RIMS Kôkyûroku Bessatsu
B75 (2019), 019–026

Intuitive representation of local cohomology groups

By

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Abstract

The theory of hyperfunctions had been introduced by M. Sato with algebraic method. A. Kaneko and M. Morimoto gave another definition which makes us easy to understand them elementarily. In this article, by generalizing their idea we construct a framework which enables us to define intuitive representation of local cohomology groups. This note is a summary of our forthcoming paper [5] by D. Komori and K. Umeta.

§ 1. Introduction

It is well-known that a hyperfunction in one variable is given by a boundary value of holomorphic functions on the upper half space and the lower half space in the complex plane. A hyperfunction in several variables is, however, defined by a local cohomology group and is not easy to understand. A. Kaneko and M. Morimoto defined it, roughly speaking, as a formal sum of holomorphic functions on infinitisimal wedges in [2]. Their idea can be applied to not only hyperfunctions but also local cohomology groups.

In this article, we generalize their idea and construct a framework which enables us to realize intuitive representation of local cohomology groups. To prove the equivalence of local cohomology groups and their intuitive representation, we study the boundary value map and its inverse map in detail.

As an application we have intuitive represenation of Laplace hyperfunctions [1], but we do not state it in this paper. We recommend the readers to make reference to our forthcoming paper [5].

Received April 19, 2017. Revised April 24, 2017.
2010 Mathematics Subject Classification(s): Primary 32A45; Secondary 32C36
Key Words: *hyperfunctions, local cohomology groups.*
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§ 2. Intuitive representation

In this section we introduce some definitions which are needed for construction of the framework, and we define intuitive representation of local cohomology groups.

§ 2.1. Preparations

Let X be a topological manifold and M a closed oriented submanifold of X. Here these manifolds are allowed to have their boundaries. We define $\widetilde{X} = M \times D^m$ and assume that $X \simeq \tilde{X}$ by the homeomorphism ι. Here $D^m = \{y \in \mathbb{R}^m; |y| < 1\}$ and $\iota : X \to \widetilde{X}$ satisfies $\iota(M) = M \times \{0\}$. Hereafter we identify $\widetilde{X}$ as X by ι. Set $S^{m-1} = \partial D^m$.

Definition 2.1. Let $k = 0, 1, \cdots, m-1$. We define a linear k-cell σ in S^{m-1} by

$$\sigma = \varphi(\cap H_i) \cap S^{m-1}. \tag{2.1}$$

Here the map $\varphi : \mathbb{R}^{k+1} \to \mathbb{R}^m$ is a linear injection and $\{H_i\}$ is a finite family of open half spaces in $\mathbb{R}^{k+1}$ whose intersection is non-empty.

Let us recall the definition of a stratification of a set U. For a set U, a partition $U = \bigsqcup_{\lambda \in \Lambda} U_\lambda$ is called a stratification of U, if it is locally finite, and it satisfies for all the pairs (U_λ, U_τ) that

$$\overline{U_\lambda} \cap U_\tau \neq \phi \;\Rightarrow\; U_\tau \subset \overline{U_\lambda}.$$

Let χ be a stratification of S^{m-1} such that each stratum is a linear cell of S^{m-1}. Moreover we denote by $\Delta(\chi)$ (resp. $\Delta_k(\chi)$) a set of all the cells of χ (resp. a set of all the k-cells of χ).

For two stratifications χ and χ' we introduce the order $\prec$.

Definition 2.2. For two stratifications χ and χ' of S^{m-1} we say that χ' is finer than χ if and only if for any $\sigma' \in \Delta(\chi')$ there exists $\sigma \in \Delta(\chi)$ such that $\sigma' \subset \sigma$, which is denoted by $\chi \prec \chi'$.

We introduce a property for two cells. Two k-cells σ_1 and σ_2 are adjacent to each other if there exists a $(k-1)$-cell τ for which $\overline{\sigma_1} \cap \overline{\sigma_2} = \overline{\tau}$ is satisfied.

Definition 2.3. For a cell $\sigma \in \Delta(\chi)$ a star open set $\mathrm{St}_\chi(\sigma)$ is defined by

$$\mathrm{St}_\chi(\sigma) = \bigsqcup_{\substack{\sigma \subset \overline{\tau} \\ \tau \in \Delta(\chi)}} \tau. \tag{2.2}$$

Definition 2.4. For a subset K of S^{m-1} we define $M * K \subset M \times D^m$ by

$$M * K = \{(x, ty) \in M \times D^m\,;\, x \in M, y \in K, 0 < t < 1\}. \tag{2.3}$$

§ 2.2. Intuitive representation and its framework

We introduce the framework which realizes intuitive representation of local cohomology groups.

Let $\mathcal{F}$ be a sheaf of $\mathbb{C}$-vector spaces on X, $\mathcal{W}$ a family of open sets in X and $\mathscr{T}$ a family of stratifications of S^{m-1}. We assume two conditions for $\mathscr{T}$ below.

1. Each $\chi \in \mathscr{T}$ is associated with partitioning by a finite family of hyperplanes in $\mathbb{R}^m$.
2. For any χ', $\chi'' \in \mathscr{T}$, there exists $\chi \in \mathscr{T}$ such that $\chi' \prec \chi$ and $\chi'' \prec \chi$.

Let T be an open neighborhood of M in X. Then we also assume that $(\mathcal{W}, \mathscr{T}, T)$ satisfies the following conditions.

(P-1) For any $\chi \in \mathscr{T}$ and for any $\sigma \in \Delta(\chi)$, we have $(M * \mathrm{St}(\sigma)) \cap T \in \mathcal{W}$ and

$$\mathrm{H}^k((M * \mathrm{St}(\sigma)) \cap T, \mathcal{F}) = 0 \quad (k \neq 0). \tag{2.4}$$

(P-2) (Existence of a finer asyclic stratification) For any $W \in \mathcal{W}$, there exist $\chi \in \mathscr{T}$ and an open neighborhood $U \subset T$ of M such that the following conditions hold.

(a) There exists $\sigma \in \Delta(\chi)$ such that $(M * \mathrm{St}(\sigma)) \cap U \subset W$.

(b) For any $\sigma \in \Delta(\chi)$, we have $(M * \mathrm{St}(\sigma)) \cap U \in \mathcal{W}$ and

$$\mathrm{H}^k((M * \mathrm{St}(\sigma)) \cap U, \mathcal{F}) = 0. \tag{2.5}$$

(P-3) (Cone connectivity of W) Let $W \in \mathcal{W}$, $\sigma_1 \in \Delta_{m-1}(\chi_1)$ and $\sigma_2 \in \Delta_{m-1}(\chi_2)$ for some χ_1, $\chi_2 \in \mathscr{T}$ and let $U \subset T$ be an open neighborhood of M satisfying

$$(M * \mathrm{St}(\sigma_k)) \cap U \in \mathcal{W} \quad (k = 1, 2). \tag{2.6}$$

Then there exist χ which is finer than χ_1 and χ_2, $(m-1)$-cells $\tau_1, \tau_2, \ldots, \tau_l \in \Delta_{m-1}(\chi)$ and an open neighborhood $U' \subset T$ of M which satisfy the following conditions.

(a) $\tau_1 \subset \sigma_1$ and $\tau_l \subset \sigma_2$.

(b) τ_k and τ_{k+1} is adjacent to each other for $k = 1, 2, \ldots, l-1$.

(c) $(M * \mathrm{St}(\tau_k)) \cap U' \subset W$ for $k = 1, 2, \ldots, l$.

Now we are ready to define intuitive representation of local cohomology groups.

Definition 2.5. We define intuitive representation $\check{H}(\mathcal{F})$ of local cohomology groups $\mathrm{H}^m_M(X,\mathcal{F}) \underset{\mathbb{Z}}{\otimes} \mathrm{H}^m_M(X,\mathbb{Z}_X)$ as follows.

$$\check{H}(\mathcal{F}) = \left(\underset{U\in\mathcal{W}}{\oplus} \mathcal{F}(U)\right)/\mathcal{R}. \tag{2.7}$$

Here $\mathcal{R}$ is a $\mathbb{C}$-vector space generated by the following elements.

$$f \oplus (-f|_V)\ (f \in \mathcal{F}(U),\ U, V \in \mathcal{W} \text{ and } V \subset U). \tag{2.8}$$

§ 3. The equivalence of local cohomology groups and their intuitive representation

The purpose of this section is to prove the following main theorem. We specifically construct the boundary value map $b_{\mathcal{W}}$ and its inverse map ϑ for that purpose.

Theorem 3.1. *There exists the boundary value map $b_{\mathcal{W}}$ from intuitive representation to a local cohomology group*

$$b_{\mathcal{W}} : \check{H}(\mathcal{F}) \xrightarrow{\sim} \mathrm{H}^m_M(X,\mathcal{F}) \underset{\mathbb{Z}}{\otimes} \mathrm{H}^m_M(X,\mathbb{Z}_X). \tag{3.1}$$

Futhermore it is isomorphic.

§ 3.1. Construction of $b_{\mathcal{W}}$

First we construct the boundary value map $b_{\mathcal{W}}$ due to Schapira's idea in [4]. Let $\mathcal{F}$ be a sheaf on X. We define the dual complex $D(\mathcal{F})$ of $\mathcal{F}$ as follows.

$$D(\mathcal{F}) = \mathrm{R}\mathcal{H}om_{\mathbb{C}_X}(\mathcal{F},\mathbb{C}_X). \tag{3.2}$$

We fix $W \in \mathcal{W}$. From the condition (P-2), we can find an open neighborhood U of M, $\chi \in \mathscr{T}$ and $\sigma \in \Delta(\chi)$ which satisfy $(M * \mathrm{St}(\sigma)) \cap U \subset W$, then we have the restriction map

$$\mathcal{F}(W) \to \mathcal{F}((M * \mathrm{St}(\sigma)) \cap U). \tag{3.3}$$

It follows from $M \subset \overline{M * \mathrm{St}(\sigma)}$ that we also have the restriction map

$$\mathbb{C}_{\overline{M*\mathrm{St}(\sigma)}} \to \mathbb{C}_M. \tag{3.4}$$

By applying $D(*)$ and $\underset{\mathbb{C}_X}{\otimes} \mathbb{C}_U$ to the above, we get

$$\mathbb{C}_{(M*\mathrm{St}(\sigma))\cap U} \leftarrow D(\mathbb{C}_M). \tag{3.5}$$

Here we note that

$$D(\mathbb{C}_M) \simeq \mathbb{C}_M[m] \underset{\mathbb{Z}_X}{\otimes} \omega_M, \tag{3.6}$$

and because M is orientable

$$\mathrm{H}^m_M(X, \mathbb{Z}_X) \simeq \Gamma(X, \omega_M), \tag{3.7}$$

where $\omega_M = \mathcal{H}^m_M(\mathbb{Z}_X)$. Then by applying $\mathrm{RHom}_{\mathbb{C}_X}(*, \mathcal{F})$ to (3.5) and taking its 0-th cohomology we obtain

$$\mathcal{F}((M * \mathrm{St}(\sigma)) \cap U) \to \mathrm{H}^m_M(X, \mathcal{F}) \underset{\mathbb{Z}}{\otimes} \mathrm{H}^m_M(X, \mathbb{Z}_X). \tag{3.8}$$

We can get b_W by composing (3.3) and (3.8):

$$\mathcal{F}(W) \to \mathcal{F}((M * \mathrm{St}(\sigma)) \cap U) \to \mathrm{H}^m_M(X, \mathcal{F}) \underset{\mathbb{Z}}{\otimes} \mathrm{H}^m_M(X, \mathbb{Z}_X). \tag{3.9}$$

It follows from the construction that b_W satisfies the proposition below.

Proposition 3.2. *b_W does not depend on the choice of U, χ and σ.*

As a corollary, we have the following.

Corollary 3.3. *For any W_1 and W_2 with $W_2 \subset W_1$ which belong to $\mathcal{W}$ and for any $f \in \mathcal{F}(W_1)$, we have*

$$b_{W_1}(f) = b_{W_2}(f|_{W_2}). \tag{3.10}$$

We get $b_{\mathcal{W}}$ by assigning $\oplus f_W \in \underset{W \in \mathcal{W}}{\oplus} \mathcal{F}(W)$ to $\sum_{W \in \mathcal{W}} b_W(f_W)$.

$$b_{\mathcal{W}} : \underset{W \in \mathcal{W}}{\oplus} \mathcal{F}(W) \to \mathrm{H}^m_M(X, \mathcal{F}) \underset{\mathbb{Z}}{\otimes} \mathrm{H}^m_M(X, \mathbb{Z}_X). \tag{3.11}$$

Moreover by Corollary 3.3 the element of $\mathcal{R}$ is sent to 0, and we have obtained $b_{\mathcal{W}}$.

$$b_{\mathcal{W}} : \left(\underset{U \in \mathcal{W}}{\oplus} \mathcal{F}(U) \right) / \mathcal{R} \to \mathrm{H}^m_M(X, \mathcal{F}) \underset{\mathbb{Z}}{\otimes} \mathrm{H}^m_M(X, \mathbb{Z}_X). \tag{3.12}$$

§ 3.2. The interpretation of $b_{\mathcal{W}}$

Next let us study the concrete correspondence of $b_{\mathcal{W}}$.

First we define the function $\mathrm{sgn}(*, *)$ which reflects the orientations of two cells. Let σ and τ be cells of χ. We assume that σ and τ satisfy one of two cases below.

1. σ and τ are k-cells.

2. σ is a k-cell and τ is a $(k+1)$-cell with $\sigma \subset \overline{\tau}$.

Definition 3.4. $\operatorname{sgn}(\sigma, \tau)$ is defined by

$$\begin{cases} 1 & (\tau \textit{ and either } \sigma \textit{ if the first case or the induced} \\ & \textit{orientation from } \sigma \textit{ if the second case have the same orientation}), \\ -1 & (\textit{otherwise}). \end{cases} \tag{3.13}$$

Now we study the concrete correspondence of $b_{\mathcal{W}}$. We fix $\chi \in \mathscr{T}$.

Definition 3.5. Let $k = 0, 1, \cdots, m-1$. We define the sheaf $\mathcal{L}_\chi^{k+1}$ by

$$\mathcal{L}_\chi^{k+1} = \left(\bigoplus_{\sigma \in \Delta_k(\chi)} \mathbb{C}_{\overline{M * \mathrm{St}(\sigma)}} \right) \underset{\mathbb{Z}_X}{\otimes} \omega_M. \tag{3.14}$$

For convenience, we set $\mathcal{L}_\chi^0 = \mathbb{C}_X \underset{\mathbb{Z}_X}{\otimes} \omega_M$. Then we have the following proposition.

Proposition 3.6. *The following sequence is exact.*

$$0 \longrightarrow \mathcal{L}_\chi^0 \longrightarrow \mathcal{L}_\chi^1 \longrightarrow \mathcal{L}_\chi^2 \longrightarrow \cdots \longrightarrow \mathcal{L}_\chi^m \longrightarrow \mathbb{C}_M \longrightarrow 0. \tag{3.15}$$

By this proposition we have the quasi-isomorphism.

$$\mathcal{L}_\chi^* \simeq \mathbb{C}_M. \tag{3.16}$$

Here $\mathcal{L}_\chi^*$ designates the complex

$$0 \longrightarrow \mathcal{L}_\chi^0 \longrightarrow \mathcal{L}_\chi^1 \longrightarrow \mathcal{L}_\chi^2 \longrightarrow \cdots \longrightarrow \mathcal{L}_\chi^m \longrightarrow 0. \tag{3.17}$$

Note that, in the complex, the leftmost term $\mathcal{L}_\chi^0$ is located at degree $-m$ and the rightmost term $\mathcal{L}_\chi^m$ is at degree 0. In particular the complex $\mathcal{L}_\chi^*$ is concentrated in degree 0. Hence for any $\sigma \in \Delta_{m-1}(\chi)$ we have the commutative diagram below.

$$\begin{array}{ccc} \mathcal{L}_\chi^m & \xrightarrow{\sim} & \mathbb{C}_M \\ {\scriptstyle\varepsilon}\uparrow & & \uparrow \\ \mathbb{C}_{\overline{M * \mathrm{St}(\sigma)}} \underset{\mathbb{Z}_X}{\otimes} \omega_M & \underset{\eta}{\xrightarrow{\sim}} & \mathbb{C}_{\overline{M * \mathrm{St}(\sigma)}}. \end{array} \tag{3.18}$$

Here ε is the embedding map and η is defined by

$$c_\sigma \otimes \omega \mapsto \operatorname{sgn}(\sigma, \omega) \cdot c_\sigma. \tag{3.19}$$

For convenience we set

$$\mathcal{F}_k^*(\Delta(\chi)) = \bigoplus_{\sigma \in \Delta_{k-1}(\chi)} \mathcal{F}(M * \mathrm{St}(\sigma)). \tag{3.20}$$

Then by (3.15) we have

Lemma 3.7.

$$\mathrm{H}^m_M(X,\mathcal{F}) \underset{\mathbb{Z}}{\otimes} \mathrm{H}^m_M(X,\mathbb{Z}_M) \simeq \frac{\mathcal{F}^*_m(\Delta(\chi))}{\mathrm{Im}(\mathcal{F}^*_{m-1}(\Delta(\chi)) \to \mathcal{F}^*_m(\Delta(\chi)))} \underset{\mathbb{Z}}{\otimes} \mathrm{H}^m_M(X,\mathbb{Z}_M). \tag{3.21}$$

By these observations we have the following conclusion.

Proposition 3.8. *The boundary value map* $b_{\mathcal{W}}$

$$\mathcal{F}(M * \mathrm{St}(\sigma)) \longrightarrow \mathrm{H}^m_M(X,\mathcal{F}) \underset{\mathbb{Z}}{\otimes} \mathrm{H}^m_M(X,\mathbb{Z}_X) \tag{3.22}$$

is induced by $f \mapsto [f \cdot 1_\sigma] \otimes [1_\sigma]$.

§ 3.3. The inverse map ϑ of $b_{\mathcal{W}}$

Let χ and $\chi' \in \mathscr{T}$ with $\chi \prec \chi'$. First we construct the map Θ_m from $\mathcal{L}^m_\chi$ to $\mathcal{L}^m_{\chi'}$. Let ψ be a choice function such that, for each $\sigma \in \Delta_{m-1}(\chi)$, there exists $\sigma' \in \Delta_{m-1}(\chi')$ with $\sigma' \subset \sigma$ and $\psi(\sigma) = \sigma'$. Then for any $f_\sigma \in \mathbb{C}_{\overline{M*\mathrm{St}(\sigma)}}$ we define the Θ_m as follows.

$$\Theta_m : f_\sigma \mapsto \mathrm{sgn}(\sigma, \psi(\sigma)) \cdot f_\sigma|_{\psi(\sigma)}. \tag{3.23}$$

And it immediately follows from the construction of Θ_m that the diagram below is commutative.

$$\begin{array}{ccc} \mathcal{L}^m_\chi & \longrightarrow & \mathbb{C}_M \\ \downarrow \Theta_m & & \downarrow id \\ \mathcal{L}^m_{\chi'} & \longrightarrow & \mathbb{C}_M. \end{array} \tag{3.24}$$

We can extend this commutative diagram to that of the complexes $\mathcal{L}^*_\chi$ and $\mathcal{L}^*_{\chi'}$ by the following proposition.

Proposition 3.9. *Let* Θ_m *be a morphism defined above. Then there exist* Θ_k $(k = 0, 1, \cdots, m-1)$ *and the following diagram is commutative.*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathcal{L}^0_\chi & \longrightarrow & \cdots & \longrightarrow & \mathcal{L}^m_\chi & \longrightarrow & \mathbb{C}_M & \longrightarrow & 0 \\ & & \downarrow \Theta_0 & & & & \downarrow \Theta_m & & \downarrow id & & \\ 0 & \longrightarrow & \mathcal{L}^0_{\chi'} & \longrightarrow & \cdots & \longrightarrow & \mathcal{L}^m_{\chi'} & \longrightarrow & \mathbb{C}_M & \longrightarrow & 0. \end{array} \tag{3.25}$$

Let us shortly explain the proof of Proposition 3.9. We construct $\Theta_{m-1} : \mathcal{L}^{m-1}_\chi \to \mathcal{L}^{m-1}_{\chi'}$ such that the diagram below is commutative.

$$\begin{array}{ccc} \mathcal{L}^{m-1}_\chi & \longrightarrow & \mathcal{L}^m_\chi \\ \downarrow \Theta_{m-1} & & \downarrow \Theta_m \\ \mathcal{L}^{m-1}_{\chi'} & \longrightarrow & \mathcal{L}^m_{\chi'}. \end{array} \tag{3.26}$$

Here the following lemma guarantees existence of Θ_{m-1}.

Lemma 3.10. *Let l be a non-negative integer and $k = 0, 1, \cdots, m$. For any $\sigma \in \Delta(\chi)$ we have*

$$\mathrm{Ext}^l(\mathbb{C}_{\overline{M * \mathrm{St}(\sigma)}}, \mathcal{L}^k_{\chi'}) = 0. \tag{3.27}$$

By the same argument we can constuct $\Theta_k (k = 0, 1, \cdots m-1)$ inductively and we have Proposition 3.9.

Then noticing Lemma 3.7, we construct ϑ from (3.21) to $\check{H}(\mathcal{F})$.

Proposition 3.11.

$$\vartheta : \frac{\mathcal{F}^*_m(\Delta(\chi))}{\mathrm{Im}(\mathcal{F}^*_{m-1}(\Delta(\chi)) \to \mathcal{F}^*_m(\Delta(\chi)))} \underset{\mathbb{Z}}{\otimes} \mathrm{H}^m_M(X, \mathbb{Z}_M) \to \check{H}(\mathcal{F}) \tag{3.28}$$

is induced by

$$\vartheta : f_\sigma \otimes \omega \mapsto \mathrm{sgn}(\sigma, \omega) \cdot f_\sigma \quad (\sigma \in \Delta_{m-1}(\chi),\ f_\sigma \in \mathcal{F}(M * \mathrm{St}(\sigma))). \tag{3.29}$$

By Proposition 3.9 we obtain the proposition below.

Proposition 3.12. *ϑ does not depend on the choice of χ.*

This proposition guarantees the well-definedness of ϑ. Furthermore by Propositions 3.8, 3.11 and 3.12 we can easily see Theorem 3.1.

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