RIMS Kôkyûroku Bessatsu
B75 (2019), 027–030

A vanishing theorem of global cohomology groups with values in the sheaf of Whitney jets with Gevery conditions

By

Kohei Umeta* and Tsubasa Shimoyama

Abstract

In this note, we announce the exactness of $\overline{\partial}$-complex with coefficients in the sheaf of Whitney jets with Gevery conditions.

§ 1. Introduction

In 1979, A. Dufresnoy established, in the paper [1], the exactness of $\overline{\partial}$-complex of the sheaf of differentiable functions in the sence of Whitney. The result plays an important role in asymptotic analysis. Recently, N. Honda and G. Morand constructed, in the paper [2], the sheaf of stratified Whitney jets of Gevrey order on the subanalytic site relative to a real analytic manifold. They had succeeded in localizing the notion of Whitney jets with Gevery conditions on the subanalytic site. In this note we establish a vanishing theorem of global cohomology groups on an analytic polyhedron with values in the sheaf of Whitney jets with Gevery conditions. We only announce the main theorem in this paper. For the detail and proof, we refer the reader to the forthcoming paper.

At the end of this section, the authors would like to express their gratitude to Professor Naofumi Honda for the valuable advises in Hokkaido University.

Received July 1, 2017. Revised April 7, 2019.
2010 Mathematics Subject Classification(s): 32C35
Key Words: *Whitney jets, Gevery order, sheaves, cohomology groups.*
*Nihon University, 24-1 Narashinodai 7-chome funabashi-shi, Chiba 274-8501, Japan.
e-mail: umeta.kouhei@nihon-u.ac.jp

§ 2. The sheaf $\mathcal{W}^{*}_{X,Z}$ of Whitney jets of class $*$

Let us introduce some notations. Let $X = \mathbb{C}^n$. We define the semi-norm $||f||_{s,h,K}$ of a $\mathcal{C}^\infty$ function f for $s > 0, h > 0$ and a compact subset K in X by

$$||f||_{s,h,K} := \sup_{x \in K, \alpha \in \mathbb{N}_0^n} \frac{|\partial^\alpha f(x)|}{\alpha!^s h^{|\alpha|}}. \tag{2.1}$$

For a relatively compact open subset V in X, the set $E^{s,h}_{X,\overline{V}}$ is given by

$$E^{s,h}_{X,\overline{V}} := \{f \in \mathcal{C}^\infty_{X,\overline{V}}; ||f||_{s,h,\overline{V}} < \infty\}, \tag{2.2}$$

where $\mathcal{C}^\infty_{X,\overline{V}}$ denotes the set of $f \in \mathcal{C}^\infty(V)$ whose arbitrary partial derivative extends to a continuous function on $\overline{V}$. The set $E^{s,h}_{X,\overline{V}}$ is a Banach space with the norm $|| \cdot ||_{s,h,\overline{V}}$. From now on, the symbol $*$ stands for (s) or $\{s\}$. For an open subset $U \subset X$, we denote by $E^*_X(U)$ the space of ultra-differentiable functions of class $*$, that is,

$$\begin{aligned} \mathcal{E}^{(s)}_X(U) &:= \varprojlim_{V \subset\subset U} \varinjlim_{h \to 0} E^{s,\,h}_{X,\overline{V}}, \\ \mathcal{E}^{\{s\}}_X(U) &:= \varprojlim_{V \subset\subset U} \varinjlim_{h \to 0} E^{s,\,h}_{X,\overline{V}}. \end{aligned} \tag{2.3}$$

Hence $f \in C^\infty(U)$ belongs to $\mathcal{E}^*_X(U)$ if and only if, when $* = (s)$, for any compact subset K in U and for any $h > 0$, the estimate $||f||_{s,h,K} < \infty$ holds, and when $* = \{s\}$, for any compact subset K in U, there exists $h > 0$ such that $||f||_{s,h,K} < \infty$.

Let Z be a closed subset in X and $\mathcal{J}_{X,Z}$ be the sheaf of jets on Z, that is, $\mathcal{J}_{X,Z}(U)$ consists of a jet $F = \{f_\alpha(x)\}_{\alpha \in \mathbb{N}_0^n}$ with $f_\alpha \in C(Z \cap U)$. We have the canonical sheaf morphism $\iota_Z : C^\infty_X \to \mathcal{J}_{X,Z}$ by

$$f \in C^\infty_X(U) \to \{\partial^\alpha f|_{Z \cap U}\}_{\alpha \in \mathbb{N}_0^n} \in \mathcal{J}_{X,Z}. \tag{2.4}$$

For a jet $F = \{f_\alpha\}_{\alpha \in \mathbb{N}_0^n}$ in $\mathcal{J}_{X,Z}(U)$, we also define the norm

$$||F||_{s,h,K} := \sup_{x \in K \cap Z,\, \alpha \in \mathbb{N}_0^n} \frac{|f_\alpha(x)|}{\alpha!^s h^{|\alpha|}}. \tag{2.5}$$

Definition 2.1. The sheaf $\mathcal{W}^*_{X,Z}$ of Whitney jets on Z of class $*$ is defined by the image sheaf of the sheaf $\mathcal{E}^*_X$ by ι_Z, that is

$$\mathscr{W}^*_{X,Z} := \iota_Z(\mathcal{E}^*_X). \tag{2.6}$$

By the definition, the sheaf $\mathcal{W}^*_{X,Z}$ satisfies the following exact sequence:

$$0 \to \mathcal{I}^*_Z \to \mathcal{E}^*_X \to \mathcal{W}^*_{X,Z} \to 0. \tag{2.7}$$

Here $\mathcal{I}^*_Z$ is the subsheaf of $\mathcal{E}^*_X$ consisting of functions which vanish on Z up to infinite order.

We also need some geometrical conditions.

Definition 2.2. Let K be a compact subset. The K is said to be 1-regular if there exists $C > 0$ for which any two points p and q in K are joined with a rectifiable curve l in K whose length $|l|$ satisfies

$$|l| \le C|p - q|. \tag{2.8}$$

Let $\theta = (\theta_1, \dots, \theta_l)$ be independent variables. Then we define the sheaf $\mathcal{W}^*_{X,Z}[[\theta]]^*$ on X as follows. For any open subset U in X, the set $\mathcal{W}^*_{X,Z}[[\theta]]^*(U)$ consists of the formal power series

$$G = \sum_{\beta \in \mathbb{N}_0^l} \frac{F_\beta}{\beta!} \theta^\beta, \tag{2.9}$$

where $F_\beta \in \mathcal{W}^*_{X,Z}(U)$ and satisfies the following estimates: when $* = (s)$, for any compact subset K in U and any $h > 0$,

$$||G||_{s,h,K} := \sup_{\beta \in \mathbb{N}_0^l} \frac{||F_\beta||_{s,h,K}}{\beta!^s h^{|\beta|}} < +\infty. \tag{2.10}$$

holds, and when $s = \{s\}$ for any compact subset K in U, there exists $h > 0$ such that $||G||_{s,h,K} < +\infty$. Note that these are also soft sheaves on X. Corollary 2.2.6 in [2] implies the following proposition.

Proposition 2.3. *Let K be a compact subset which is 1-regular. Let* $G = \sum_{\beta \in \mathbb{N}_0^l} \frac{F_\beta}{\beta!} \theta^\beta \in \mathcal{W}^*_{X,Z}[[\theta]]^*(X)$ *with* $F_\beta = \{f_{\beta,\alpha}(x)\}_{\alpha \in \mathbb{N}_0^n} \in \mathcal{W}^*_{X,K}(X)$. *Then there exists* $\varphi(x,\theta) \in \mathcal{E}^*_{X \times \mathbb{R}^l_\theta}$ *such that* $f_{\beta,\alpha}(x) = \partial_\theta^\beta \partial_x^\alpha \varphi|_{\theta=0, x \in K}$.

§ 3. A vanishing theorem of global cohomology groups with values in the sheaf $\mathcal{WO}^*_{X,K}$

Set

$$\mathcal{WO}^*_{X,K} := R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \mathcal{W}^*_{X,K}), \tag{3.1}$$

where $\overline{X}$ denotes the complex conjugate of X and $\mathcal{O}_{\overline{X}}$ is the sheaf of antiholomorphic functions on X. That is, $\mathcal{WO}^*_{X,K}$ is nothing but the $\overline{\partial}$-complex with coefficients in the sheaf $\mathcal{W}^*_{X,K}$:

$$0 \to \mathcal{W}^*_{X,K}{}^{(0,0)} \xrightarrow{\overline{\partial}} \mathcal{W}^*_{X,K}{}^{(0,1)} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{W}^*_{X,K}{}^{(0,n)} \to 0. \tag{3.2}$$

Let $f_1, \ldots, f_l$ be a sequence of l-holomorphic functions on X. Set, for $j = 1, \ldots, l$,

$$K_j := \{z \in X; |f_1(z)| \leq 1, \ldots, |f_j(z)| \leq 1\}. \tag{3.3}$$

For convenience, we set $K_0 := X$ and $K := K_l$. We introduce the following conditions.

A1 K is compact.

The condition below guarantees that $K_1, \ldots, K_l$ are regularly situated.

A2 There exists relatively compact open subanalytic subset V containing K and a constant $C > 0$ such that

$$C||f_j(z)| - 1| \geq \text{dist}(z,\, K_j) \qquad (z \in K_{j-1} \cap \overline{V})$$

for any $j = 1, 2, \ldots, l$.

A3 There exists relatively compact open subanalytic subset V containing K such that $\overline{V} \cap K_j$ is 1-regular for any $j = 1, 2, \ldots, l$.

We have the following vanishing theorem.

Theorem 3.1. *Let K be an analytic polyhedron. Assume that there exists a sequence $f_1, \ldots, f_l$ of holomorphic functions on X satisfying the conditions* A1, A2 *and* A3 *described above and that K is given by* (3.3). *Then we have*

$$\mathrm{H}^k(X,\, \mathcal{WO}^*_{X,K}) = 0 \qquad (k \neq 0). \tag{3.4}$$

References

[1] A. Dufresnoy, Sur I'opérateur d" et les fonctions différentiables au sens de Whitney, Annales de I'institut Fourier, tome 29, no 1, 229-238 (1979)

[2] N. Honda and G. Morando, Stratified Whitney jets and tempered ultradistributions on the subanalytic site, Bulletin de la SociétéMathématique de France, 139(3), 389-435 (2011)

[3] H. Komatsu, Grothendieck spaces and kernel theorems, Jyouchi Daigaku Suugaku Koukyuroku, No.9, (1981) written in Japanese

[4] P. Schapira, Microdifferential systems in the complex domain, Grundleher m der Math., vol.269, Springer-Verlag, Berlin (1985)

[5] H. Whitney, On the extension of differentiable functions, Bull. Amer. Math. Soc., 50, 76-81, (1944)