^\varphi}^r\right)^{\frac{1}{r}} \text{ and } \|\{f_j\}_{j=0}^\infty\|_{\mathcal{M}_q^\varphi(\ell^r)} \equiv \left\|\left(\sum_{j=0}^\infty |f_j|^r\right)^{\frac{1}{r}}\right\|_{\mathcal{M}_q^\varphi}.$$

We also use $\|\cdot\|_{\ell^\infty(\mathcal{M}_q^\varphi)}$ and $\|\cdot\|_{\mathcal{M}_q^\varphi(\ell^\infty)}$ with an easy modification.

7. For $q, r \in (0,\infty)$, we define $\sigma_q \equiv n\left(\dfrac{1}{\min(1,q)} - 1\right)$ and $\sigma_{q,r} \equiv n\left(\dfrac{1}{\min(1,q,r)} - 1\right)$.

Now let us introduce generalized Besov-Morrey spaces and generalized Triebel-Lizorkin-Morrey spaces defined by [11, Definition 1.3].

Definition 2.2. Let $0 < q < \infty$, $0 < r \le \infty$, $s \in \mathbb{R}$ and $\varphi \in \mathcal{G}_q$. Let τ_0 and τ be compactly supported functions satisfying

$$0 \notin \mathrm{supp}(\tau), \quad \tau_0(\xi) > 0 \text{ if } \xi \in Q(2), \quad \tau(\xi) > 0 \text{ if } \xi \in Q(2) \setminus Q(1).$$

Define $\tau_k(\xi) \equiv \tau(2^{-k}\xi)$ for $\xi \in \mathbb{R}^n$ and $k \in \mathbb{N}$.

1. *The (nonhomogeneous) generalized Besov–Morrey space* $\mathcal{N}^s_{\mathcal{M}_q^\varphi,r}(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which the quasi-norm

$$\|f\|_{\mathcal{N}^s_{\mathcal{M}_q^\varphi,r}} \equiv \left\|\{2^{js}|\tau_j(D)f|\}_{j=0}^\infty\right\|_{\ell^r(\mathcal{M}_q^\varphi)}.$$

is finite.

2. Assume that there exist constants ε, $C > 0$ such that

$$\frac{t^\varepsilon}{\varphi(t)} \le \frac{Cu^\varepsilon}{\varphi(u)} \quad (t \ge u) \tag{2.2}$$

when $r < \infty$. *The (nonhomogeneous) generalized Triebel–Lizorkin–Morrey space* $\mathcal{E}^s_{\mathcal{M}_q^\varphi,r}(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which the quasi-norm

$$\|f\|_{\mathcal{E}^s_{\mathcal{M}_q^\varphi,r}} \equiv \left\|\{2^{js}|\tau_j(D)f|\}_{j=0}^\infty\right\|_{\mathcal{M}_q^\varphi(\ell^r)}$$

is finite.

3. The space $\mathcal{A}^s_{\mathcal{M}_q^\varphi,r}(\mathbb{R}^n)$ denotes either $\mathcal{N}^s_{\mathcal{M}_q^\varphi,r}(\mathbb{R}^n)$ or $\mathcal{E}^s_{\mathcal{M}_q^\varphi,r}(\mathbb{R}^n)$.

Remark 2. There exist some examples of functions $\varphi \in \mathcal{G}_q$ satisfying (2.2):

1. Let $0 < q \le p < \infty$. Then $\varphi(t) = t^{\frac{n}{p}} \in \mathcal{G}_q$ satisfies (2.2) when $0 < \epsilon < \frac{n}{p}$.

2. Let $0 < \beta \le \alpha < \dfrac{n}{q}$. Then $\varphi(t) = \begin{cases} t^{\alpha} & (0 < t \le 1) \\ t^{\beta} & (1 < t < \infty) \end{cases}$ belongs to $\mathcal{G}_q$ and satisfies (2.2) when $0 < \epsilon < \beta$.

The quasi-norm of $\mathcal{A}^s_{\mathcal{M}^{\varphi}_q,r}(\mathbb{R}^n)$ can be also defined by the following partition of unity $\{\theta_j\}_{j=0}^{\infty}$.

Definition 2.3. Let $\Theta(\mathbb{R}^n)$ be the collection of all systems $\{\theta_j\}_{j=0}^{\infty} \subset \mathcal{S}(\mathbb{R}^n)$ satisfying (i), (ii) and (iii):

(i) $\operatorname{supp}\theta_0 \subset \{x \in \mathbb{R}^n : |x| \le 2\}$ and $\operatorname{supp}\theta_j \subset \{x \in \mathbb{R}^n : 2^{j-1} \le |x| \le 2^{j+1}\}$ if $j \ge 1$.

(ii) For every multi-index α there exists a positive constant c_α such that $2^{j|\alpha|}\,|\partial^\alpha \theta_j(x)| \le c_\alpha$ holds for all $j = 0, 1, \ldots$ and all $x \in \mathbb{R}^n$

(iii) $\displaystyle\sum_{j=0}^{\infty} \theta_j(x) = 1$ holds for every $x \in \mathbb{R}^n$.

Remark 3. It is well known that $\Theta(\mathbb{R}^n)$ is not empty. In fact, there do exist $\theta_0, \theta \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\begin{aligned} |\theta_0(x)| > 0 \quad &\text{on} \quad \{x \in \mathbb{R}^n : |x| < 1\}, \\ |\theta(x)| > 0 \quad &\text{on} \quad \{x \in \mathbb{R}^n : 1/2 < |x| < 2\} \end{aligned}$$

and $\{\theta_j\}_{j=0}^{\infty} \in \Theta(\mathbb{R}^n)$ hold, where $\theta_j(x) \equiv \theta(2^{-j}x)$ for $j \in \mathbb{N}$. See [16, Remark 1, p45].

Nakamura, Noi and Sawano [11, Theorem 2.19] proved a boundedness of Fourier multiplier. Using the result, we can replace compactly supported functions $\{\tau_j\}_{j=0}^{\infty}$ in the definition of the quasi-norm of $\mathcal{A}^s_{\mathcal{M}^{\varphi}_q,r}(\mathbb{R}^n)$ by $\{\theta_j\}_{j=0}^{\infty} \in \Theta(\mathbb{R}^n)$. We omit the proof because it is similar to the one of [16, Proposition 1, p46].

Theorem 2.4. *Let $0 < q < \infty$, $0 < r \le \infty$, $s \in \mathbb{R}$, $\varphi \in \mathcal{G}_q$ and $\{\theta_j\}_{j=0}^{\infty} \in \Theta(\mathbb{R}^n)$.*

(i) *For $f \in \mathcal{N}^s_{\mathcal{M}^{\varphi}_q,r}(\mathbb{R}^n)$, we have*

$$\|f\|_{\mathcal{N}^s_{\mathcal{M}^{\varphi}_q,r}} \sim \left\|\{2^{js}|\theta_j(D)f|\}_{j=0}^{\infty}\right\|_{\ell^r(\mathcal{M}^{\varphi}_q)}. \tag{2.3}$$

(ii) *Assume that φ satisfies* (2.2) *when $r < \infty$. For $f \in \mathcal{E}^s_{\mathcal{M}^{\varphi}_q,r}(\mathbb{R}^n)$, we have*

$$\|f\|_{\mathcal{E}^s_{\mathcal{M}^{\varphi}_q,r}} \sim \left\|\{2^{js}|\theta_j(D)f|\}_{j=0}^{\infty}\right\|_{\mathcal{M}^{\varphi}_q(\ell^r)}. \tag{2.4}$$

In this paper, we use (2.3) and (2.4) when we consider the quasi norm of $\mathcal{A}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)$.

At the rest of this section, we mention the two reason why the spaces $\mathcal{A}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)$ are important.

The first reason is that the spaces $\mathcal{A}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)$ recover many spaces. By letting $\varphi(t) = t^{n/q}$ for $t > 0$, we obtain the well known fact $\mathcal{M}^\varphi_q(\mathbb{R}^n) = L^q(\mathbb{R}^n)$. On the other hand, letting $0 < q \le p < \infty$ and $\varphi(t) = t^{n/p}$ for $t > 0$, we additionally obtain the fact $\mathcal{M}^\varphi_q(\mathbb{R}^n) = \mathcal{M}^p_q(\mathbb{R}^n)$. Therefore, the spaces $\mathcal{A}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)$ can recover very well known classical function spaces (for example, Besov–Morrey spaces and Triebel–Lizorkin–Morrey spaces). Furthermore, we introduce the result obtained by Nakamura, Noi and Sawano [11]. We refer [11, Definition 6.1, Section 6.6] for the precise definitions of homogeneous generalized Triebel–Lizorkin–Morrey spaces $\dot{\mathcal{E}}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)$ and generalized Hardy–Morrey spaces $H\mathcal{M}^\varphi_q(\mathbb{R}^n)$.

Theorem 2.5 ([11, Corollary 6.17]). *Let $0 < q < \infty$ and $\varphi \in \mathcal{G}_q$ satisfy (2.2).*

1. *If $0 < q \le 1$, then $\dot{\mathcal{E}}^0_{\mathcal{M}^\varphi_q,2}(\mathbb{R}^n) = H\mathcal{M}^\varphi_q(\mathbb{R}^n)$ holds.*
2. *If $1 < q < \infty$, then $\dot{\mathcal{E}}^0_{\mathcal{M}^\varphi_q,2}(\mathbb{R}^n) = \mathcal{M}^\varphi_q(\mathbb{R}^n)$ holds.*
3. *If $q > 1$, then $\mathcal{E}^0_{\mathcal{M}^\varphi_q,2}(\mathbb{R}^n) = \mathcal{M}^\varphi_q(\mathbb{R}^n)$ holds.*

The second reason is that we may improve known results. For example, Nakamura, Noi and Sawano investigated the limiting case of the Sobolev embedding by taking advantage of the function φ [11, Proposition A.3 and Remark A.4].

§ 3. Preliminaries

In this section, we summarize some preliminary facts from [11]. We first note that the following $\min(1, q, r)$-triangle inequality holds. We omit the proof because it follows by a standard argument.

Lemma 3.1. *Let $0 < q < \infty$, $0 < r \le \infty$, $s \in \mathbb{R}$ and $\varphi : (0,\infty) \to (0,\infty)$ be a function belonging to $\mathcal{G}_q$. Assume (2.2) in the case when $\mathcal{A} = \mathcal{E}$ with $r < \infty$. Then we have that for all functions $f_1, f_2 \in \mathcal{A}^s_{\mathcal{M}^\varphi_q,r}$,*

$$(\|f_1 + f_2\|_{\mathcal{A}^s_{\mathcal{M}^\varphi_q,r}})^{\min(1,q,r)} \le (\|f_1\|_{\mathcal{A}^s_{\mathcal{M}^\varphi_q,r}})^{\min(1,q,r)} + (\|f_2\|_{\mathcal{A}^s_{\mathcal{M}^\varphi_q,r}})^{\min(1,q,r)}.$$

Lemma 3.2 ([1, Proposition 2.6], [4, Theorem 3.3], [11, Lemma 2.5]). *Let $0 < q < \infty$.*

(i) If $\varphi \in \mathcal{G}_q$ and $r \in (0, q)$, then $\|f\|_{\mathcal{M}^\varphi_r} \le \|f\|_{\mathcal{M}^\varphi_q}$ holds for all $f \in \mathcal{M}^\varphi_q(\mathbb{R}^n)$.

(ii) *If* $0 < u < \infty$ *and* $\varphi : (0,\infty) \to (0,\infty)$*, then* $\| \, |f|^u \, \|_{\mathcal{M}_q^\varphi} = \left(\|f\|_{\mathcal{M}_{uq}^{\varphi^{1/u}}} \right)^u$ *holds for all* $f \in \mathcal{M}_{uq}^{\varphi^{1/u}}(\mathbb{R}^n)$.

§ 3.1. Boundedness of the Hardy-Littlewood maximal operator

In the sequel $\mathcal{M}$ denotes the *Hardy–Littlewood maximal operator* defined by:

$$\mathcal{M}f(x) \equiv \sup_{Q \in \mathcal{Q}} \frac{\chi_Q(x)}{|Q|} \int_Q |f(y)| \, dy \tag{3.1}$$

for $f \in L^1_{loc}(\mathbb{R}^n)$. Here we recall the following vector-valued inequality:

Theorem 3.3 ([11, Theorem 2.9])**.** *Let* $1 < q < \infty$*,* $1 < r < \infty$ *and* $\varphi \in \mathcal{G}_q$.

1. *For all measurable functions* $f : \mathbb{R}^n \to \mathbb{C}$ *we have*

$$\|\mathcal{M}f\|_{\mathcal{M}_q^\varphi} \lesssim \|f\|_{\mathcal{M}_q^\varphi}. \tag{3.2}$$

In addition, for any sequence $\{f_j\}_{j=1}^\infty$ *of* $\mathcal{M}_q^\varphi(\mathbb{R}^n)$*-functions we have*

$$\left\| \sup_{j \in \mathbb{N}} \mathcal{M}f_j \right\|_{\mathcal{M}_q^\varphi} \lesssim \left\| \sup_{j \in \mathbb{N}} |f_j| \right\|_{\mathcal{M}_q^\varphi}. \tag{3.3}$$

2. *If we assume* (2.2)*, then we have that for any sequence* $\{f_j\}_{j=1}^\infty$ *of* $\mathcal{M}_q^\varphi(\mathbb{R}^n)$*-functions,*

$$\left\| \left(\sum_{j=1}^\infty (\mathcal{M}f_j)^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}_q^\varphi} \lesssim \left\| \left(\sum_{j=1}^\infty |f_j|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}_q^\varphi}. \tag{3.4}$$

§ 3.2. Boundedness of the Peetre maximal function

Given $\{\theta_k\}_{k=0}^\infty \subset \mathcal{S}(\mathbb{R}^n)$, $f \in \mathcal{S}'(\mathbb{R}^n)$, $a > 0$ and $k \in \mathbb{N}_0$, we define the Peetre maximal function by

$$(\theta_k^* f)_a(x) \equiv \sup_{y \in \mathbb{R}^n} \frac{|(\mathcal{F}^{-1}[\theta_k \mathcal{F} f])(x-y)|}{1 + |2^k y|^a} = \sup_{y \in \mathbb{R}^n} \frac{|\theta_k(D) f(x-y)|}{1 + |2^k y|^a} \quad (x \in \mathbb{R}^n).$$

The following theorem is a direct corollary of [1, Theorem 5.4 and 5.5]. So we omit the proof

Theorem 3.4 (Boundedness of Peetre maximal function)**.** *Let* $0 < q < \infty$*,* $0 < r \le \infty$*,* $s \in \mathbb{R}^n$*,* $\varphi \in \mathcal{G}_q$*,* $\{\theta_k\}_{k=0}^\infty \subset \mathcal{S}(\mathbb{R}^n)$ *and* $f \in \mathcal{S}'(\mathbb{R}^n)$*. Also let* $a > 0$ *and* $(\theta_k^* f)_a$ *(*$k = 0, 1, \dots$*) be the Peetre maximal functions.*

(*i*) *If* $a > \dfrac{n}{q}$, *then we have*

$$\left\| \{2^{ks}(\theta_k^* f)_a\}_{k=0}^{\infty} \right\|_{\ell^r(\mathcal{M}_q^\varphi)} \lesssim \| \{2^{ks}\theta_k(D)f\}_{k=0}^{\infty} \|_{\ell^r(\mathcal{M}_q^\varphi)}.$$

(*ii*) *If we assume* $a > \dfrac{n}{\min(q,r)}$ *and* (2.2), *then we have*

$$\left\| \{2^{ks}(\theta_k^* f)_a\}_{k=0}^{\infty} \right\|_{\mathcal{M}_q^\varphi(\ell^r)} \lesssim \| \{2^{ks}\theta_k(D)f\}_{k=0}^{\infty} \|_{\mathcal{M}_q^\varphi(\ell^r)}.$$

§ 3.3. Embeddings

Next we verify the embedding properties.

Proposition 3.5. *Let* $0 < q < \infty$, $0 < r_1, r_2 \le \infty$, $s \in \mathbb{R}$ *and* $\varphi \in \mathcal{G}_q$. *Assume* (2.2) *in the case when* $\mathcal{A} = \mathcal{E}$ *with* $r < \infty$.

(i) *If* $r_1 \le r_2$, *then we have* $\mathcal{A}^s_{\mathcal{M}_q^\varphi, r_1}(\mathbb{R}^n) \hookrightarrow \mathcal{A}^s_{\mathcal{M}_q^\varphi, r_2}(\mathbb{R}^n)$.

(ii) *If* $\epsilon > 0$, *then we have* $\mathcal{A}^s_{\mathcal{M}_q^\varphi, r_1}(\mathbb{R}^n) \hookrightarrow \mathcal{A}^{s-\varepsilon}_{\mathcal{M}_q^\varphi, r_2}(\mathbb{R}^n)$.

(iii) *Let* $0 < r < \infty$. *If* φ *satisfies* (2.2), *then we have*

$$\mathcal{N}^s_{\mathcal{M}_q^\varphi, \min(q,r)}(\mathbb{R}^n) \hookrightarrow \mathcal{E}^s_{\mathcal{M}_q^\varphi, r}(\mathbb{R}^n) \hookrightarrow \mathcal{N}^s_{\mathcal{M}_q^\varphi, \infty}(\mathbb{R}^n).$$

Proof. (i) and (ii) have been proved by [11, Proposition 3.3]. We omit the proof of (iii) because it is obtained by an argument similar to [16, Section 2.3.2, Proposition 2]. □

Lemma 3.6 ([11, Lemma 3.4]). *Let* $0 < q < \infty, 0 < r \le \infty$ *and* $\varphi \in \mathcal{G}_q$. *Assume that* $s > 0$ *satisfies*

$$\sum_{j=1}^{\infty} \frac{1}{2^{js}\varphi(2^{-j})} < \infty. \tag{3.5}$$

Then we have $\mathcal{A}^s_{\mathcal{M}_q^\varphi, r}(\mathbb{R}^n) \hookrightarrow B^0_{\infty 1}(\mathbb{R}^n)$. *In particular, we obtain*

$$\mathcal{A}^s_{\mathcal{M}_q^\varphi, r}(\mathbb{R}^n) \hookrightarrow B^0_{\infty 1}(\mathbb{R}^n) \hookrightarrow \mathrm{BUC}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

Proposition 3.7 ([11, Proposition 3.6]). *Let* $0 < q < \infty$, $0 < r \le \infty$, $s \in \mathbb{R}$ *and* $\varphi : (0,\infty) \to (0,\infty)$ *be a function belonging to* $\mathcal{G}_q$. *Assume in addition that* φ *satisfies* (2.2) *when* $\mathcal{A} = \mathcal{E}$ *with* $r < \infty$. *Then we have* $\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{A}^s_{\mathcal{M}_q^\varphi, r}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ *in the sense of continuous embeddings.*

§ 3.4. Characterization by local means

In this section, we recall the characterization of $\mathcal{A}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)$ by local means. We start with two given functions $\psi_0, \psi \in \mathcal{S}(\mathbb{R}^n)$. We define $\psi_j(x) \equiv \psi(2^{-j+1}x)$ for $x \in \mathbb{R}^n$ and $j \in \mathbb{N}$.

Theorem 3.8. *Let $\psi_0, \psi \in \mathcal{S}(\mathbb{R}^n)$, $\phi_0, \phi \in \mathcal{S}(\mathbb{R}^n)$, $R \in \mathbb{N}_0$ and $s \in \mathbb{R}$ with $R > s$. For $x \in \mathbb{R}^n$ and $j \in \mathbb{N}$ we define $\phi_j(x) \equiv \phi(2^{-j+1}x)$. Suppose that*

$$\partial^\beta \psi(0) = 0 \tag{3.6}$$

holds for all $0 \le |\beta| < R$ and that

$$|\phi_0(x)| > 0 \quad \text{on} \quad \{x \in \mathbb{R}^n : |x| < \epsilon'\},$$
$$|\phi(x)| > 0 \quad \text{on} \quad \{x \in \mathbb{R}^n : \epsilon'/2 < |x| < 2\epsilon'\}$$

hold for some $\epsilon' > 0$. Let $(\psi_j^* f)_a$ and $(\phi_j^* f)_a$ be Peetre maximal functions. Then we have

$$\|\{2^{js}(\psi_j^* f)_a\}_{j=0}^\infty\|_{\ell^r(\mathcal{M}^\varphi_q)} \lesssim \|\{2^{js}(\phi_j^* f)_a\}_{j=0}^\infty\|_{\ell^r(\mathcal{M}^\varphi_q)}, \tag{3.7}$$

$$\|\{2^{js}(\psi_j^* f)_a\}_{j=0}^\infty\|_{\mathcal{M}^\varphi_q(\ell^r)} \lesssim \|\{2^{js}(\phi_j^* f)_a\}_{j=0}^\infty\|_{\mathcal{M}^\varphi_q(\ell^r)}. \tag{3.8}$$

Before proving Theorem 3.8, we need the following lemma which is a corollary of [6, Lemma 8].

Lemma 3.9 (Hardy type inequality). *Let $0 < q < \infty$, $0 < r \le \infty$ and $\delta > 0$. Let $\{g_\nu\}_{\nu=-\infty}^\infty$ be a sequence of non-negative measurable functions on $\mathbb{R}^n$ and define*

$$G_j(x) \equiv \sum_{\nu \in \mathbb{Z}} 2^{-|j-\nu|\delta} g_\nu(x)$$

for $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$. Then we have

$$\|\{G_j\}_{j=-\infty}^\infty\|_{\ell^r(\mathcal{M}^\varphi_q)} \lesssim_{q,r,\delta} \|\{g_\nu\}_{\nu=-\infty}^\infty\|_{\ell^r(\mathcal{M}^\varphi_q)},$$
$$\|\{G_j\}_{j=-\infty}^\infty\|_{\mathcal{M}^\varphi_q(\ell^r)} \lesssim_{q,r,\delta} \|\{g_\nu\}_{\nu=-\infty}^\infty\|_{\mathcal{M}^\varphi_q(\ell^r)}.$$

Proof of Theorem 3.8 . By the argument in [6, Proof of Theorem 12] with $w_k = 2^{ks}$, we have

$$2^{vs}(\psi_v^* f)_a(x) \lesssim \sum_{k=0}^\infty 2^{-|k-v|\delta} 2^{ks}(\phi_k^* f)_a(x)$$

for $x \in \mathbb{R}^n$, where $\delta \equiv \min(1, R-s) > 0$. Therefore, we see that (3.7) and (3.8) hold by Lemma 3.9. □

By using a similar argument in [6, 3.1 Proof of Local Means] with Theorem 3.8, we have the following theorem.

Theorem 3.10. *Let* $0 < q < \infty$, $0 < r \le \infty$, $R \in \mathbb{N}_0$ *and* $s \in \mathbb{R}$ *with* $R > s$. *Furthermore suppose that* $\psi_0, \psi \in \mathcal{S}(\mathbb{R}^n)$ *satisfy*

$$\partial^\beta \psi(0) = 0, \tag{3.9}$$

for all $\beta \in \mathbb{N}_0^n$ *with* $0 \le |\beta| < R$, *and that*

$$|\psi_0(x)| > 0 \quad on\ \{x \in \mathbb{R}^n : |x| < \epsilon\}, \tag{3.10}$$

$$|\psi(x)| > 0 \quad on\ \{x \in \mathbb{R}^n : \epsilon/2 < |x| < 2\epsilon\} \tag{3.11}$$

for some $\epsilon > 0$. *Let* $(\psi_j^* f)_a$ *be Peetre maximal functions.*

(i) For $a > \frac{n}{q}$ *and for all* $f \in \mathcal{S}'(\mathbb{R}^n)$, *we have*

$$\|f\|_{\mathcal{N}^s_{\mathcal{M}^\varphi_q, r}} \sim \|\{2^{js}\mathcal{F}^{-1}[\psi_j] * f\}_{j=0}^\infty\|_{\ell^r(\mathcal{M}^\varphi_q)} \sim \|\{2^{js}(\psi_j^* f)_a\}_{j=0}^\infty\|_{\ell^r(\mathcal{M}^\varphi_q)}.$$

(ii) Assume in addition (2.2) *when* $r < \infty$. *For* $a > \frac{n}{\min(q,r)}$ *and for all* $f \in \mathcal{S}'(\mathbb{R}^n)$, *we have*

$$\|f\|_{\mathcal{E}^s_{\mathcal{M}^\varphi_q, r}} \sim \|\{2^{js}\mathcal{F}^{-1}[\psi_j] * f\}_{j=0}^\infty\|_{\mathcal{M}^\varphi_q(\ell^r)} \sim \|\{2^{js}(\psi_j^* f)_a\}_{j=0}^\infty\|_{\mathcal{M}^\varphi_q(\ell^r)}.$$

Remark 4.

1. Conditions (3.10) and (3.11) are the so-called Tauberian conditions.
2. If $R = 0$, then no moment condition (3.9) on ψ is required.

§ 4. Ball means of differences

It is well known that there are several definitions of Besov spaces. Two of the most prominent approaches are the Fourier analytic approach using the Fourier transforms and the classical approach via higher order differences involving the modulus of smoothness order. These two definitions are equivalent under certain restriction on the parameters. In this section, we consider the characterization of $\mathcal{A}^s_{\mathcal{M}^\varphi_q, r}(\mathbb{R}^n)$ by ball means of differences [6, Section 4] due to Kempka and Vybíral.

Let f be a function on $\mathbb{R}^n$ and $h \in \mathbb{R}^n$. Then we define

$$\Delta_h^1 f(x) \equiv f(x+h) - f(x) \ (x \in \mathbb{R}^n).$$

The higher order differences are defined inductively by

$$\Delta_h^M f(x) = \Delta_h^1(\Delta_h^{M-1} f)(x), \quad M = 2, 3, \ldots.$$

This definition also allows a direct formula

$$\Delta_h^M f(x) \equiv \sum_{j=0}^{M} (-1)^j \binom{M}{j} f(x+(M-j)h) =: \sum_{j=0}^{M} c_{j,M} f(x+jh). \tag{4.1}$$

By ball mean of differences we mean the quantity

$$d_t^M f(x) \equiv t^{-n} \int_{|h|\le t} \left|\Delta_h^M f(x)\right| \,\mathrm{d}h = \int_{\mathbb{B}} \left|\Delta_{th}^M f(x)\right| \,\mathrm{d}h,$$

where $t>0$ is a real number and M is a natural number.

Now we define the quasi-norms corresponding to generalized Triebel–Lizorkin Morrey spaces. Let $f \in \mathcal{M}_q^\varphi(\mathbb{R}^n)$. Then we define

$$\|f\|^*_{\mathcal{E}^s_{\mathcal{M}_q^\varphi,r}(\mathbb{R}^n)} \equiv \|f\|_{\mathcal{M}_q^\varphi} + \left\| \left(\int_0^\infty t^{-sr} (d_t^M f)^r \frac{\mathrm{d}t}{t} \right)^{1/r} \right\|_{\mathcal{M}_q^\varphi} \tag{4.2}$$

(modification if $r=\infty$) and its partially discretized counterpart

$$\begin{aligned} \|f\|^{**}_{\mathcal{E}^s_{\mathcal{M}_q^\varphi,r}(\mathbb{R}^n)} &\equiv \|f\|_{\mathcal{M}_q^\varphi} + \left\| \left(\sum_{k=-\infty}^{\infty} 2^{ksr} (d_{2^{-k}}^M f)^r \right)^{1/r} \right\|_{\mathcal{M}_q^\varphi} \\ &= \|f\|_{\mathcal{M}_q^\varphi} + \left\| \left\{ 2^{ks} d_{2^{-k}}^M f \right\}_{k=-\infty}^{\infty} \right\|_{\mathcal{M}_q^\varphi(\ell^r)} \end{aligned} \tag{4.3}$$

(modification if $r=\infty$). For the generalized Besov–Morrey spaces, we define

$$\|f\|^*_{\mathcal{N}^s_{\mathcal{M}_q^\varphi,r}(\mathbb{R}^n)} \equiv \|f\|_{\mathcal{M}_q^\varphi} + \left(\int_0^\infty t^{-sr} \|d_t^M f\|^r_{\mathcal{M}_q^\varphi} \frac{\mathrm{d}t}{t} \right)^{1/r}, \tag{4.4}$$

$$\|f\|^{**}_{\mathcal{N}^s_{\mathcal{M}_q^\varphi,r}(\mathbb{R}^n)} \equiv \|f\|_{\mathcal{M}_q^\varphi} + \left\| \left\{ 2^{ks} d_{2^{-k}}^M f \right\}_{k=-\infty}^{\infty} \right\|_{\ell^r(\mathcal{M}_q^\varphi)} \tag{4.5}$$

(modification if $r=\infty$). Then the following theorem holds.

Theorem 4.1. *Let $0<q<\infty$, $0<r\le\infty$, $s\in\mathbb{R}$ and $\varphi\in\mathcal{G}_q$. Furthermore, let $M\in\mathbb{N}$ with $M>s$.*

(i) If $r<\infty$ and $s>\sigma_{q,r}$, then $f\in\mathcal{E}^s_{\mathcal{M}_q^\varphi,r}(\mathbb{R}^n)$ holds if and only if $f\in L^1_{loc}(\mathbb{R}^n)\cap\mathcal{S}'(\mathbb{R}^n)$ and $\|f\|^*_{\mathcal{E}^s_{\mathcal{M}_q^\varphi,r}(\mathbb{R}^n)}<\infty$. Furthermore, $\|f\|^*_{\mathcal{E}^s_{\mathcal{M}_q^\varphi,r}(\mathbb{R}^n)}$ and $\|f\|_{\mathcal{E}^s_{\mathcal{M}_q^\varphi,r}(\mathbb{R}^n)}$ are equivalent on $\mathcal{E}^s_{\mathcal{M}_q^\varphi,r}(\mathbb{R}^n)$. The same statement holds for $\|f\|^{**}_{\mathcal{E}^s_{\mathcal{M}_q^\varphi,r}(\mathbb{R}^n)}$.

(ii) If $s>\sigma_q$, then $f\in\mathcal{N}^s_{\mathcal{M}_q^\varphi,r}(\mathbb{R}^n)$ holds if and only if $f\in L^1_{loc}(\mathbb{R}^n)\cap\mathcal{S}'(\mathbb{R}^n)$ and $\|f\|^*_{\mathcal{N}^s_{\mathcal{M}_q^\varphi,r}(\mathbb{R}^n)}<\infty$. Furthermore, $\|f\|^*_{\mathcal{N}^s_{\mathcal{M}_q^\varphi,r}(\mathbb{R}^n)}$ and $\|f\|_{\mathcal{N}^s_{\mathcal{M}_q^\varphi,r}(\mathbb{R}^n)}$ are equivalent on $\mathcal{N}^s_{\mathcal{M}_q^\varphi,r}(\mathbb{R}^n)$. The same statement holds for $\|f\|^{**}_{\mathcal{N}^s_{\mathcal{M}_q^\varphi,r}(\mathbb{R}^n)}$.

Before proving Theorem 4.1, we need the following results in the next section.

§ 4.1. Key results

Lemma 4.2 ([6, Lemma 22]). *Let $a, b > 0$, $M \in \mathbb{N}$ and $h \in \mathbb{R}^n$. Define*

$$P_{b,a}f(x) \equiv \sup_{z \in \mathbb{R}^n} \frac{|f(x-z)|}{1+|bz|^a}$$

for $f \in \mathcal{S}'(\mathbb{R}^n)$ with $\operatorname{supp} \hat{f} \subset \{\xi \in \mathbb{R}^n : |\xi| < b\}$. Then there is a constant $C > 0$ independent of f, b and h such that

$$\left|\Delta_h^M f(x)\right| \le C \max(1, |bh|^a) \min(1, |bh|^M) P_{b,a}f(x)$$

holds for every $x \in \mathbb{R}^n$.

If $q \ge 1$, then all elements of $\mathcal{A}^s_{\mathcal{M}^\varphi_q, r}(\mathbb{R}^n)$ can be regard as $L^1_{loc}(\mathbb{R}^n)$-functions as well as Besov and Triebel–Lizorkin space cases. This assertion is also true when $0 < q < 1$ and $s > \sigma_q$.

Proposition 4.3. *If $0 < q < 1$, $0 < r \le \infty$, $\varphi \in \mathcal{G}_q$ and $s > \sigma_q$, then we have*

$$\sup_{y \in \mathbb{R}^n} \|f\|_{L^1(B(y,1))} \lesssim \|f\|_{\mathcal{A}^s_{\mathcal{M}^\varphi_q, r}}. \tag{4.6}$$

Proof. Note that $\mathcal{A}^s_{\mathcal{M}^\varphi_q, r}(\mathbb{R}^n) \hookrightarrow \mathcal{N}^s_{\mathcal{M}^\varphi_q, \infty}(\mathbb{R}^n)$ holds for all $0 < r \le \infty$ by Proposition 3.5. Thus it suffices to prove

$$\sup_{y \in \mathbb{R}^n} \|f\|_{L^1(B(y,1))} \lesssim \|f\|_{\mathcal{N}^s_{\mathcal{M}^\varphi_q, \infty}} \tag{4.7}$$

in order to get (4.6). Let $f \in \mathcal{A}^s_{\mathcal{M}^\varphi_q, r}(\mathbb{R}^n)$ and $\varphi \in \mathcal{G}_q$. Then $\psi \equiv \varphi^q \in \mathcal{G}_1$. Take $g \in \mathcal{M}^\varphi_q(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ arbitrarily. Using the following obvious inequality

$$\frac{\varphi(r)^q}{r^n} \int_{B(x,r)} |g(y)| \, \mathrm{d}y \le \|g\|_{L^\infty}^{1-q} \left[\varphi(r) \left(\frac{1}{r^n} \int_{B(x,r)} |g(y)|^q \, \mathrm{d}y \right)^{1/q} \right]^q,$$

we first note that

$$\|g\|_{\mathcal{M}^\psi_1} \le \|g\|^q_{\mathcal{M}^\varphi_q} \|g\|^{1-q}_{L^\infty}. \tag{4.8}$$

We next estimate $\|\theta_j(D)f\|^q_{\mathcal{M}^\varphi_q}$ and $\|\theta_j(D)f\|^{1-q}_{L^\infty}$ for each $j \in \mathbb{N}_0$. From the definition of the quasi-norm of $\mathcal{N}^s_{\mathcal{M}^\varphi_q, \infty}(\mathbb{R}^n)$ we have

$$2^{jsq} \|\theta_j(D)f\|^q_{\mathcal{M}^\varphi_q} \le \|f\|^q_{\mathcal{N}^s_{\mathcal{M}^\varphi_q, \infty}}. \tag{4.9}$$

Additionally by the proof of [11, Lemma 3.4] we get $|\theta_j(D)f(x)| \lesssim \dfrac{2^{-js}}{\varphi(2^{-j})}\|f\|_{\mathcal{N}^s_{\mathcal{M}^\varphi_q,\infty}}$. Hence we obtain $2^{js}\varphi(2^{-j})\|\theta_j(D)f\|_{L^\infty} \lesssim \|f\|_{\mathcal{N}^s_{\mathcal{M}^\varphi_q,\infty}}$. Therefore we have

$$2^{js(1-q)}\varphi(2^{-j})^{1-q}\|\theta_j(D)f\|_{L^\infty}^{1-q} \lesssim \|f\|^{1-q}_{\mathcal{N}^s_{\mathcal{M}^\varphi_q,\infty}}. \tag{4.10}$$

Finally we prove (4.7) using (4.9) and (4.10). Note that we can use (4.8) as $g = \theta_j(D)f$ for any $j \in \mathbb{N}_0$. By virtue of $\varphi(2^{-j}) \gtrsim 2^{-jn/q}\varphi(1)$ we obtain

$$\sum_{j=0}^\infty \frac{2^{-js}}{\varphi(2^{-j})^{1-q}} \lesssim \sum_{j=0}^\infty 2^{-js}2^{jn(1-q)/q} = \sum_{j=0}^\infty 2^{-j(s-n(\frac{1}{q}-1))} < \infty.$$

Thus we get $\|f\|_{\mathcal{M}^\psi_1} \le \sum_{j=0}^\infty \|\theta_j(D)f\|_{\mathcal{M}^\psi_1} \lesssim \sum_{j=0}^\infty \frac{2^{-js}}{\varphi(2^{-j})^{1-q}}\|f\|_{\mathcal{N}^s_{\mathcal{M}^\varphi_q,\infty}} \lesssim \|f\|_{\mathcal{N}^s_{\mathcal{M}^\varphi_q,\infty}}$. This implies that (4.7) holds. □

Let $M \in \mathbb{N}$ be given. Triebel [19, p. 173, Lemma 3.3.1] proved that there exist two smooth functions φ and ψ defined on $\mathbb{R}$ satisfying $\operatorname{supp}\varphi \subset (0,1)$, $\operatorname{supp}\psi \subset (0,1)$, $\int_{\mathbb{R}} \varphi(\tau)\,\mathrm{d}\tau = 1$ and $\varphi(t) - \frac{1}{2}\varphi\left(\frac{t}{2}\right) = \psi^{(M)}(t)$ for all $t \in \mathbb{R}$. Let $\rho(x) \equiv \prod_{\ell=1}^n \varphi(x_\ell)$. We additionally define

$$\begin{aligned} T_0(x) &\equiv \sum_{m'=1}^M \sum_{m=1}^M \frac{(-1)^{M+m+m'+1}}{M!}\binom{M}{m'}\binom{M}{m} m^M (mm')^{-n}\rho\left(\frac{x}{mm'}\right), \\ T(x) &\equiv T_0(x) - 2^{-n}T_0(\frac{x}{2}), \\ T_j(x) &\equiv 2^{nj}T(2^jx) \end{aligned}$$

for $x \in \mathbb{R}^n$ and $j \in \mathbb{N}$, where every $\binom{M}{m}$ denotes the binomial coefficient.

Proposition 4.4 ([17, p174, Section 3.3.2, Proposition]). *Let T_0 and T be the functions defined as above.*

(i) The functions T_0 and T are compactly supported on $\mathbb{R}^n$ satisfying

$$\operatorname{supp} T_0 \subset \mathbb{R}^n_+ \quad \text{and} \quad \operatorname{supp} T \subset \mathbb{R}^n_+,$$

where $\mathbb{R}^n_+ \equiv \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$. Furthermore we have $\mathcal{F}T_0(0) = 1$ and $\mathcal{F}T(\xi) = O(|\xi|^M)$ near the origin.

(ii) For all $f \in \mathcal{S}(\mathbb{R}^n)$, f can be represented as $f = \sum_{j=0}^\infty T_j * f$ in the sense of $\mathcal{S}'(\mathbb{R}^n)$.

(iii) *For any* $f \in \mathcal{S}'(\mathbb{R}^n) \cap L^1_{loc}(\mathbb{R}^n)$, *we have that*

$$(4.11)\quad (T_0 * f)(x) = \frac{(-1)^M}{M!}\sum_{m=1}^{M}(-1)^{M-m}\binom{M}{m} m^{M-n}\times \\ \times \int_{\mathbb{R}^n}\rho(-\frac{y}{m})\sum_{m'=1}^{M}(-1)^{M-m'}\binom{M}{m'} f(x+m'y)\,\mathrm{d}y$$

and that for all $j \in \mathbb{N}$,

$$(4.12)\quad (T_j * f)(x) = \frac{(-1)^{M+1}}{M!}\sum_{m=1}^{M}(-1)^{M-m}\binom{M}{m} m^{M-n}\times \\ \times \int_{\mathbb{R}^n}\left[\rho(-\frac{y}{m}) - 2^{-n}\rho(-\frac{y}{2m})\right]\Delta^M_{2^{-j}y}f(x)\,\mathrm{d}y.$$

Proposition 4.4 shows that the function $\mathcal{F}T$ satisfies condition (3.6) with $\psi = \mathcal{F}T$. Then we have the following lemma by Theorem 3.10 with a similar argument as in [17, Section 3.3.3, Step 1 in the proof of Theorem].

Lemma 4.5. *Let* $0 < q < \infty$, $0 < r \le \infty$, $s \in \mathbb{R}$, $\varphi \in \mathcal{G}_q$ *and* $M \in \mathbb{N}$ *such that* $M > s$.

(i) *For all* $f \in \mathcal{N}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)$ *we have*

$$(4.13)\quad \left(\sum_{j=0}^{\infty}\left\|2^{js}T_j * f\right\|^r_{\mathcal{M}^\varphi_q}\right)^{1/r} \sim \|f\|_{\mathcal{N}^s_{\mathcal{M}^\varphi_q,r}}.$$

(ii) *Assume in addition* (2.2) *when* $r < \infty$. *Then for all* $f \in \mathcal{E}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)$ *we have*

$$(4.14)\quad \left\|\left(\sum_{j=0}^{\infty}2^{jsr}(T_j * f)^r\right)^{1/r}\right\|_{\mathcal{M}^\varphi_q} \sim \|f\|_{\mathcal{E}^s_{\mathcal{M}^\varphi_q,r}}.$$

§ 4.2. Proof of Theorem 4.1

We rely on an argument similar to the proof of [6, Lemma 16]. We divide the proof into 3 parts as follows: Firstly, we prove

$$(4.15)\quad \|f\|^*_{\mathcal{A}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)} \sim \|f\|^{**}_{\mathcal{A}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)}$$

for $f \in L^1_{loc}(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$ in Part 1. We next prove $\|f\|^{**}_{\mathcal{A}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{A}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)}$ for $f \in \mathcal{A}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)$ in Part 2. Finally, we prove $\|f\|_{\mathcal{A}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)} \lesssim \|f\|^{**}_{\mathcal{A}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)}$ for $f \in L^1_{loc}(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$ in Part 3.

Part 1. We have only to prove (4.15) in the case $\mathcal{A}=\mathcal{E}$ and $0<r<\infty$ because the case $\mathcal{A}=\mathcal{N}$ and the case $r=\infty$ are proved by a similar argument. It is easy to see that

$$\left[\int_0^\infty t^{-sr}\left(\int_B |\Delta_{th}^M f(x)|\,\mathrm{d}h\right)^r \frac{\mathrm{d}t}{t}\right]^{1/r} = \left[\sum_{k=-\infty}^{\infty}\int_{2^{-k-1}}^{2^{-k}} t^{-sr}\left(t^{-n}\int_{t\mathbb{B}} |\Delta_v^M f(x)|\,\mathrm{d}v\right)^r \frac{\mathrm{d}t}{t}\right]^{1/r}. \tag{4.16}$$

If $2^{-k-1}\le t\le 2^{-k}$, then we have $2^{ksr}\le t^{-sr}\le 2^{(k+1)sr}$ and

$$2^{kn}\int_{2^{-(k+1)}\mathbb{B}} \left|\Delta_v^M f(x)\right|\,\mathrm{d}v \lesssim t^{-n}\int_{t\mathbb{B}}\left|\Delta_v^M f(x)\right|\,\mathrm{d}v \lesssim 2^{(k+1)n}\int_{2^{-k}\mathbb{B}}\left|\Delta_v^M f(x)\right|\,\mathrm{d}v. \tag{4.17}$$

(4.16) and the right-hand side of (4.17) yield

$$\left[\int_0^\infty t^{-sr}\left(\int_{\mathbb{B}} |\Delta_{th}^M f(x)|\,\mathrm{d}h\right)^r \frac{\mathrm{d}t}{t}\right]^{1/r} \lesssim \left[\sum_{k=-\infty}^{\infty}\int_{2^{-k-1}}^{2^{-k}} 2^{ksr}\left(\int_{\mathbb{B}} |\Delta_{2^{-k}v}^M f(x)|\,\mathrm{d}v\right)^r \frac{\mathrm{d}t}{t}\right]^{1/r}.$$

This implies that $\|f\|^*_{\mathcal{E}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)} \lesssim \|f\|^{**}_{\mathcal{E}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)}$ holds. The opposite inequality is easily obtained by virtue of the same argument using (4.16) and the left-hand side of (4.17).

Part 2. We prove $\|f\|^{**}_{\mathcal{A}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{A}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)}$.

(*Step 1.*) We first prove the estimate $\|f\|_{\mathcal{M}^\varphi_q} \lesssim \|f\|_{\mathcal{A}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)}$. By $s>0$ and Proposition 3.5, we have

$$\begin{aligned}\|f\|_{\mathcal{M}^\varphi_q}^{\min(1,q,r)} &\le \sum_{k=0}^\infty \|\theta_k(D)f\|_{\mathcal{M}^\varphi_q}^{\min(1,q,r)}\\ &\le \sum_{k=0}^\infty 2^{-s\min(1,q,r)k}\|f\|^{\min(1,q,r)}_{\mathcal{N}^s_{\mathcal{M}^\varphi_q,\infty}(\mathbb{R}^n)}\\ &\lesssim \|f\|^{\min(1,q,r)}_{\mathcal{A}^s_{\mathcal{M}^\varphi_{q,r}}(\mathbb{R}^n)}.\end{aligned}$$

(*Step 2.*) From this step through Step 4, we prove $\|f\|^{**}_{\mathcal{E}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{E}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)}$. Let $\kappa_0\in\mathcal{S}(\mathbb{R}^n)$ with $\kappa_0(x)=1$ for $|x|\le 1$ and $\operatorname{supp}\kappa_0\subset\{x\in\mathbb{R}^n : |x|\le 2\}$. For $j\in\mathbb{N}$, we define

$$\kappa_j(x)=\kappa_0(2^{-j}x)-\kappa_0(2^{-j+1}x).$$

We use the decomposition

$$f = \sum_{l=-\infty}^{\infty} f_{(k+l)}, \quad k \in \mathbb{Z},$$

where $f_{(k+l)} = \kappa_{k+l}(D)f$, or $= 0$ if $k + l < 0$. Recall that

$$\|f\|^{**}_{\mathcal{E}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)} \equiv \|f\|_{\mathcal{M}^\varphi_q} + \left\| \{2^{ks} d^M_{2^{-k}} f\}_{k=-\infty}^{\infty} \right\|_{\mathcal{M}^\varphi_q(\ell^r)}.$$

Therefore, we consider the $\mathcal{M}^\varphi_q$ quasi-norm of

$$\sum_{k=-\infty}^{\infty} 2^{ksr} \left(\int_B |\Delta^M_{2^{-k}h} f(x)| \, \mathrm{d}h \right)^r = \sum_{k=-\infty}^{\infty} 2^{ksr} \left(\int_B |\Delta^M_{2^{-k}h} \left(\sum_{l=-\infty}^{\infty} f_{(k+l)} \right)(x)| \, \mathrm{d}h \right)^r.$$

If $r \le 1$, then we have

$$\sum_{k=-\infty}^{\infty} 2^{ksr} \left(\int_B |\Delta^M_{2^{-k}h} f(x)| \, \mathrm{d}h \right)^r \le \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} 2^{ksr} \left(\int_B |\Delta^M_{2^{-k}h} f_{(k+l)}(x)| \, \mathrm{d}h \right)^r.$$

We split the above right-hand side into two parts as below:

(4.18)

$$\begin{aligned}
&\sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} 2^{ksr} \left(\int_B |\Delta^M_{2^{-k}h} f_{(k+l)}(x)| \, \mathrm{d}h \right)^r \\
&= \sum_{l=-\infty}^{0} \sum_{k=-\infty}^{\infty} 2^{ksr} \left(\int_B |\Delta^M_{2^{-k}h} f_{(k+l)}(x)| \, \mathrm{d}h \right)^r + \sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} 2^{ksr} \left(\int_B |\Delta^M_{2^{-k}h} f_{(k+l)}(x)| \, \mathrm{d}h \right)^r \\
&=: I_1 + I_2.
\end{aligned}$$

In this case, to obtain the desired inequality, it suffices to prove $\|I_1^{1/r}\|_{\mathcal{M}^\varphi_q} \lesssim \|f\|_{\mathcal{E}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)}$ and $\|I_2^{1/r}\|_{\mathcal{M}^\varphi_q} \lesssim \|f\|_{\mathcal{E}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)}$.

If $r > 1$, by using Minkowski's inequality, we get

$$\begin{aligned}
&\left(\sum_{k=-\infty}^{\infty} 2^{ksr} \left(\int_B |\Delta^M_{2^{-k}h} f(x)| \, \mathrm{d}h \right)^r \right)^{1/r} \\
&\le \left(\sum_{k=-\infty}^{\infty} 2^{ksr} \left(\int_B \sum_{l=-\infty}^{\infty} |\Delta^M_{2^{-k}h} f_{(k+l)}(x)| \, \mathrm{d}h \right)^r \right)^{1/r} \\
&\lesssim \sum_{l=-\infty}^{0} \left(\sum_{k=-\infty}^{\infty} 2^{ksr} \left(\int_B |\Delta^M_{2^{-k}h} f_{(k+l)}(x)| \, \mathrm{d}h \right)^r \right)^{1/r} \\
&\quad + \sum_{l=1}^{\infty} \left(\sum_{k=-\infty}^{\infty} 2^{ksr} \left(\int_B |\Delta^M_{2^{-k}h} f_{(k+l)}(x)| \, \mathrm{d}h \right)^r \right)^{1/r} \\
&=: I_3 + I_4.
\end{aligned} \tag{4.19}$$

In this case, to obtain the desired inequality, it suffices to prove $\|I_3\|_{\mathcal{M}^\varphi_q} \lesssim \|f\|_{\mathcal{E}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)}$ and $\|I_4\|_{\mathcal{M}^\varphi_q} \lesssim \|f\|_{\mathcal{E}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)}$.

(*Step 3.*) We estimate I_1 and I_3. We use Lemma 4.2 in the form

$$|\Delta^M_h f_{(k+l)}(x)| \lesssim \max(1, |bh|^a) \min\left(1, |bh|^M\right) P_{b,a} f_{(k+l)}(x),$$

where $a > 0$ is arbitrary, $b = 2^{k+l}$ and

$$P_{b,a} f(x) = \sup_{z\in\mathbb{R}^n} \frac{|f(x-z)|}{1+|bz|^a}.$$

We use this estimate with $2^{-k}h$ instead of h, we obtain

$$\begin{aligned}\int_B |\Delta^M_{2^{-k}h} f_{(k+l)}(x)|\,\mathrm{d}h &\lesssim \int_B \max\left(1, |b2^{-k}h|^a\right) \min\left(1, |b2^{-k}h|^M\right) P_{b,a} f_{(k+l)}(x)\,\mathrm{d}h \\ &\lesssim 2^{lM} P_{2^{k+l},a} f_{(k+l)}(x),\end{aligned} \tag{4.20}$$

where we used the fact that $\max(1, |b2^{-k}h|^a) \le 1$ and $\min(1, |b2^{-k}h|^M) \le 2^{lM}$ (recall that $l \le 0$ and $|h| \le 1$).

If $r \le 1$, by $M > s$ and $f_{(k+l)}(x) = 0$ when $k + l < 0$, we see that

$$\begin{aligned}I_1 &\lesssim \sum_{l=-\infty}^{0} \sum_{k=-\infty}^{\infty} 2^{ksr} \left(2^{lM} P_{2^{k+l},a} f_{(k+l)}(x)\right)^r \\ &= \sum_{l=-\infty}^{0} 2^{l(M-s)r} \sum_{k=-\infty}^{\infty} 2^{(k+l)sr} (P_{2^{k+l},a} f_{(k+l)}(x))^r \\ &\lesssim \sum_{k=0}^{\infty} 2^{ksr} (P_{2^k,a} f_{(k)}(x))^r.\end{aligned}$$

By the boundedness of Peetre maximal function (Theorem 3.4), we get

$$\begin{aligned}\|I_1^{1/r}\|_{\mathcal{M}^\varphi_q} &\lesssim \left\| \left\{2^{ks} P_{2^k,a} f_{(k)}(\cdot)\right\}_{k=0}^{\infty} \right\|_{\mathcal{M}^\varphi_q(\ell^r)} \\ &\lesssim \|\{2^{ks} f_{(k)}\}_{k=0}^{\infty}\|_{\mathcal{M}^\varphi_q(\ell^r)} \\ &\lesssim \|f\|_{\mathcal{E}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)}.\end{aligned}$$

If $r > 1$, by using the similar argument as above, we have

$$
\begin{aligned}
I_3 &\lesssim \sum_{l=-\infty}^{0} \left(\sum_{k=-\infty}^{\infty} 2^{ksr} \left(2^{lM} P_{2^{k+l},a} f_{(k+l)}(x) \right)^r \right)^{1/r} \\
&= \sum_{l=-\infty}^{0} 2^{l(M-s)} \left(\sum_{k=-\infty}^{\infty} 2^{(k+l)sr} \left(P_{2^{k+l},a} f_{(k+l)}(x) \right)^r \right)^{1/r} \\
&\lesssim \left(\sum_{k=-\infty}^{\infty} 2^{ksr} \left(P_{2^k,a} f_{(k)}(x) \right)^r \right)^{1/r}.
\end{aligned}
$$

Therefore, we get

$$
\|I_3\|_{\mathcal{M}_q^\varphi} \lesssim \|f\|_{\mathcal{E}^s_{\mathcal{M}_q^\varphi,r}(\mathbb{R}^n)}.
$$

(*Step 4.*) In this step, we consider the I_2 and I_4. Let $\lambda = 1$ when $\min(q, r) > 1$. Otherwise, by virtue of the condition of s, there exists a $\lambda \in (0, 1)$ such that $s > \dfrac{n}{\min(q,r)}(1-\lambda)$. This implies that there exists a real number $a > 0$ such that $a > \dfrac{n}{\min(q,r)}$ and $s > a(1-\lambda)$.

By Lemma 4.2 and (4.1), we see that

$$
\begin{aligned}
\text{(4.21)} \quad & \int_B |\Delta^M_{2^{-k}h} f_{(k+l)(x)}| \,\mathrm{d}h \\
&= \int_B |\Delta^M_{2^{-k}h} f_{(k+l)(x)}|^{1-\lambda} \cdot |\Delta^M_{2^{-k}h} f_{(k+l)(x)}|^{\lambda} \,\mathrm{d}h \\
&\lesssim \left(2^{la} P_{2^{k+l},a} f_{(k+l)}(x) \right)^{1-\lambda} \int_B |\Delta^M_{2^{-k}h} f_{(k+l)}(x)|^{\lambda} \,\mathrm{d}h \\
&\lesssim \left(2^{la} P_{2^{k+l},a} f_{(k+l)}(x) \right)^{1-\lambda} \sum_{j=0}^{M} c_{j,M} \int_B |f_{(k+l)}(x + j2^{-k}h)|^{\lambda} \,\mathrm{d}h \\
&\lesssim \left(2^{la} P_{2^{k+l},a} f_{(k+l)}(x) \right)^{1-\lambda} \sum_{j=0}^{M} c_{j,M} \mathcal{M}[|f_{(k+l)}|^{\lambda}],
\end{aligned}
$$

where the constants $c_{j,M}$ are given by (4.1).

Firstly, we estimate I_4. That is, we consider the case $r > 1$. We denote

$$
F(x) \equiv \left(\sum_{k=0}^{\infty} \left(2^{ks} P_{2^k,a} f_{(k)}(x) \right)^r \right)^{1/r}, \quad x \in \mathbb{R}^n
$$

and

$$
B_{k+l}(x) \equiv |2^{(k+l)s} f_{(k+l)}(x)|, \quad x \in \mathbb{R}^n.
$$

Let $\delta \equiv -(a(1-\lambda)-s) > 0$. By Hölder inequality, we obtain

$$\begin{aligned} I_4 &= \sum_{l=1}^{\infty} \left(\sum_{k=-\infty}^{\infty} 2^{ksr} \left(\int_B |\Delta^M_{2^{-k}h} f_{(k+l)}(x)| \, \mathrm{d}h \right)^r \right)^{1/r} \\ &\lesssim \sum_{l=1}^{\infty} 2^{-l\delta} \left(\sum_{k=-\infty}^{\infty} \left(2^{(k+l)s} P_{2^{k+l},a} f_{(k+l)}(x) \right)^{(1-\lambda)r} (\mathcal{M}[|B_{(k+l)}|^{\lambda}])^r(x) \right)^{1/r} \\ &\lesssim F(x)^{1-\lambda} \sum_{l=1}^{\infty} 2^{-l\delta} \left(\sum_{k=-\infty}^{\infty} (\mathcal{M}[|B_{(k+l)}|^{\lambda}])^{r/\lambda}(x) \right)^{\lambda/r}. \end{aligned}$$

Therefore we have

(4.22)

$$\begin{aligned} \|I_4\|^{\min(1,q,r)}_{\mathcal{M}^{\varphi}_q} &\lesssim \left\| F(\cdot)^{1-\lambda} \sum_{l=1}^{\infty} 2^{-l\delta} \left(\sum_{k=-\infty}^{\infty} \mathcal{M}[B^{\lambda}_{(k+l)}]^{r/\lambda}(\cdot) \right)^{\lambda/r} \right\|^{\min(1,q,r)}_{\mathcal{M}^{\varphi}_q} \\ &\lesssim \sum_{l=1}^{\infty} 2^{-l\delta\min(1,q,r)} \left\| F(\cdot)^{1-\lambda} \left(\sum_{k=-\infty}^{\infty} \mathcal{M}[B^{\lambda}_{(k+l)}]^{r/\lambda}(\cdot) \right)^{\lambda/r} \right\|^{\min(1,q,r)}_{\mathcal{M}^{\varphi}_q}. \end{aligned}$$

The Hölder inequality implies that

$$\|F_1^{1-\lambda} F_2^{\lambda}\|_{\mathcal{M}^{\varphi}_q} \le \|F_1\|^{1-\lambda}_{\mathcal{M}^{\varphi}_q} \|F_2\|^{\lambda}_{\mathcal{M}^{\varphi}_q} \tag{4.23}$$

holds for any $F_1, F_2 \in L^q(E)$. Thus we have $\|F_1^{1-\lambda} F_2^{\lambda}\|_{\mathcal{M}^{\varphi}_q} \le \|F_1\|^{1-\lambda}_{\mathcal{M}^{\varphi}_q} \|F_2\|^{\lambda}_{\mathcal{M}^{\varphi}_q}$ for all $F_1, F_2 \in \mathcal{M}^{\varphi}_q$. Therefore, By (4.22), (4.23), Lemma 3.1 and Theorem 3.3, we see that

(4.24)

$$\begin{aligned} \|I_4\|^{\min(1,q,r)}_{\mathcal{M}^{\varphi}_q} &\lesssim \sum_{l=1}^{\infty} 2^{-l\delta\min(1,q,r)} \left\| F(\cdot)^{1-\lambda} \left(\sum_{k=-\infty}^{\infty} \mathcal{M}[B^{\lambda}_{(k+l)}]^{r/\lambda}(\cdot) \right)^{\lambda/r} \right\|^{\min(1,q,r)}_{\mathcal{M}^{\varphi}_q} \\ &\lesssim \|f\|^{\min(1,q,r)(1-\lambda)}_{\mathcal{E}^s_{\mathcal{M}^{\varphi}_{q,r}}} \sum_{l=1}^{\infty} 2^{-l\delta\min(1,q,r)} \left\| \left\{ \mathcal{M}[B^{\lambda}_{(k+l)}](\cdot) \right\}_{k=-\infty}^{\infty} \right\|^{\min(1,q,r)}_{\mathcal{M}^{\varphi^{\lambda}}_{q/\lambda}(\ell^{r/\lambda})} \\ &\lesssim \|f\|^{\min(1,q,r)(1-\lambda)}_{\mathcal{E}^s_{\mathcal{M}^{\varphi}_{q,r}}} \sum_{l=1}^{\infty} 2^{-l\delta\min(1,q,r)} \left\| \left\{ B^{\lambda}_{(k+l)}(\cdot) \right\}_{k=-\infty}^{\infty} \right\|^{\min(1,q,r)}_{\mathcal{M}^{\varphi^{\lambda}}_{q/\lambda}(\ell^{r/\lambda})} \\ &\lesssim \|f\|^{\min(1,q,r)}_{\mathcal{E}^s_{\mathcal{M}^{\varphi}_{q,r}}}. \end{aligned}$$

Next, we estimate I_2. That is, we consider the case $r \le 1$. By using the similar way, we obtain

$$I_2 \lesssim F(x)^{r(1-\lambda)} \sum_{l=1}^{\infty} 2^{(la(1-\lambda)-ls)r} \left(\sum_{k=-\infty}^{\infty} (\mathcal{M}[|B_{(k+l)}|^{\lambda}])^{r/\lambda}(x) \right)^{\lambda}.$$

Therefore, by Hölder inequality, we get

$$I_2^{1/r} \lesssim F(x)^{(1-\lambda)/r} \sum_{l=1}^{\infty} 2^{(la(1-\lambda)-ls)/2} \left(\sum_{k=-\infty}^{\infty} (\mathcal{M}[B_{(k+l)}^{\lambda}])^{r/\lambda}(x) \right)^{\lambda/r}.$$

Hence, by using the similar argument as in (4.24), we have

$$\|I_2^{1/r}\|_{\mathcal{M}_q^{\varphi}}^{\min(1,q,r)} \lesssim \|f\|_{\mathcal{E}^s_{\mathcal{M}_{q,r}^{\varphi}}}^{\min(1,q,r)}.$$

(Step 5) In this step, we prove $\|f\|^{**}_{\mathcal{N}^s_{\mathcal{M}_q^{\varphi},r}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{N}^s_{\mathcal{M}_q^{\varphi},r}(\mathbb{R}^n)}$. Recall that

$$\|f\|^{**}_{\mathcal{N}^s_{\mathcal{M}_q^{\varphi},r}(\mathbb{R}^n)} \equiv \|f\|_{\mathcal{M}_q^{\varphi}} + \left\| \{2^{ks} d^M_{2^{-k}} f\}_{k=-\infty}^{\infty} \right\|_{\ell^r(\mathcal{M}_q^{\varphi})}.$$

Therefore, we consider the $\ell^r(\mathcal{M}_q^{\varphi})$ quasi-norm of

$$f^{(k)}(x) \equiv 2^{ks} \int_B |\Delta^M_{2^{-k}h} f(x)| \,\mathrm{d}h = 2^{ks} \int_B |\Delta^M_{2^{-k}h} \left(\sum_{l=-\infty}^{\infty} f_{(k+l)} \right)(x)| \,\mathrm{d}h,$$

where we use again the decomposition

$$f = \sum_{l=-\infty}^{\infty} f_{(k+l)}, \quad k \in \mathbb{Z}.$$

We split $f^{(k)}(x)$ into two parts as below:

$$\begin{aligned} f^{(k)}(x) &= 2^{ks} \int_B |\Delta^M_{2^{-k}h} \left(\sum_{l=-\infty}^{\infty} f_{(k+l)} \right)(x)| \,\mathrm{d}h \\ &\le \sum_{l=-\infty}^{0} 2^{ks} \int_B |\Delta^M_{2^{-k}h} \left(f_{(k+l)} \right)(x)| \,\mathrm{d}h + \sum_{l=1}^{\infty} 2^{ks} \int_B |\Delta^M_{2^{-k}h} \left(f_{(k+l)} \right)(x)| \,\mathrm{d}h \\ &=: f^{(k),I} + f^{(k),II}. \end{aligned}$$

Firstly, we shall prove $\|\{f^{(k),I}\}_{k=-\infty}^{\infty}\|_{\ell^r(\mathcal{M}_q^{\varphi})} \lesssim \|f\|_{\mathcal{N}^s_{\mathcal{M}_q^{\varphi}}}$. Let

$$g_u^1(x) \equiv 2^{us} P_{2^u,a} f_{(u)}(x).$$

Then, by (4.20), it is easy to see that

$$f^{(k),I} \lesssim \sum_{l=-\infty}^{0} 2^{l(M-s)} g^1_{k+l} = \sum_{u=-\infty}^{k} 2^{-|k-u|(M-s)} g^1_u.$$

Thanks to Lemma 3.9 and the boundedness of Peetre maximal function (Theorem 3.4) with $a > n/\min(1,q)$, we obtain

$$\left\| \left\{ f^{(k),I} \right\}_{k=-\infty}^{\infty} \right\|_{\ell^r(\mathcal{M}_q^\varphi)} \lesssim \|\{g_u\}_{u=0}^\infty\|_{\ell^r(\mathcal{M}_q^\varphi)} \lesssim \|f\|_{\mathcal{N}^s_{\mathcal{M}_q^\varphi}}.$$

Finally, we shall prove $\|\{f^{(k),II}\}_{k=-\infty}^\infty\|_{\ell^r(\mathcal{M}_q^\varphi)} \lesssim \|f\|_{\mathcal{N}^s_{\mathcal{M}_q^\varphi}}$. Let again $\lambda = 1$ when $q > 1$. Otherwise we set real parameters $0 < \lambda < 1$ and we set a real number $a > 0$ such that

$$a > \frac{n}{q}$$

and $a(1-\lambda) < s$. Due to conditions $s > \sigma_q$, we can choose such a real number a by using the similar argument in Step 4. Let

$$g_u^2(x) \equiv |2^{us} f_{(u)}(x)|.$$

By (4.21), we get

$$\begin{aligned} f^{(k),II} &\lesssim \sum_{l=1}^\infty 2^{ks} \int_B |\Delta^M_{2^{-k}h} f_{(k+l)(x)}| \, \mathrm{d}h \\ &\lesssim \sum_{l=1}^\infty 2^{la(1-\lambda)-ls} \left(g^1_{k+l}\right)^{1-\lambda} \mathcal{M}[|g^2_{k+l}|^\lambda](x). \end{aligned}$$

Let $\delta \equiv -(a(1-\lambda)-s) > 0$. By Lemma 3.1, (4.23) and Hölder inequality, we see that

$$\begin{aligned} &\|\{f^{(k),II}\}_{k=-\infty}^\infty\|^{\min(1,q,r)}_{\ell^r(\mathcal{M}_q^\varphi)} \\ &\lesssim \sum_{l=1}^\infty 2^{-l\delta \min(1,q,r)} \left\| \left\{ \left(g^1_{k+l}\right)^{1-\lambda} \mathcal{M}[|g^2_{k+l}|^\lambda] \right\}_{k=-\infty}^\infty \right\|^{\min(1,q,r)}_{\ell^r(\mathcal{M}_q^\varphi)} \\ &\lesssim \sum_{l=1}^\infty 2^{-l\delta \min(1,q,r)} \left\| \left\{ \|g^1_{k+l}\|^{1-\lambda}_{\mathcal{M}_q^\varphi} \left\| (\mathcal{M}[|g^2_{k+l}|^\lambda])^{1/\lambda} \right\|^\lambda_{\mathcal{M}_q^\varphi} \right\}_{k=-\infty}^\infty \right\|^{\min(1,q,r)}_{\ell^r} \\ &\lesssim \sum_{l=1}^\infty 2^{-l\delta \min(1,q,r)} \left\| \left\{ g^1_{k+l} \right\}_{k=-\infty}^\infty \right\|^{(1-\lambda)\min(1,q,r)}_{\ell^r(\mathcal{M}_q^\varphi)} \left\| \left\{ (\mathcal{M}[|g^2_{k+l}|^\lambda])^{1/\lambda} \right\}_{k=-\infty}^\infty \right\|^{\lambda \min(1,q,r)}_{\ell^r(\mathcal{M}_q^\varphi)}. \end{aligned}$$

Therefore, by Theorem 3.3, we obtain

$$\begin{aligned} &\|\{f^{(k),II}\}_{k=0}^\infty\|^{\min(1,q,r)}_{\ell^r(\mathcal{M}_q^\varphi)} \\ &\lesssim \sum_{l=1}^\infty 2^{\{la(1-\lambda)-ls\}\min(1,q,r)} \left\| \left\{ g^1_u \right\}_{u=0}^\infty \right\|^{(1-\lambda)\min(1,q,r)}_{\ell^r(\mathcal{M}_q^\varphi)} \left\| \left\{ M[|g^2_u|^\lambda] \right\}_{u=0}^\infty \right\|^{\min(1,q,r)}_{\ell^{r/\lambda}(\mathcal{M}^{\varphi^\lambda}_{q/\lambda})} \\ &\lesssim \|f\|^{\min(1,q,r)}_{\mathcal{N}^s_{\mathcal{M}_q^\varphi}}. \end{aligned}$$

Part 3. We prove $\|f\|_{\mathcal{A}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)} \lesssim \|f\|^{**}_{\mathcal{A}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)}$ for $f \in L^1_{loc}(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\psi(x) = 1$ when $|x| \le 1$ and $\psi(x) = 0$ when $|x| > 3/2$. We define $\kappa_0(x) \equiv (-1)^{M+1} \sum_{\mu=0}^{M-1} (-1)^\mu \binom{M}{\mu} \psi((M-\mu)x)$. Note that $\kappa_0 \in C_0^\infty(\mathbb{R}^n)$ with $\kappa_0(x) = 0$ when $|x| > 3/2$ and $\kappa_0(x) = 1$ when $|x| < 1/M$. We define $\kappa_j(x) = \kappa_0(2^{-j}x) - \kappa_0(2^{-j+1}x)$ for $j \in \mathbb{N}$. Then we see that $\{\kappa_j\}_{j=0}^\infty$ is a decomposition of unity. Observe that $\kappa_0(x) = (-1)^{M+1}\left(\Delta_x^M \psi(0) - (-1)^M\right)$ and that

$$(4.25) \qquad (\kappa_j(D)f)(x) = \begin{cases} \left(\mathcal{F}^{-1}\Delta_\xi^M \psi(0)\mathcal{F}f\right)(x) + (-1)^{M+1} f(x) & (j=0) \\ \left(\mathcal{F}^{-1}(\Delta^M_{2^{-j}\xi}\psi(0) - \Delta^M_{2^{-j+1}\xi}\psi(0))\mathcal{F}f\right)(x) & (j \in \mathbb{N}) \end{cases}.$$

Kempka and Vybíral [6, (49)] proved that for all $j \in \mathbb{N}_0$,

$$(4.26) \qquad \left|\left(\mathcal{F}^{-1}\left(\Delta^M_{2^{-j}\xi}\psi(0)\right)\mathcal{F}f\right)(x)\right| \lesssim \int_{\mathbb{R}^n} |\hat{\psi}(h)| \cdot |\Delta^M_{2^{-j}h} f(x)| \, \mathrm{d}h.$$

Firstly, we shall prove the case $\mathcal{A} = \mathcal{E}$. We put $g = \hat{\psi} \in \mathcal{S}(\mathbb{R}^n)$ and obtain

$$(4.27) \qquad \begin{aligned} \|f\|_{\mathcal{E}^s_{\mathcal{M}^\varphi_q,r}} &\sim \|\{2^{js}\kappa_j(D)f\}_{j=0}^\infty\|_{\mathcal{M}^\varphi_q(\ell^r)} \\ &\lesssim \|f\|_{\mathcal{M}^\varphi_q} + \left\|\left\{2^{js}\int_{\mathbb{R}^n} |g(h)| \cdot \left|\Delta^M_{2^{-j}h} f(x)\right| \, \mathrm{d}h\right\}_{j=0}^\infty\right\|_{\mathcal{M}^\varphi_q(\ell^r)}. \end{aligned}$$

Let $I_0 \equiv \mathbb{B}$ and $I_k \equiv 2^k\mathbb{B}\backslash 2^{k-1}\mathbb{B}$ for $k \in \mathbb{N}$. Take $t > s+n$ arbitrarily. By $g \equiv \hat{\psi} \in \mathcal{S}(\mathbb{R}^n)$, $|g(h)| \lesssim 2^{-kt}$ holds for all $h \in I_k$. Then we can estimate

$$(4.28) \qquad \begin{aligned} \int_{\mathbb{R}^n} |g(h)| \cdot \left|\Delta^M_{2^{-j}h} f(x)\right| \, \mathrm{d}h &= \sum_{k=0}^\infty \int_{I_k} |g(h)| \cdot \left|\Delta^M_{2^{-j}h} f(x)\right| \, \mathrm{d}h \\ &\le \sum_{k=0}^\infty 2^{-kt} \int_{I_k} \left|\Delta^M_{2^{-j}h} f(x)\right| \, \mathrm{d}h \\ &\lesssim \sum_{k=0}^\infty 2^{k(n-t)} d^M_{2^{k-j}} f. \end{aligned}$$

We put $g_l(x) \equiv 2^{ls} d^M_{2^{-l}} f(x)$ for $l \in \mathbb{Z}$. By (4.25), (4.26) and (4.28), we see that

$$(4.29) \qquad \begin{aligned} 2^{js}\int_{\mathbb{R}^n} |g(h)| \cdot \left|\Delta^M_{2^{-j}h} f(x)\right| \, \mathrm{d}h &\lesssim 2^{js}\sum_{k=0}^\infty 2^{k(n-t)} d^M_{2^{k-j}} f(x) \\ &\lesssim \sum_{l=-\infty}^\infty 2^{|j-l|(s+n-t)} g_l(x). \end{aligned}$$

By virtue of Lemma 3.9, we have

$$\begin{aligned}
\|f\|_{\mathcal{E}^s_{\mathcal{M}^\varphi_q,r}} &\lesssim \|f\|_{\mathcal{M}^\varphi_q} + \left\| \left\{ 2^{js} \int_{\mathbb{R}^n} |g(h)| \cdot \left|\Delta^M_{2^{-j}h} f(x)\right| \, \mathrm{d}h \right\}_{j=0}^{\infty} \right\|_{\mathcal{M}^\varphi_q(\ell^r)} \\
&\lesssim \|f\|_{\mathcal{M}^\varphi_q} + \left\| \left\{ \sum_{k=-\infty}^{\infty} 2^{|j-k|(s+n-t)} g_k \right\}_{j=0}^{\infty} \right\|_{\mathcal{M}^\varphi_q(\ell^r)} \\
&\lesssim \|f\|_{\mathcal{M}^\varphi_q} + \left\| \{g_j\}_{j=-\infty}^{\infty} \right\|_{\mathcal{M}^\varphi_q(\ell^r)} \\
&= \|f\|^{**}_{\mathcal{E}^s_{\mathcal{M}^\varphi_q,r}}.
\end{aligned}$$

Finally we shall prove the case $\mathcal{A} = \mathcal{N}$. Following the argument as above, we have

$$\begin{aligned}
\|f\|_{\mathcal{N}^s_{\mathcal{M}^\varphi_q,r}} &\sim \|\{2^{js}\kappa_j(D)f\}_{j=0}^{\infty}\|_{\mathcal{M}^\varphi_q(\ell^r)} \\
&\lesssim \|f\|_{\mathcal{M}^\varphi_q} + \left\| \left\{ 2^{js} \int_{\mathbb{R}^n} |g(h)| \cdot \left|\Delta^M_{2^{-j}h} f(x)\right| \, \mathrm{d}h \right\}_{j=0}^{\infty} \right\|_{\ell^r(\mathcal{M}^\varphi_q)}.
\end{aligned}$$

Using (4.29) with $t > s+n$ and applying Lemma 3.9, we have

$$\begin{aligned}
\|f\|_{\mathcal{N}^s_{\mathcal{M}^\varphi_q,r}} &\lesssim \|f\|_{\mathcal{M}^\varphi_q} + \left\| \left\{ 2^{js} \int_{\mathbb{R}^n} |g(h)| \cdot \left|\Delta^M_{2^{-j}h} f(x)\right| \, \mathrm{d}h \right\}_{j=0}^{\infty} \right\|_{\ell^r(\mathcal{M}^\varphi_q)} \\
&\lesssim \|f\|_{\mathcal{M}^\varphi_q} + \left\| \left\{ \sum_{k=-\infty}^{\infty} 2^{|j-k|(s+n-t)} g_k \right\}_{j=0}^{\infty} \right\|_{\ell^r(\mathcal{M}^\varphi_q)} \\
&\lesssim \|f\|_{\mathcal{M}^\varphi_q} + \left\| \{g_j\}_{j=-\infty}^{\infty} \right\|_{\ell^r(\mathcal{M}^\varphi_q)} \\
&= \|f\|^{**}_{\mathcal{N}^s_{\mathcal{M}^\varphi_q,r}}.
\end{aligned}$$

§ 5. Characterization of $\mathcal{A}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)$ by differences

Theorem 5.1. *Let $0 < q < \infty$, $0 < r \le \infty$ and $s > \dfrac{n}{\min(1,q,r)}$. If M is an integer such that $M > s$, then the two quasi-norms*

$$\|f\|^{(1)}_{\mathcal{E}^s_{\mathcal{M}^\varphi_q,r}} = \|f\|_{\mathcal{M}^\varphi_q} + \left\| \left(\int_{\mathbb{R}^n} |h|^{-sr} \sup_{\substack{|\rho|\le|h| \\ \rho\in\mathbb{R}^n}} |(\Delta^M_\rho f)(\cdot)|^r \frac{\mathrm{d}h}{|h|^n} \right)^{1/r} \right\|_{\mathcal{M}^\varphi_q}, \tag{5.1}$$

$$\|f\|^{(2)}_{\mathcal{E}^s_{\mathcal{M}^\varphi_q,r}} = \|f\|_{\mathcal{M}^\varphi_q} + \left\| \left(\int_{\mathbb{R}^n} |h|^{-sr} |(\Delta^M_h f)(\cdot)|^r \frac{\mathrm{d}h}{|h|^n} \right)^{1/r} \right\|_{\mathcal{M}^\varphi_q} \tag{5.2}$$

are equivalent on $\mathcal{E}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)$.

§ 5.1. Proof of Theorem 5.1

We rely on the proof of [16, Section 2.5.10, Theorem] and [8, Theorem 8.2].

Step 1. Firstly, we prove that $\|f\|^{(1)}_{\mathcal{E}^s_{\mathcal{M}^\varphi_{q,r}}} \lesssim \|f\|_{\mathcal{E}^s_{\mathcal{M}^\varphi_{q,r}}}$ holds for all $f \in \mathcal{E}^s_{\mathcal{M}^\varphi_{q,r}}(\mathbb{R}^n)$. Let $f \in \mathcal{E}^s_{\mathcal{M}^\varphi_{q,r}}(\mathbb{R}^n)$. Note that $\|f\|_{\mathcal{M}^\varphi_q} \lesssim \|f\|_{\mathcal{E}^s_{\mathcal{M}^\varphi_{q,r}}}$. By Lemma 3.6 we see that f is bounded and continuous on $\mathbb{R}^n$ under the conditions for s. We have

$$\left\| \left(\int_{\mathbb{R}^n} |h|^{-sr} \sup_{\substack{|\rho|\le|h| \\ \rho\in\mathbb{R}^n}} |(\Delta^M_\rho f)(\cdot)|^r \frac{\mathrm{d}h}{|h|^n} \right)^{1/r} \right\|_{\mathcal{M}^\varphi_q} \lesssim \left\| \left(\sum_{k=-\infty}^{\infty} 2^{ksr} \sup_{0<|h|\le 2^{-k}} |(\Delta^M_h f)(\cdot)|^r \right)^{1/r} \right\|_{\mathcal{M}^\varphi_q}. \tag{5.3}$$

Let $\{\theta_k(x)\}_{k=0}^\infty \in \Theta(\mathbb{R}^n)$ and $\theta_k(x) = 0$ if $k = -1, -2, \ldots$. Then, if $0 < r \le 1$, by Jensen's inequality, we obtain

$$\begin{aligned} \sum_{k=-\infty}^{\infty} 2^{ksr} \sup_{0<|h|\le 2^{-k}} |(\Delta^M_h f)(x)|^r &\le \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} 2^{ksr} \sup_{0<|h|\le 2^{-k}} |(\Delta^M_h \mathcal{F}^{-1}[\theta_{k+m}\mathcal{F}f])(x)|^r \\ &= \sum_{m=-\infty}^{-1} \sum_{k=-\infty}^{\infty} 2^{ksr} \sup_{0<|h|\le 2^{-k}} |(\Delta^M_h \mathcal{F}^{-1}[\theta_{k+m}\mathcal{F}f])(x)|^r \\ &\quad + \sum_{m=0}^{\infty} \sum_{k=-\infty}^{\infty} 2^{ksr} \sup_{0<|h|\le 2^{-k}} |(\Delta^M_h \mathcal{F}^{-1}[\theta_{k+m}\mathcal{F}f])(x)|^r \\ &=: I_{m\le -1} + I_{m\ge 0}. \end{aligned} \tag{5.4}$$

We estimate $I_{m\le-1}$. For any $|h| \le 2^{-k}$, by using the mean value theorem, there exists an appropriate positive constant c such that

$$|\Delta^M_h \mathcal{F}^{-1}[\theta_{k+m}\mathcal{F}f](x)| \le 2^{-kM} \sup_{|x-y|\le c2^{-k}} \sum_{|\alpha|=M} |D^\alpha \mathcal{F}^{-1}[\theta_{k+m}\mathcal{F}f](y)|. \tag{5.5}$$

Let $(\theta^*_k f)(x)$ be the Peetre maximal function. By virtue of [16, (7) in Section 2.5.10], we have

$$|D^\alpha \mathcal{F}^{-1}[\theta_{k+m}\mathcal{F}f](y)| \lesssim 2^{(k+m)M} (\theta^*_{k+m} f)(y). \tag{5.6}$$

Putting this estimate into (5.5), we have

$$I_{m\le -1} = \sum_{m=-\infty}^{-1}\sum_{k=-\infty}^{\infty} 2^{ksr}\sup_{0<|h|\le 2^{-k}}|(\Delta_h^M\mathcal{F}^{-1}[\theta_{k+m}\mathcal{F}f])(x)|^r$$
$$\lesssim \sum_{m=-\infty}^{-1}\sum_{k=-\infty}^{\infty} 2^{ksr}2^{-kMr}2^{(k+m)Mr}\sup_{|x-y|\le c2^{-k}}(\theta_{k+m}^*f)^r(y)$$
$$\lesssim \sum_{m=-\infty}^{-1}\sum_{k=-\infty}^{\infty} 2^{ksr}2^{-kMr}2^{(k+m)Mr}(\theta_{k+m}^*f)^r(x)$$
$$\lesssim \sum_{j=0}^{\infty} 2^{sjr}(\theta_j^*f)(x)^r. \tag{5.7}$$

We estimate $I_{m\ge 0}$. It is easy to see that $|(\Delta_h^M\mathcal{F}^{-1}[\theta_{k+m}\mathcal{F}f])(x)| \lesssim 2^{ma}(\theta_{k+m}^*f)(x)$, where a is the same as in definition of Peetre maximal function. Let $s > a > \dfrac{n}{\min(1,q,r)}$. Then we have

$$I_{m\ge 0} = \sum_{m=0}^{\infty}\sum_{k=-\infty}^{\infty} 2^{ksr}\sup_{0<|h|<2^{-k}}|(\Delta_h^M\mathcal{F}^{-1}[\theta_{k+m}\mathcal{F}f])(x)|^r$$
$$\lesssim \sum_{m=0}^{\infty}\sum_{k=-\infty}^{\infty} 2^{s(k+m)r}(\theta_{k+m}^*f)^r(x)2^{-m(s-a)r}$$
$$\lesssim \sum_{j=0}^{\infty} 2^{jsr}(\theta_j^*f)^r(x). \tag{5.8}$$

The sum of the left hand sides of (5.7) and (5.8) yields the right hand side of (5.4). By (5.3) and (5.4) we have

$$\left\| \left(\int_{\mathbb{R}^n} |h|^{-sr} \sup_{\substack{|\rho|\le|h|\\ \rho\in\mathbb{R}^n}} |(\Delta_\rho^M f)(\cdot)|^r \frac{\mathrm{d}h}{|h^n|} \right)^{1/r} \right\|_{\mathcal{M}_q^\varphi} \lesssim \|2^{sj}\theta_j^*f\|_{\mathcal{M}_q^\varphi(\ell^r)}. \tag{5.9}$$

If $1 < r \le \infty$, the left hand side of (5.4) is estimated as follows:

$$\left(\sum_{k=-\infty}^{\infty} 2^{ksr}\sup_{0<|h|\le 2^{-k}}|(\Delta_h^M f)(x)|^r\right)^{1/r}$$
$$\le \left(\sum_{k=-\infty}^{\infty} (\sum_{m=-\infty}^{\infty}\sup_{0<|h|\le 2^{-k}}|(2^{ks}\Delta_h^M\theta_{k+m}(D)f)(x)|)^r\right)^{1/r}$$
$$\le \sum_{m=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty}\sup_{0<|h|\le 2^{-k}}|(2^{ks}\Delta_h^M\theta_{k+m}(D)f)(x)|^r\right)^{1/r},$$

where we have used the triangle inequality for the ℓ^r-norm in order to obtain the last inequality. We split the sum $\sum_{m=-\infty}^{\infty}$ into $\sum_{m=-\infty}^{-1}$ and $\sum_{m=0}^{\infty}$. Using an argument similar to the cases $I_{m\le -1}$ and $I_{m\ge 0}$, we obtain (5.9) for $1<r\le\infty$. By virtue of $a>\dfrac{n}{\min(1,q,r)}$ and Theorem 3.4, the right hand side of (5.9) can be dominated by $\|f\|_{\mathcal{E}^s_{\mathcal{M}^{\varphi}_{q,r}}}$.

Step 2. We prove that $\|f\|_{\mathcal{E}^s_{\mathcal{M}^{\varphi}_{q,r}}}\lesssim \|f\|^{(2)}_{\mathcal{E}^s_{\mathcal{M}^{\varphi}_{q,r}}}$ holds for all $f\in L^1_{loc}(\mathbb{R}^n)\cap\mathcal{S}'(\mathbb{R}^n)$ with $\|f\|^{(2)}_{\mathcal{E}^s_{\mathcal{M}^{\varphi}_{q,r}}}<\infty$. Let $\psi\in C_0^\infty(\mathbb{R}^n)$ with $\psi(x)=1$ when $|x|\le 1$ and $\psi(x)=0$ when $|x|>3/2$. We define $\kappa_0(x)=(-1)^{M+1}\sum_{\mu=0}^{M-1}(-1)^{\mu}\binom{M}{\mu}\psi((M-\mu)x)$. Note that $\kappa_0\in C_0^\infty(\mathbb{R}^n)$ with $\kappa_0(x)=0$ when $|x|>3/2$ and $\kappa_0(x)=1$ when $|x|<1/M$. We define $\kappa_j(x)=\kappa_0(2^{-j}x)-\kappa_0(2^{-j+1}x)$ for $j\in\mathbb{N}$. Note that $\{\kappa_j\}_{j=0}^\infty$ is a decomposition of unity. Kempka and Vybíral [6, (49)] proved that $\left|\left(\mathcal{F}^{-1}\left(\Delta^M_{2^{-j}\xi}\psi(0)\right)\mathcal{F}f\right)(x)\right|\lesssim \int_{\mathbb{R}^n}|\hat{\psi}(h)|\cdot|\Delta^M_{2^{-j}h}f(x)|\,\mathrm{d}h$ holds for any $j\in\mathbb{N}_0$. We put $g=\hat{\psi}\in\mathcal{S}(\mathbb{R}^n)$ and obtain

$$
\begin{aligned}
(5.10)\qquad \|f\|_{\mathcal{E}^s_{\mathcal{M}^{\varphi}_{q,r}}} &\sim \|\{2^{js}\kappa_j(D)f\}_{j=0}^\infty\|_{\mathcal{M}^{\varphi}_q(\ell^r)}\\
&\lesssim \|f\|_{\mathcal{M}^{\varphi}_q}+\left\|\left\{2^{js}\int_{\mathbb{R}^n}|g(h)|\cdot\left|\Delta^M_{2^{-j}h}f(x)\right|\,\mathrm{d}h\right\}_{j=0}^\infty\right\|_{\mathcal{M}^{\varphi}_q(\ell^r)}.
\end{aligned}
$$

Let $\gamma_n=\{x\in\mathbb{R}^n : |x|=1\}$. By using the polar coordinate system, we have

(5.11)

$$
\begin{aligned}
\int_{\mathbb{R}^n}|g(h)|\cdot\left|\Delta^M_{2^{-j}h}f(x)\right|\,\mathrm{d}h &\lesssim \int_0^\infty\int_{\gamma_n}R^{n-1}|g(Rz)|\cdot|\Delta^M_{2^{-j}Rz}f(x)|\,\mathrm{d}z\mathrm{d}R\\
&=\sum_{\ell=-\infty}^{\infty}\int_{2^\ell}^{2^{\ell+1}}\int_{\gamma_n}R^{n-1}|g(Rz)|\cdot|\Delta^M_{2^{-j}Rz}f(x)|\,\mathrm{d}z\mathrm{d}R\\
&=\sum_{\ell=-\infty}^{\infty}\int_1^2 2^\ell\int_{\gamma_n}(2^\ell R)^{n-1}|g(2^\ell Rz)|\cdot|\Delta^M_{2^{-j+\ell}Rz}f(x)|\,\mathrm{d}z\mathrm{d}R,
\end{aligned}
$$

where, in the last equality, we changed the variable R to $2^\ell R$. Let $\Gamma_n=\{x\in\mathbb{R}^n : 1\le$

$|x| \le 2\}$. By substituting Rz to z, we obtain that for all $\ell \in \mathbb{Z}$,

$$\begin{aligned}
(5.12)\qquad & \sum_{\ell=-\infty}^{\infty} \int_1^2 2^{\ell} \int_{\gamma_n} (2^{\ell}R)^{n-1} |g(2^{\ell}Rz)| \cdot |\Delta^M_{2^{-j+\ell}Rz} f(x)| \,\mathrm{d}z\mathrm{d}R \\
&= \sum_{\ell=-\infty}^{\infty} \int_1^2 2^{\ell} \int_{\Gamma_n} 2^{\ell(n-1)} R^{-1} |g(2^{\ell}z)| \cdot |\Delta^M_{2^{-j+\ell}z} f(x)| \,\mathrm{d}z\mathrm{d}R \\
&\lesssim \sum_{\ell=-\infty}^{\infty} \int_{\Gamma_n} 2^{n\ell} |g(2^{\ell}z)| \cdot |\Delta^M_{2^{-j+\ell}z} f(x)| \,\mathrm{d}z.
\end{aligned}$$

Note that $|g(2^{\ell}z)| \lesssim 2^{-\ell\sigma}$ holds for all $z \in \Gamma_n$, $\sigma > 0$ and $\ell \in \mathbb{N}$. Let $\sigma > s$. Thus we get

$$\begin{aligned}
(5.13)\qquad & \sum_{\ell=-\infty}^{\infty} \int_{\Gamma_n} 2^{n\ell} |g(2^{\ell}z)| \cdot |\Delta^M_{2^{-j+\ell}z} f(x)| \,\mathrm{d}z \\
&\lesssim \sum_{\ell=-\infty}^{0} \int_{\Gamma_n} |\Delta^M_{2^{-j+\ell}z} f(x)| \,\mathrm{d}z + \sum_{\ell=1}^{\infty} 2^{-\ell\sigma} \int_{\Gamma_n} \cdot |\Delta^M_{2^{-j+\ell}z} f(x)| \,\mathrm{d}z
\end{aligned}$$

If $0 < r \le 1$ then we have

(5.14)

$$\begin{aligned}
& \sum_{j=0}^{\infty} \left(2^{js} \int_{\mathbb{R}^n} |g(h)| \cdot \left|\Delta^M_{2^{-j}h} f(x)\right| \,\mathrm{d}h \right)^r \\
&= \sum_{j=0}^{\infty} \left(2^{js} \sum_{\ell=-\infty}^{0} \int_{\Gamma_n} |\Delta^M_{2^{-j+\ell}z} f(x)| \,\mathrm{d}z + 2^{js} \sum_{\ell=1}^{\infty} 2^{-\ell\sigma} \int_{\Gamma_n} |\Delta^M_{2^{-j+\ell}z} f(x)| \,\mathrm{d}z \right)^r \\
&\lesssim \sum_{j=0}^{\infty} \sum_{\ell=-\infty}^{0} \left(2^{js} \int_{\Gamma_n} |\Delta^M_{2^{-j+\ell}z} f(x)| \,\mathrm{d}z \right)^r + \sum_{j=0}^{\infty} \sum_{\ell=1}^{\infty} \left(2^{js} 2^{-\ell\sigma} \int_{\Gamma_n} |\Delta^M_{2^{-j+\ell}z} f(x)| \,\mathrm{d}z \right)^r \\
&\lesssim \sum_{j=-\infty}^{\infty} 2^{sjr} \left(\int_{\Gamma_n} \left|\Delta^M_{2^{-j}z} f(x)\right| \,\mathrm{d}z \right)^r,
\end{aligned}$$

where, in the last inequality, we substitute $j - \ell$ to j. If $1 < r \le \infty$, then we use the triangle inequality for ℓ^r-norm and obtain a corresponding inequality. Hence, (5.14) holds for $0 < r \le \infty$. If $1 \le r \le \infty$, then we have

$$(5.15)\qquad \left(\int_{\Gamma_n} |(\Delta^M_{2^{-j}z} f)(x)| \,\mathrm{d}z \right)^r \lesssim \int_{\Gamma_n} |(\Delta^M_{2^{-j}z} f)(x)|^r \,\mathrm{d}z.$$

Putting (5.15) into (5.14), taking the $1/r$-power and applying the $\mathcal{M}^{\varphi}_q$-quasi-norm, we obtain

$$\left\| \left\{ 2^{js} \int_{\mathbb{R}^n} |g(h)| \cdot \left|\Delta^M_{2^{-j}h} f(\cdot)\right| \,\mathrm{d}h \right\}_{j=0}^{\infty} \right\|_{\mathcal{M}^{\varphi}_q(\ell^r)} \lesssim \|f\|^{(2)}_{\mathcal{E}^s_{\mathcal{M}^{\varphi}_{q,r}}}.$$

If $0<r<1$, then (5.14) yields

$$\begin{aligned}(5.16)\quad &\sum_{j=0}^{\infty}\left(2^{js}\int_{\mathbb{R}^n}|g(h)|\cdot\left|\Delta^M_{2^{-j}h}f(x)\right|\,\mathrm{d}h\right)^r\\ &\lesssim\sum_{j=-\infty}^{\infty}2^{sjr}\left(\int_{\Gamma_n}\left|\Delta^M_{2^{-j}z}f(x)\right|\,\mathrm{d}z\right)^r\\ &\lesssim\sum_{j=-\infty}^{\infty}2^{sjr}\left(\int_{\Gamma_n}\left|\Delta^M_{2^{-j}z}f(x)\right|^r\,\mathrm{d}z\right)^r\sup_{z\in\Gamma_n}|(\Delta^M_{2^{-j}z}f)(x)|^{r(1-r)}.\end{aligned}$$

By Hölder's inequality, we have

$$\begin{aligned}(5.17)\quad &\sum_{j=0}^{\infty}\left(2^{js}\int_{\mathbb{R}^n}|g(h)|\cdot\left|\Delta^M_{2^{-j}h}f(x)\right|\,\mathrm{d}h\right)^r\\ &\lesssim\sum_{j=-\infty}^{\infty}2^{jsr^2}\left(\int_{\Gamma_n}\left|\Delta^M_{2^{-j}z}f(x)\right|^r\,\mathrm{d}z\right)^r\sup_{z\in\Gamma_n}2^{jsr(1-r)}|(\Delta^M_{2^{-j}z}f)(x)|^{r(1-r)}\\ &\lesssim\left(\sum_{j=-\infty}^{\infty}2^{jsr}\int_{\Gamma_n}\left|\Delta^M_{2^{-j}z}f(x)\right|^r\,\mathrm{d}z\right)^r\left(\sum_{j=-\infty}^{\infty}\sup_{z\in\Gamma_n}2^{jsr}|(\Delta^M_{2^{-j}z}f)(x)|^r\right)^{1-r}.\end{aligned}$$

Taking $1/r$-power, applying the $\mathcal{M}^{\varphi}_q$-quasi-norm, and using Hölder's inequality again we obtain $\left\|\left\{2^{js}\int_{\mathbb{R}^n}|g(h)|\cdot\left|\Delta^M_{2^{-j}h}f(\cdot)\right|\,\mathrm{d}h\right\}_{j=0}^{\infty}\right\|_{\mathcal{M}^{\varphi}_q(\ell^r)}\lesssim\left(\|f\|^{(2)}_{\mathcal{E}^s_{\mathcal{M}^{\varphi}_{q,r}}}\right)^r\left(|f\|^{(1)}_{\mathcal{E}^s_{\mathcal{M}^{\varphi}_{q,r}}}\right)^{1-r}$. This proves $\|f\|_{\mathcal{E}^s_{\mathcal{M}^{\varphi}_{q,r}}}\lesssim\|f\|^{(2)}_{\mathcal{E}^s_{\mathcal{M}^{\varphi}_{q,r}}}$.

§ 5.2. Besov cases

Theorem 5.2. *Let $0<q<\infty$, $0<r\le\infty$ and $s>\sigma_q$. If M is an integer such that $M>s$, then*

$$(5.18)\quad \|f\|^{(1),M}_{\mathcal{N}^s_{\mathcal{M}^{\varphi}_{q,r}}}=\|f\|_{\mathcal{M}^{\varphi}_q}+\left(\int_{\mathbb{R}^n}|h|^{-sr}\sup_{\substack{|\rho|\le|h|\\ \rho\in\mathbb{R}^n}}\left\|\Delta^M_{\rho}f\right\|^r_{\mathcal{M}^{\varphi}_q}\frac{\mathrm{d}h}{|h|^n}\right)^{1/r}$$

is an equivalent quasi-norm in $\mathcal{N}^s_{\mathcal{M}^{\varphi}_{q,r}}(\mathbb{R}^n)$. Furthermore,

$$(5.19)\quad \|f\|^{(2),M}_{\mathcal{N}^s_{\mathcal{M}^{\varphi}_{q,r}}}=\|f\|_{\mathcal{M}^{\varphi}_q}+\left(\int_{\mathbb{R}^n}|h|^{-sr}\left\|\Delta^M_h f\right\|^r_{\mathcal{M}^{\varphi}_q}\frac{\mathrm{d}h}{|h|^n}\right)^{1/r}$$

is also an equivalent quasi-norm in $\mathcal{N}^s_{\mathcal{M}^{\varphi}_{q,r}}(\mathbb{R}^n)$.

One can replace $\int_{\mathbb{R}^n}$ in (5.18) and (5.19) by $\int_{|h|\le1}$.

We simply write $\|f\|^{(1)}_{\mathcal{N}^s_{\mathcal{M}^\varphi_{q,r}}} = \|f\|^{(1),M}_{\mathcal{N}^s_{\mathcal{M}^\varphi_{q,r}}}$ and $\|f\|^{(2)}_{\mathcal{N}^s_{\mathcal{M}^\varphi_{q,r}}} = \|f\|^{(2),M}_{\mathcal{N}^s_{\mathcal{M}^\varphi_{q,r}}}$, respectively when we do not need to emphasize M.

Proof. We rely on the proof of [16, p 110, Section 2.5.12, Theorem] and [2, Theorem 3.18].

Step 1. If $f \in \mathcal{N}^s_{\mathcal{M}^\varphi_{q,r}}(\mathbb{R}^n)$ with $s > \sigma_q$, then we have $\|f\|_{\mathcal{M}^\varphi_q} \lesssim \|f\|_{\mathcal{N}^s_{\mathcal{M}^\varphi_{q,r}}}$. In this step, we prove $\|f\|^{(1)}_{\mathcal{N}^s_{\mathcal{M}^\varphi_{q,r}}} \lesssim \|f\|_{\mathcal{N}^s_{\mathcal{M}^\varphi_{q,r}}}$ for any $f \in \mathcal{N}^s_{\mathcal{M}^\varphi_{q,r}}(\mathbb{R}^n)$. Let $f \in \mathcal{N}^s_{\mathcal{M}^\varphi_{q,r}}(\mathbb{R}^n)$ and $\{\theta_j(x)\}_{j=0}^\infty \in \Theta(\mathbb{R}^n)$. Then $f = \sum_{j=0}^\infty \theta_j(D)f$ holds not only in the sense of $\mathcal{S}'(\mathbb{R}^n)$ but also in the sense of $\mathcal{M}^\varphi_q(\mathbb{R}^n)$. We have

$$|(\Delta^M_\rho \theta_j(D)f)(x)| \lesssim 2^{(j-k)M}(\theta^*_j f)(x) \quad (x \in \mathbb{R}^n) \tag{5.20}$$

if $|\rho| \le 2^{-k}$ and $j = 0, 1, \ldots, k$. Hence we obtain

$$\sup_{|\rho|\le 2^{-k}} \left\|\Delta^M_\rho \theta_j(D)f\right\|_{\mathcal{M}^\varphi_q} \lesssim \min(1, 2^{(j-k)M}) \left\|\theta^*_j f\right\|_{\mathcal{M}^\varphi_q}. \tag{5.21}$$

Let $q^* = \min(1, q)$. Then we get

$$\begin{aligned}\sup_{|\rho|\le 2^{-k}} \left\|\Delta^M_\rho f\right\|^{q^*}_{\mathcal{M}^\varphi_q} &\lesssim \sum_{j=0}^\infty \sup_{|\rho|\le 2^{-k}} \left\|\Delta^M_\rho \theta_j(D)f\right\|^{q^*}_{\mathcal{M}^\varphi_q} \\ &\lesssim \sum_{j=0}^k 2^{(j-k)Mq}\|\theta^*_j f\|^{q^*}_{\mathcal{M}^\varphi_q} + \sum_{j=k+1}^\infty \|\theta^*_j f\|^{q^*}_{\mathcal{M}^\varphi_q}.\end{aligned}$$

It follows that

(5.22)

$$\begin{aligned}&\|f\|^{(1)r}_{\mathcal{N}^s_{\mathcal{M}^\varphi_{q,r}}} \\ &\lesssim \|f\|^r_{\mathcal{M}^\varphi_q} + \sum_{k=0}^\infty 2^{ksr} \sup_{|\rho|\le 2^{-k}} \|\Delta^M_\rho f\|^r_{\mathcal{M}^\varphi_q} \\ &\lesssim \|f\|^r_{\mathcal{M}^\varphi_q} + \sum_{k=0}^\infty 2^{ksr}\left(\sum_{j=0}^k 2^{(j-k)Mq^*}\|\theta^*_j f\|^{q^*}_{\mathcal{M}^\varphi_q}\right)^{r/q^*} + \sum_{k=0}^\infty 2^{ksr}\left(\sum_{j=k+1}^\infty \|\theta^*_j f\|^{q^*}_{\mathcal{M}^\varphi_q}\right)^{r/q^*}.\end{aligned}$$

Let ϵ be a positive number such that $0 < \epsilon < s < s + \epsilon < M$. Then we have

$$\begin{aligned}\sum_{k=0}^\infty 2^{ksr}\left(\sum_{j=0}^k 2^{(j-k)Mq^*}\|\theta^*_j f\|^{q^*}_{\mathcal{M}^\varphi_q}\right)^{r/q^*} &\lesssim \sum_{k=0}^\infty \sum_{j=0}^k 2^{(j-k)(M-\epsilon-s)r} 2^{jsr}\|\theta^*_j f\|^r_{\mathcal{M}^\varphi_q} \\ &\lesssim \sum_{j=0}^\infty 2^{jsr}\|\theta^*_j f\|^r_{\mathcal{M}^\varphi_q}\end{aligned} \tag{5.23}$$

and

$$\sum_{k=0}^{\infty} 2^{ksr} \left(\sum_{j=k+1}^{\infty} \|\theta_j^* f\|_{\mathcal{M}_q^\varphi}^{q^*} \right)^{r/q^*} \lesssim \sum_{k=0}^{\infty} \sum_{j=k+1}^{\infty} 2^{(k-j)(s-\epsilon)r} 2^{jsr} \|\theta_j^* f\|_{\mathcal{M}_q^\varphi}^{r} \lesssim \sum_{j=0}^{\infty} 2^{jsr} \|\theta_j^* f\|_{\mathcal{M}_q^\varphi}^{r}. \tag{5.24}$$

Substituting (5.23) and (5.24) into (5.22), we get

$$\|f\|_{\mathcal{N}^s_{\mathcal{M}^\varphi_{q,r}}}^{(1)r} \lesssim \|f\|_{\mathcal{M}_q^\varphi}^{r} + \sum_{j=0}^{\infty} 2^{jsr} \|\theta_j^* f\|_{\mathcal{M}_q^\varphi}^{r}.$$

By the boundedness of the Peetre maximal function (Theorem 3.4), we have $\|f\|_{\mathcal{N}^s_{\mathcal{M}^\varphi_{q,r}}}^{(1)} \lesssim \|f\|_{\mathcal{N}^s_{\mathcal{M}^\varphi_{q,r}}}$.

Step 2. We prove $\|f\|_{\mathcal{N}^s_{\mathcal{M}^\varphi_{q,r}}} \lesssim \|f\|_{\mathcal{N}^s_{\mathcal{M}^\varphi_{q,r}}}^{(1)}$. Because we assume that ρ satisfies $\operatorname{supp} \rho \subset B(0, \frac{1}{2^2 M})$, by (4.11) and (4.12) we have

$$\begin{aligned}
|(T_0 * f)(x)| &= \left| \frac{(-1)^M}{M!} \sum_{m=1}^{M} (-1)^{M-m} \binom{M}{m} m^{M-n} \int_{\mathbb{R}^n} \rho(-\frac{y}{m}) \Delta_y f(x) \, \mathrm{d}y + f(x) \right| \\
&\lesssim \sup_{|h| \le 2^{-1}} |\Delta_h^M f(x)| + |f(x)|
\end{aligned}$$

and

$$\begin{aligned}
&|(T_j * f)(x)| \\
&= \left| \frac{(-1)^{M+1}}{M!} \sum_{m=1}^{M} (-1)^{M-m} \binom{M}{m} m^{M-n} \int_{\mathbb{R}^n} \left[\rho(-\frac{y}{m}) - 2^{-n} \rho(-\frac{y}{2m}) \right] \Delta_{2^{-j}y}^M f(x) \, \mathrm{d}y \right| \\
&\lesssim \sup_{|h| \le 2^{-j-1}} |\Delta_h^M f(x)|.
\end{aligned}$$

Therefore, for each $j \in \mathbb{N}$, we obtain $\|T_0 * f\|_{\mathcal{M}_q^\varphi} \lesssim \|f\|_{\mathcal{M}_q^\varphi} + \sup_{|h| \le 2^{-1}} \|\Delta_h^M f\|_{\mathcal{M}_q^\varphi}$ and

$\|T_j * f\|_{\mathcal{M}_q^\varphi} \lesssim \sup_{|h|\le 2^{-j-1}} \|\Delta_h^M f\|_{\mathcal{M}_q^\varphi}$. Hence by Lemma 4.5 we have

$$\begin{aligned}
\|f\|_{\mathcal{N}^s_{\mathcal{M}^\varphi_{q},r}} &= \left(\sum_{j=0}^\infty 2^{jsr}\|T_j * f\|^r_{\mathcal{M}_q^\varphi}\right)^{1/r}\\
&\lesssim \|f\|_{\mathcal{M}_q^\varphi} + \left(\sum_{j=0}^\infty 2^{jsr} \sup_{|h|\le 2^{-j-1}} \|\Delta_h^M f\|^r_{\mathcal{M}_q^\varphi}\right)^{1/r}\\
&\lesssim \|f\|_{\mathcal{M}_q^\varphi} + \left(\sum_{j=0}^\infty \int_{2^{-j-1}}^{2^{-j}} t^{-sr} \sup_{|h|\le 2^{-j-1}} \|\Delta_h^M f\|^r_{\mathcal{M}_q^\varphi} \frac{\mathrm{d}t}{t}\right)^{1/r}\\
&\lesssim \|f\|_{\mathcal{M}_q^\varphi} + \left(\int_0^1 t^{-sr} \sup_{|h|\le t} \|\Delta_h^M f\|^r_{\mathcal{M}_q^\varphi} \frac{\mathrm{d}t}{t}\right)^{1/r}.
\end{aligned}$$

Additionally using the polar coordinate transformation we have

$$\|f\|_{\mathcal{N}^s_{\mathcal{M}^\varphi_{q},r}} \lesssim \|f\|_{\mathcal{M}_q^\varphi} + \left(\int_{|h|\le 1} |h|^{-sr} \sup_{\substack{|\rho|\le|h|\\ \rho\in\mathbb{R}^n}} \left\|\Delta_\rho^M f\right\|^r_{\mathcal{M}_q^\varphi} \frac{\mathrm{d}h}{|h|^n}\right)^{1/r} \lesssim \|f\|^{(1)}_{\mathcal{N}^s_{\mathcal{M}^\varphi_{q},r}}. \tag{5.25}$$

Step 3. We prove $\|f\|_{\mathcal{N}^s_{\mathcal{M}^\varphi_{q},r}} \lesssim \|f\|^{(2)}_{\mathcal{N}^s_{\mathcal{M}^\varphi_{q},r}}$. It is easy to see that

$$I \equiv \int_{\mathbb{R}^n} |h|^{-sr} \sup_{\substack{|\rho|\le|h|\\ \rho\in\mathbb{R}^n}} \left\|\Delta_\rho^M f\right\|^r_{\mathcal{M}_q^\varphi} \frac{\mathrm{d}h}{|h|^n} \le \int_{\mathbb{R}^n} |h|^{-sr} \sup_{\substack{\frac{|h|}{2}\le|\rho|\le|h|\\ \rho\in\mathbb{R}^n}} \left\|\Delta_\rho^M f\right\|^r_{\mathcal{M}_q^\varphi} \frac{\mathrm{d}h}{|h|^n} + 2^{-sr} I. \tag{5.26}$$

This implies that I can be dominated by the first term on the right-hand side of (5.26). Hence, replacing $\sup\limits_{|\rho|\le|h|}$ by $\sup\limits_{\frac{|h|}{2}\le|\rho|\le|h|}$, we see that the right-hand side of (5.18) is also an equivalent quasi-norm in $\mathcal{N}^s_{\mathcal{M}^\varphi_{q},r}(\mathbb{R}^n)$. Let $\overline{O\rho}$ be the segment from the origin $O \in \mathbb{R}^n$ to ρ. Note that the length of the segment $\overline{O\rho}$ is in $[\frac{|h|}{2}, |h|]$. Then we define a ball K such that the center is on $\overline{O\rho}$ and the distance from origin O is $\dfrac{|h|}{8} + \dfrac{|h|}{16}$, and that the radius is $\dfrac{|h|}{16}$. We see that $K \subseteq \{x \in \mathbb{R}^n : \frac{|h|}{8} \le |x| \le \frac{|h|}{4}\}$. See Figure 1. Let $\rho = \rho_0 + \rho_1$ with $\rho_0 \in K$. Then, by using the same methods as [18, Section 2.5.12], we obtain $\|\Delta_\rho^{2M} f\|^r_{\mathcal{M}_q^\varphi} \lesssim \|\Delta_{\rho_0}^M f\|^r_{\mathcal{M}_q^\varphi} + \|\Delta_{\rho_1}^M f\|^r_{\mathcal{M}_q^\varphi}$. By taking integration over K we get

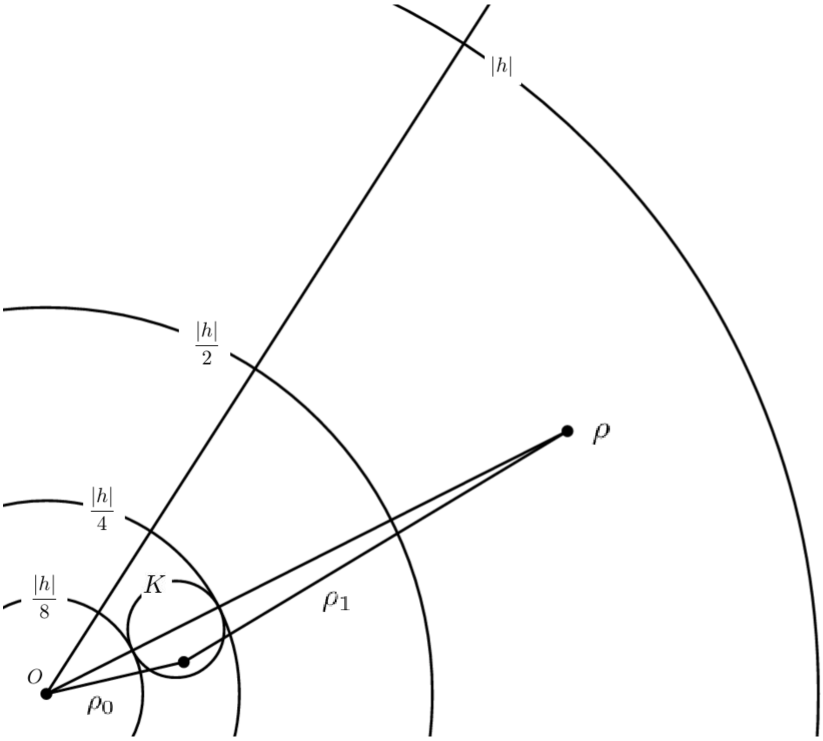

Figure 1.

$\|\Delta_\rho^{2M} f\|_{\mathcal{M}_q^\varphi}^r \lesssim \int_{\frac{|h|}{8} \le |\lambda| \le |h|} \|\Delta_\lambda^M f\|_{\mathcal{M}_q^\varphi}^r \,\mathrm{d}\lambda \cdot |h|^{-n}$. Therefore, we see that

$$\begin{aligned}
\|f\|_{\mathcal{N}^s_{\mathcal{M}_q^\varphi,r}}^{(1),2M} &= \int_{\mathbb{R}^n} |h|^{-sq-n} \sup_{\frac{|h|}{2} \le |\rho| \le |h|} \|\Delta_\rho^{2M} f\|_{\mathcal{M}_q^\varphi}^r \,\mathrm{d}h \\
&\lesssim \int_{\mathbb{R}^n} \int_{\frac{|h|}{8} \le |\lambda| \le |h|} |h|^{-sq-n} \|\Delta_\lambda^M f\|_{\mathcal{M}_q^\varphi}^r \,\mathrm{d}\lambda \, \frac{\mathrm{d}h}{|h|^n} \\
&\lesssim \sum_{k=-\infty}^{\infty} \int_{2^k \le |h| \le 2^{k+1}} \int_{\frac{|h|}{8} \le |\lambda| \le |h|} |\lambda|^{-sq-n} \|\Delta_\lambda^M f\|_{\mathcal{M}_q^\varphi}^r \,\mathrm{d}\lambda \, \frac{\mathrm{d}h}{|h|^n} \\
&\lesssim \sum_{k=-\infty}^{\infty} \int_{2^{k-1} \le |h| \le 2^k} \int_{2^{k-3} \le |\lambda| \le 2^k} |\lambda|^{-sq-n} \|\Delta_\lambda^M f\|_{\mathcal{M}_q^\varphi}^r \,\mathrm{d}\lambda \, \frac{\mathrm{d}h}{|h|^n} \\
&\lesssim \sum_{k=-\infty}^{\infty} \int_{2^{k-3} \le |\lambda| \le 2^k} |\lambda|^{-sq-n} \|\Delta_\lambda^M f\|_{\mathcal{M}_q^\varphi}^r \,\mathrm{d}\lambda \\
&\lesssim \int_{\mathbb{R}^n} |\lambda|^{-sq-n} \|\Delta_\lambda^M f\|_{\mathcal{M}_q^\varphi}^r \,\mathrm{d}\lambda \\
&= \|f\|_{\mathcal{N}^s_{\mathcal{M}_q^\varphi,r}}^{(2),M}.
\end{aligned}$$

This implies that $\|f\|_{\mathcal{N}^s_{\mathcal{M}_q^\varphi,r}} \lesssim \|f\|_{\mathcal{N}^s_{\mathcal{M}_q^\varphi,r}}^{(2)}$ by (5.25). □

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