RIMS Kôkyûroku Bessatsu
B75 (2019), 031–039

Asymptotics for the focusing integrable discrete nonlinear Schrödinger equation

By

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Abstract

We investigate the long-time asymptotics for the focusing integrable discrete nonlinear Schrödinger equation. The soliton resolution conjecture is valid for this equation. Namely, under generic assumptions on the initial value, the solution is approximated by a sum of 1-solitons.

§ 1. Introduction

In this article we announce our result about the long-time behavior of the solutions to the focusing integrable discrete nonlinear Schrödinger equation (IDNLS) introduced by Ablowitz and Ladik ([1]):

$$i\frac{d}{dt}R_n + (R_{n+1} - 2R_n + R_{n-1}) + |R_n|^2(R_{n+1} + R_{n-1}) = 0 \quad (n \in \mathbb{Z}). \tag{1.1}$$

It is a discrete version of the focusing nonlinear Schrödinger equation (NLS)

$$iu_t + u_{xx} + 2u|u|^2 = 0. \tag{1.2}$$

As is the case with (1.2), the equation (1.1) can be solved by the inverse scattering transform. Eigenvalues appear in quartets of the form $(\pm z_j, \pm \bar{z}_j^{-1})$ (see §2).

It is well known ([1, 2]) that (1.1) admits multi-soliton solutions in the reflectionless case. When there is only one quartet of eigenvalues including $z_1 = \exp(\alpha_1 + i\beta_1)$ with $\alpha_1 > 0$, $R_n(t)$ is a (bright) 1-soliton solution, namely,

$$R_n(t) = \mathrm{BS}\,(n, t; z_1, C_1(0))\,,$$

Received Octorber 27, 2016. Revised May 27, 2017.
2010 Mathematics Subject Classification(s): Primary 35Q55; Secondary 35Q15
Key Words: *discrete nonlinear Schrödinger equation, Riemann-Hilbert problem, soliton.*
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where $C_1(0)$ is the norming constant and

$$\begin{aligned}\mathrm{BS}(n,t;z_1,C_1(0)) &= \exp\Big(-i[2\beta_1(n+1)-2w_1t+\arg C_1(0)]\Big)\\ &\qquad\times\sinh(2\alpha_1)\mathrm{sech}[2\alpha_1(n+1)-2v_1t-\theta_1],\\ v_1 &= -\sinh(2\alpha_1)\sin(2\beta_1),\quad w_1=\cosh(2\alpha_1)\cos(2\beta_1)-1,\\ \theta_1 &= \log|C_1(0)|-\log\sinh(2\alpha_1).\end{aligned}$$

Changing the value of $C_1(0)$ implies phase shifts in the exp and/or the sech factors. The velocity of $\mathrm{BS}(n,t;z_1,C_1(0))$ is, by definition, that of its sech factor denoted by $\mathrm{tw}(z_1)$. We have

$$\mathrm{tw}(z_1)=\mathrm{tw}(\exp(\alpha_1+i\beta_1))=\alpha_1^{-1}v_1=-\alpha_1^{-1}\sinh(2\alpha_1)\sin(2\beta_1).$$

The soliton resolution conjecture (see [14] for a brief survey) about (not necessarily integrable) nonlinear dispersive equations states that any reasonable solution splits into a sum of solitons up to a small error term as t tends to infinity. In the case of integrable equations, the difficulty lies in the perturbed case where the reflection coefficient does not vanish identically. In the present paper, we deal with such a problem concerning (1.1). Our main result is as follows: if the quartets of eigenvalues are $(\pm z_j,\pm\bar{z}_j^{-1})$ with $\mathrm{tw}(z_j)<\mathrm{tw}(z_{j'})\,(j<j')$, then we have, formally,

$$\begin{aligned}R_n(t) &\sim \sum_{j\in G_1}\mathrm{BS}\left(n,t;z_j,\delta_{n/t}(0)\delta_{n/t}(z_j)^2p_jT(z_j)^{-2}C_j(0)\right)\\ &\qquad+\sum_{j\in G_2}\mathrm{BS}\left(n,t;z_j,p_jT(z_j)^{-2}C_j(0)\right),\\ p_j &= \prod_{k>j} z_k^2\bar{z}_k^{-2},\\ T(z_j) &= \prod_{k>j}\frac{z_k^2(z_j^2-\bar{z}_k^{-2})}{z_j^2-z_k^2}\end{aligned}$$

under generic assumptions. Here we have denoted $G_1=\{j;\,|\mathrm{tw}(z_j)|<2\}$ and $G_2=\{j;\,|\mathrm{tw}(z_j)|\geq 2\}$. The function $\delta_{n/t}(z)=\delta(z)$ is defined in terms of the reflection coefficient. In the reflectionless case we have $\delta(z)=1$ and recover the known formula about a multi-soliton.

This result has a significant difference from those about the continuous NLS (1.2) in [7, 10, 11]. In the case of (1.2), the effect of the reflection coefficient can be felt in the entire half-plane $t>0$, but in the case of (1.1) it is irrelevant in $|n|/t\geq 2$.

We review some known results about the long-time asymptotics of integrable equations based on the method of nonlinear steepest descent. This method was established in [5] in order to study the MKdV equation and has been employed in many papers including ones by the present author. The defocusing NLS was studied in [3]. The soliton resolution for the focusing NLS (1.2) was studied in [7, 10, 11]. The present author studied the defocusing IDNLS in [16, 17]. The Toda lattice was studied in [9] (solitonless), [12] (soliton resolution included). In [8], the asymptotics for the KdV equation was studied in several regions.

Notice that the focusing IDNLS was investigated in the solitonless case in [15] by using some ansatz. It is possible, although now unpopular, to study the long-time asymptotics by using the Gelfand-Levitan-Marchenko integral equation (e.g. the study of the KdV equation in [13]).

§ 2. Inverse scattering transform

We review the inverse scattering transform for (1.1) following [1] and [2, Chap. 3]. The n-part of the Lax pair is

$$X_{n+1} = \begin{bmatrix} z & -\bar{R}_n \\ R_n & z^{-1} \end{bmatrix} X_n, \tag{2.1}$$

where the bar denotes the complex-conjugate. The t-part is

$$\frac{d}{dt} X_n = \begin{bmatrix} -iR_{n-1}\bar{R}_n - \frac{i}{2}(z - z^{-1})^2 & i(z\bar{R}_n - z^{-1}\bar{R}_{n-1}) \\ i(z^{-1}R_n - zR_{n-1}) & iR_n\bar{R}_{n-1} + \frac{i}{2}(z - z^{-1})^2 \end{bmatrix} X_n \tag{2.2}$$

and (1.1) is equivalent to the compatibility condition $\frac{d}{dt}X_{n+1} = (\frac{d}{dt}X_m)_{m=n+1}$.

We can construct eigenfunctions of (2.1) for any fixed t in $|z| \geq 1$ and $|z| \leq 1$. More precisely, one can define the eigenfunctions $\phi_n(z,t), \psi_n(z,t) \in \mathcal{O}(|z| > 1) \cap \mathcal{C}^0(|z| \geq 1)$ and $\psi_n^*(z,t), \phi_n^*(z,t) \in \mathcal{O}(|z| < 1) \cap \mathcal{C}^0(|z| \leq 1)$ such that

$$\phi_n(z,t) \sim z^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \phi_n^*(z,t) \sim z^{-n} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{as } n \to -\infty,$$
$$\psi_n(z,t) \sim z^{-n} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \psi_n^*(z,t) \sim z^{n} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{as } n \to \infty.$$

On the unit circle C: $|z| = 1$, there exist unique functions $a(z)$, $a^*(z)$, $b(z,t)$, $b^*(z,t)$ for which

$$\phi_n(z,t) = b(z,t)\psi_n(z,t) + a(z)\psi_n^*(z,t),$$
$$\phi_n^*(z,t) = a^*(z)\psi_n(z,t) + b^*(z,t)\psi_n^*(z,t)$$

holds. It can be proved that a and a^* are independent of t. Moreover we have

$$\begin{aligned}
&a(z) \in \mathcal{O}(|z|>1) \cap \mathcal{C}^0(|z| \geq 1),\ a^*(z) \in \mathcal{O}(|z|<1) \cap \mathcal{C}^0(|z| \leq 1),\\
&a^*(z) = \bar{a}(1/\bar{z})\ (0<|z|\leq 1)\\
&b(z), b^*(z) \in \mathcal{C}^0(|z|=1),\ b^*(z) = -\bar{b}(1/\bar{z})\ (|z|=1).
\end{aligned}$$

It is well known that if a function on $\mathbf{R}$ decays rapidly then its Fourier transform is smooth. There is an analogous fact in our context: if the potential R_n decays rapidly as $|n| \to \infty$, then a, a^*, b and b^* are smooth functions on $|z|=1$. They are $\mathcal{C}^\infty$ if R_n decays faster than any negative power of n and are analytic if R_n decays exponentially.

We assume that $a(z)$ and $a^*(z)$ never vanish on the unit circle. Their zeros in $|z|>1$ and $|z|<1$ are called *eigenvalues*. They appear in quartets of the form $(\pm z_j, \pm \bar{z}_j^{-1})$ $(j = 1, 2, \dots, J)$. They are time-independent. We assume that the eigenvalues are all simple. If $a(z_j) = 0$ we have

$$\phi_n(z_j) = b_j \psi_n(z_j)$$

for some constant b_j. We introduce the *the norming constant* C_j associated with z_j by

$$C_j = C_j(t) = \frac{b_j}{\frac{d}{dz}a(z_j)}.$$

Set $\omega_j = (z_j - z_j^{-1})^2/2$. Then the time evolution of the norming constant is given by

$$C_j(t) = C_j(0)\exp(2i\omega_j t). \tag{2.3}$$

We can define the *reflection coefficient* $r(z,t)$ by

$$r(z,t) = \frac{b(z,t)}{a(z)},\ |z|=1. \tag{2.4}$$

Assume $\{R_n(0)\}$ is rapidly decreasing in the sense that $\{R_n(0)\} \in \ell^{1,p}$ for any $p \in \mathbb{N}$. Then $\{R_n(t)\}$ is also rapidly decreasing for any t. Due to the construction in [2, pp.49-56], the eigenfunctions ϕ_n, ϕ_n^*, ψ_n and ψ_n^* are smooth on C: $|z|=1$. Hence a, b and $r = r(z,t)$ are also smooth there.

The time evolution of $r(z,t)$ according to (2.2) is given by

$$r(z,t) = r(z)\exp\left(it(z-z^{-1})^2\right) = r(z)\exp\left(it(z-\bar{z})^2\right), \tag{2.5}$$

where $r(z) = r(z,0)$. Notice that $(z-z^{-1})^2$ is real if $|z|=1$.

Set $c_n = \prod_{k=n}^{\infty}(1+|R_k|^2)$. Following [2, (3.2.94)], we set

$$m(z) = m(z;n,t) = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & c_n \end{bmatrix} \left[\frac{1}{a(z)} z^{-n}\phi_n(z,t),\, z^n\psi_n(z,t)\right] & \text{in } |z|>1, \\ \begin{bmatrix} 1 & 0 \\ 0 & c_n \end{bmatrix} \left[z^{-n}\psi_n^*(z,t),\, \frac{1}{a^*(z)} z^n\phi_n^*(z,t)\right] & \text{in } |z|<1. \end{cases}$$

It is meromorphic in $|z| \neq 1$ with poles $\pm z_j$ and $\pm \bar{z}_j^{-1}$. We have $m(z) \to I$ as $z \to \infty$. In terms of $m(z)$, the pole conditions [2, (3.2.93)] are, in view of [2, (3.2.87)],

$$\text{Res}(m(z); \pm z_j) = \lim_{z \to \pm z_j} m(z) \begin{bmatrix} 0 & 0 \\ z_j^{-2n} C_j(t) & 0 \end{bmatrix}, \tag{2.6}$$

$$\text{Res}(m(z); \pm \bar{z}_j^{-1}) = \lim_{z \to \pm \bar{z}_j^{-1}} m(z) \begin{bmatrix} 0 & \bar{z}_j^{-2n-2} \bar{C}_j(t) \\ 0 & 0 \end{bmatrix} \tag{2.7}$$

for $j = 1, 2, \ldots, J$. The jump condition is given by

$$m_+(z) = m_-(z) v(z) \quad \text{on } C\colon |z| = 1, \tag{2.8}$$

$$\begin{aligned} v(z) = v(z,t) &= \begin{bmatrix} 1 + |r(z,t)|^2 & z^{2n} \bar{r}(z,t) \\ z^{-2n} r(z,t) & 1 \end{bmatrix} \\ &= e^{-(it/2)(z - z^{-1})^2 \operatorname{ad} \sigma_3} \begin{bmatrix} 1 + |r(z)|^2 & z^{2n} \bar{r}(z) \\ z^{-2n} r(z) & 1 \end{bmatrix}, \end{aligned} \tag{2.9}$$

$$m(z) \to I \quad \text{as } z \to \infty. \tag{2.10}$$

Here m_+ and m_- are the boundary values from the *outside* and *inside* of C respectively (C is oriented clockwise following the convention in [2].) We employ the usual notation $\sigma_3 = \text{diag}\,(1, -1)$, $a^{\operatorname{ad} \sigma_3} Q = a^{\sigma_3} Q a^{-\sigma_3}$. The solution $\{R_n\} = \{R_n(t)\}$ to (1.1) can be reconstructed from the $(2,1)$-component of $m(z)$. We have ([2, (3.2.91c)])

$$R_n(t) = -\left. \frac{d}{dz} m(z)_{21} \right|_{z=0}. \tag{2.11}$$

In the reflectionless case, $R_n(t)$ is a multi-soliton. When there is only one quartet of eigenvalues, it is a 1-soliton as in the following proposition ([2, p.83]).

Proposition 2.1. *Assume $r(z) \equiv 0$ and that there is only one quartet of eigenvalues including $z_1 = \exp(\alpha_1 + i\beta_1), \alpha_1 > 0$. Then the RHP* (2.6)-(2.10) *has a unique solution. We denote it by $m_0(z)$. The solution $R_n(t)$ to* (1.1) *obtained from $m_0(z)$ through* (2.11) *is the bright 1-soliton solution $R_n(t) = \text{BS}(n, t; z_1, C_1(0))$ given in the introduction.*

Let us introduce the phase function

$$\varphi = \varphi(z) = \varphi(z; n, t) = \frac{1}{2} it(z - z^{-1})^2 - n \log z,$$

so that the jump matrix $v(z)$ in (2.9) is given by

$$v = v(z) = e^{-\varphi(z) \operatorname{ad} \sigma_3} \begin{bmatrix} 1 + |r(z)|^2 & \bar{r}(z) \\ r(z) & 1 \end{bmatrix} = \begin{bmatrix} 1 + |r(z)|^2 & e^{-2\varphi(z)} \bar{r}(z) \\ e^{2\varphi(z)} r(z) & 1 \end{bmatrix}. \tag{2.12}$$

Moreover, we have $\varphi(z_j) = i\omega_j t - n \log z_j$ and

$$z_j^{-2n} C_j(t) = C_j(0) \exp[2\varphi(z_j)], \tag{2.13}$$

$$\operatorname{Re} \varphi(z_j) = \alpha_j t \big[\operatorname{tw}(z_j) - n/t\big], \tag{2.14}$$

$$\operatorname{tw}(z_j) = -\alpha_j^{-1} \sinh(2\alpha_j) \sin(2\beta_j), \tag{2.15}$$

where

$$z_j = \exp(\alpha_j + i\beta_j),\ \alpha_j > 0.$$

Notice the equivalence

$$\pm \operatorname{Re} \varphi(z_j) > 0 \Leftrightarrow \pm(\operatorname{tw}(z_j) - n/t) > 0.$$

If the j-th soliton is slower than the observer with velocity n/t, then $z_j^{-2n} C_j(t)$ is exponentially decreasing as $t \to \infty$. By (2.6) and (2.7), the residues at $\pm z_j$ and $\pm \bar{z}_j^{-1}$ are infinitely small. In other words, these poles can be neglected.

The situation is complicated if the j-th soliton is faster than the observer. Assume that the observer is chasing the s-th soliton. One can replace $z_j^{-2n} C_j$ (which is growing) with its reciprocal (decreasing) by using a trick, which changes the value of the s-th norming constant. The implication of this trick is twofold. One is that the j-th soliton can be neglected and the other is that the s-th soliton undergoes phase shift.

§ 3. Main results

In this section we state our main results. See [18] for technical details.

Throughout this section, we make the following three generic assumptions:

- $a(z)$ never vanishes on the unit circle. It implies that $a^*(z)$ never vanishes there either.
- The eigenvalues are all simple.
- $\operatorname{tw}(z_j)$'s are mutually distinct. We may assume that $\operatorname{tw}(z_j) < \operatorname{tw}(z_{j+1})$ for any j without loss of generality.

The first and the second are assumed in [2].

We derive asymptotic expansions by using the method of nonlinear steepest descent. Stationary points of $\varphi(z)$ play important roles. They have different orders and configurations in different regions of the (n, t)-plane. In each region, we choose a contour that is compatible with the geometry of stationary points.

In the region $|n| < 2t$, the function $\varphi(z)$ has four saddle points (stationary points of the first order) on the unit circle $|z| = 1$. They are

$$S_1 = e^{-\pi i/4}A,\ S_2 = e^{-\pi i/4}\bar{A},\ S_3 = -S_1,\ S_4 = -S_2,$$
$$A = 2^{-1}\left(\sqrt{2+n/t} - i\sqrt{2-n/t}\,\right).$$

We draw steepest descent paths passing through the saddle points. Moreover we introduce

$$\delta(z) = \exp\left(\frac{-1}{2\pi i}\left[\int_{S_1}^{S_2} + \int_{S_3}^{S_4}\right](\tau - z)^{-1}\log(1+|r(\tau)|^2)\,d\tau\right), \tag{3.1}$$

where the contours are the arcs $\subset \{|z| = 1\}$. We have the following result.

Theorem 3.1. *Let V_0 be a constant with $0 < V_0 < 2$. Assume that the initial value satisfies the rapid decrease condition $\{R_n(0)\} \in \bigcap_{p=0}^{\infty} \ell^{1,p}$. Then in the region $|n| \le (2 - V_0)t$, the asymptotic behavior of the solution to (1.1) is as follows:*
(soliton case) *In the region $-d \le \mathrm{tw}(z_j) - n/t \le d$, $j \in \{1, \dots, J\}$, with sufficiently small d, we have*

$$R_n(t) = \mathrm{BS}\left(n, t; z_j, \delta(0)\delta(z_j)^{-2}p_jT(z_j)^{-2}C_j(0)\right) + O(t^{-1/2}),$$
$$p_j = \prod_{k>j} z_k^2\bar{z}_k^{-2}, \qquad T(z_j) = \prod_{k>j}\frac{z_k^2(z_j - \bar{z}_k^{-2})}{z_j - z_k^2}.$$

(solitonless case) *If $\{\mathrm{tw}(z_j); j = 1, \dots, J\} \cap [n/t - d, n/t + d] = \emptyset$, then there exist $C_j = C_j(n/t) \in \mathbb{C}$ and $p_j = p_j(n/t), q_j = q_j(n/t) \in \mathbb{R}$ $(j = 1, 2)$ depending only on the ratio n/t such that*

$$R_n(t) = \sum_{j=1}^{2} C_j t^{-1/2} e^{-i(p_jt + q_j\log t)} + O(t^{-1}\log t) \ \text{ as } \ t \to \infty. \tag{3.2}$$

The symbol O represents an asymptotic estimate which is uniform with respect to (t, n) satisfying $|n| \le (2 - V_0)t$.

Next, we study other regions. Since the equation (1.1) is invariant under $n \mapsto -n$, we may assume $n > 0$ without loss of generality.

If $n = 2t$, the function $\varphi(z)$ has two stationary points of the second order on the unit circle. We have

Theorem 3.2. *Assume that* $\mathrm{tw}(z_s) = 2$ *for some eigenvalue* z_s. *Then in the region* $2t - Mt^{1/3} < n < 2t + M't^{1/3}$ $(M > 0)$, *we have*

$$R_n(t) = \mathrm{BS}\left(n, t; z_s, p_s T(z_s)^{-2} C_s(0)\right) + O(t^{-1/3}) \quad as\, t \to \infty.$$

In the solitonless case, i.e. if $\mathrm{tw}(z_j) \neq 2$ *for any* j, *then the behavior is as follows: let* t_0 *be such that* $\pi^{-1}\left[\arg r(e^{-\pi i/4})\overline{T(e^{-\pi i/4})}^2 - 2t_0\right]$ *is an integer. Set* $t' = t - t_0$, $p' = d + i(-4t' + \pi n)/4$, $\alpha' = [12t'/(6t' - n)]^{1/3}$, $q' = -2^{-4/3}3^{1/3}(6t' - n)^{-1/3}(2t' - n)$ *and* $\hat{r} = r(e^{-\pi i/4})\overline{T(e^{-\pi i/4})}^2$. *Then we have*

$$R_n(t) = \frac{e^{2p' - \pi i/4}\alpha'}{(3t')^{1/3}} u\Big(\frac{4q'}{3^{1/3}}; \hat{r}, -\hat{r}, 0\Big) + O(t'^{-2/3}).$$

Here $u(s; \mathrm{p}, \mathrm{q}, \mathrm{r})$ *is a solution of the Painlevé II equation* $u''(s) - su(s) - 2u^3(s) = 0$. *Its parametrization is given in [5] (and is repeated in [17]).*

If $n > 2t$, the function $\varphi(z)$ has four saddle points off the unit circle. We have

Theorem 3.3. *In the region* $2 < \mathrm{tw}(z_s) - d \leq n/t \leq \mathrm{tw}(z_s) + d$ *with sufficiently small* d,

$$R_n(t) = \mathrm{BS}\left(n, t; z_s, p_s T(z_s)^{-2} C_s(0)\right) + O(n^{-k}) \quad as\, |n| \to \infty$$

for any positive integer k.

In the solitonless case, i.e. if $\mathrm{tw}(z_j) \notin [n/t - d, n/t + d]$ *for any* j, *then*

$$R_n(t) = O(n^{-k}) \quad as\, |n| \to \infty$$

for any positive integer k.

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