RIMS Kôkyûroku Bessatsu
B75 (2019), 041–060

An algorithmic study on the integration of holonomic hyperfunctions — oscillatory integrals and a phase space integral associated with a Feynman diagram

By

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Abstract

Let $u(x, y)$ be a generalized function which satisfies a holonomic system M of linear differential equations with polynomial coefficients. Suppose that $u(x, y)$ is integrable with respect to x and let $v(y)$ be its integral. We give a sufficient condition for $v(y)$ to satisfy the D-module theoretic integration module of M, which can be computed algorithmically. We present some examples related to oscillatory integrals and Cutkosky-type phase space integrals associated with Feynman diagrams

§ 1. Introduction

In this paper, we call a distribution, or more generally, a hyperfunction *holonomic* for short, if it satisfies a holonomic system of linear differential equations with *polynomial* coefficients. The integration of a holonomic function with respect to some of its variables is again holonomic if the integrand is 'rapidly decreasing' with respect to the integration variables. Moreover, a holonomic system for the integral is defined naturally as a D-module and is computable, at least in theory, under this condition.

First we give a sufficient condition for the integral to be well-defined and to satisfy the D-module theoretic integration module, or the direct image. This allows us to treat, e.g., the oscillatory integral with a polynomial phase and a holonomic distribution as the amplitude function which is 'rapidly decreasing' with respect to the integration variables.

Received May 19, 2017. Revised February 15, 2018.
2010 Mathematics Subject Classification(s): 14F10, 42B20, 81Q30
Key Words: *holonomic system, hyperfunction, algorithm, oscillatory integral, Feynman integral.*
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Then we give some examples of computation of holonomic systems for such oscillatory integrals and their Fourier transforms, as well as what are called Cutkosky-type phase space integrals associated with Feynman diagrams.

The author would like to thank Professors Takahiro Kawai and Naofumi Honda for valuable suggestions on the phase space integral. All examples in this article were computed by using a computer algebra system Risa/Asir [7]. Especially, the author would like to acknowledge the assistance of Professor Masayuki Noro in computing the characteristic cycle associated with the phase space integral of Example 5.4.

§ 2. Integration of generalized functions

Let $\varpi : \mathbb{R}^{n+d} \ni (x, y) \mapsto y \in \mathbb{R}^d$ be the projection with the standard coordinates $x = (x_1, \dots, x_n)$ of $\mathbb{R}^n$ and $y = (y_1, \dots, y_d)$ of $\mathbb{R}^d$.

Then the integration along the fibers of ϖ gives a sheaf homomorphism $\varpi_! \mathcal{B}_{\mathbb{R}^{n+d}} \to \mathcal{B}_{\mathbb{R}^d}$, and hence, in particular, a homomorphism

$$\Gamma(U, \varpi_! \mathcal{B}_{\mathbb{R}^{n+d}}) \longrightarrow \Gamma(U, \mathcal{B}_{\mathbb{R}^d})$$

for an open set U of $\mathbb{R}^d$. Here $\mathcal{B}_{\mathbb{R}^n}$ stands for the sheaf of hyperfunctions on $\mathbb{R}^n$ and $\varpi_!$ the sheaf-theoretic direct image with proper supports.

For example, for a real polynomial f in x, and a distribution φ on $\mathbb{R}^n$, we are interested in the integrals

$$I(f, \varphi)(t) = \int_{\mathbb{R}^n} \delta(t - f(x))\varphi(x)\, dx, \qquad \hat{I}(f, \varphi)(t) = \int_{\mathbb{R}^n} e^{itf(x)}\varphi(x)\, dx.$$

$I(f, \varphi)$ and $\hat{I}(f, \varphi)$ are related by Fourier transformation. If φ is a probability density function, then $I(f, \varphi)$ is that of the random variable $f(x)$, and $\hat{I}(f, \varphi)$ is the characteristic function. We do not assume that φ has a compact support. Hence the integrands do not belong to $\Gamma(\mathbb{R}, \varpi_! \mathcal{B}_{\mathbb{R}^{n+1}})$ in general.

Definition 2.1. We call a pair of classes $(\mathcal{F}_{n,d}, \mathcal{F}_{0,d})$ adapted to the projection $\varpi : \mathbb{R}^{n+d} \to \mathbb{R}^d$ if the following conditions are satisfied:

1. $\mathcal{F}_{n,d}$ is a left module over the ring $D_{n+d} = \mathbb{C}\langle x, y, \partial_x, \partial_y\rangle$ of differential operators with polynomial coefficients in the variables (x, y) with the notation $\partial_x = (\partial_{x_1}, \dots, \partial_{x_n})$, $\partial_{x_j} = \partial/\partial x_j$.

2. $\mathcal{F}_{0,d}$ is a left module over $D_d = \mathbb{C}\langle y, \partial_y\rangle$.

3. There exists a $\mathbb{C}$-linear map $\varpi_* : \mathcal{F}_{n,d} \longrightarrow \mathcal{F}_{0,d}$.

4. For any $u \in \mathcal{F}_{n,d}$, $P \in D_d$, and $j = 1, \dots, n$, one has

$$P\varpi_*(u) = \varpi_*(Pu), \qquad \varpi_*(\partial_{x_j} u) = 0.$$

The first example of a pair adapted to ϖ is $\Gamma(U, \varpi_! \mathcal{B}_{\mathbb{R}^{n+d}})$ and $\Gamma(U, \mathcal{B}_{\mathbb{R}^d})$ with $\varpi_*(u(x,y)) = \int_{\mathbb{R}^n} u(x,y)\,dy$ for $u \in \Gamma(U, \varpi_! \mathcal{B}_{\mathbb{R}^{n+d}})$.

As the second example, let $\mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d)$ be the subspace of $\mathcal{S}'(\mathbb{R}^{n+d})$ consisting of distributions of the form

$$u(x,y) = \sum_{j=1}^{m} u_j(x) v_j(x,y) \qquad (m \in \mathbb{N},\ u_j \in \mathcal{S}(\mathbb{R}^n),\ v_j \in \mathcal{S}'(\mathbb{R}^{n+d})), \tag{2.1}$$

where $\mathcal{S}$ and $\mathcal{S}'$ denote the space of rapidly decreasing functions and that of tempered distributions respectively. Then $\mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d)$ is a left D_{n+d}-submodule of $\mathcal{S}'(\mathbb{R}^{n+d})$.

As a special case $d = 0$, we denote by $\mathcal{SS}'(\mathbb{R}^n)$ the subspace of $\mathcal{S}'(\mathbb{R}^n)$ consisting of distributions of the form

$$u(x) = \sum_{j=1}^{m} u_j(x) v_j(x) \qquad (m \in \mathbb{N},\ u_j \in \mathcal{S}(\mathbb{R}^n),\ v_j \in \mathcal{S}'(\mathbb{R}^n)).$$

For a distribution $u(x,y)$ in $\mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d)$, the integral $\varpi_*(u(x,y)) = \int_{\mathbb{R}^n} u(x,y)\,dx$ is defined by the pairing

$$\left\langle \int_{\mathbb{R}^n} u(x,y)\,dx,\ \varphi(y) \right\rangle = \sum_{j=1}^{m} \langle v_j(x,y),\ u_j(x)\varphi(y) \rangle \qquad (\forall \varphi \in \mathcal{S}(\mathbb{R}^d)).$$

It does not depend on the choice of expression (2.1). In fact, assume $u(x,y) = 0$ in (2.1) and take $\chi(x)$ which belongs to the space $C_0^\infty(\mathbb{R}^n)$ of infinitely differentiable functions with compact support such that $\chi(x) = 1$ if $|x| \le 1$. Then for an arbitrary constant $r > 0$, we have an equality

$$0 = \left\langle \sum_{j=1}^{m} u_j(x) v_j(x,y),\ \chi(x/r)\varphi(y) \right\rangle = \sum_{j=1}^{m} \langle v_j(x,y),\ \chi(x/r) u_j(x) \varphi(y) \rangle$$

for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Since $\chi(x/r)u_j(x)\varphi(y)$ converges to $u_j(x)\varphi(y)$ in $\mathcal{S}(\mathbb{R}^{n+d})$ as $r \to \infty$, we get

$$\sum_{j=1}^{m} \langle v_j(x,y),\ u_j(x)\varphi(y) \rangle = 0.$$

Proposition 2.2 (differentiation under the integral sign). *Suppose that $u(x,y)$ belongs to $\Gamma(U, \varpi_! \mathcal{B}_{\mathbb{R}^{n+d}})$ with an open set U of $\mathbb{R}^d$, or else to $\mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d)$. Then*

$$P(y, \partial_y) \int_{\mathbb{R}^n} u(x,y)\,dx = \int_{\mathbb{R}^n} P(y, \partial_y) u(x,y)\,dx$$

holds for any $P = P(y, \partial_y) \in D_d$.

Proof. First we suppose $u(x,y)$ belongs to $\mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d)$ and is defined by

$$u(x,y) = \sum_{j=1}^{m} u_j(x) v_j(x,y)$$

with $u_j \in \mathcal{S}(\mathbb{R}^n)$ and $v_j \in \mathcal{S}'(\mathbb{R}^{n+d})$. Then for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\begin{aligned}
&\left\langle \partial_{y_i} \int_{\mathbb{R}^n} u(x,y)\,dx,\ \varphi(y) \right\rangle = -\left\langle \int_{\mathbb{R}^n} u(x,y)\,dx,\ \partial_{y_i}\varphi(y) \right\rangle \\
&= -\sum_{j=1}^{m} \langle v_j(x,y),\ u_j(x)\partial_{y_i}\varphi(y)\rangle = -\sum_{j=1}^{m} \langle v_j(x,y),\ \partial_{y_i}(u_j(x)\varphi(y))\rangle \\
&= \sum_{j=1}^{m} \langle \partial_{y_i} v_j(x,y),\ u_j(x)\varphi(y)\rangle = \left\langle \int_{\mathbb{R}^n} \partial_{y_i} u(x,y)\,dx,\ \varphi(y) \right\rangle
\end{aligned}$$

and

$$\begin{aligned}
&\left\langle y_i \int_{\mathbb{R}^n} u(x,y)\,dx,\ \varphi(y) \right\rangle = \sum_{j=1}^{m} \langle v_j(x,y),\ y_i u_j(x)\varphi(y)\rangle \\
&= \sum_{j=1}^{m} \langle y_i v_j(x,y),\ u_j(x)\varphi(y)\rangle = \left\langle \int_{\mathbb{R}^n} y_i u(x,y)\,dx,\ \varphi(y) \right\rangle.
\end{aligned}$$

Next, let us assume $u(x,y)$ to belong to $\Gamma(U, \varpi_! \mathcal{B}_{\mathbb{R}^{n+d}})$. Moreover, by induction on n, we may assume $n = 1$. Since the statement is local in U, we may suppose that U is convex and the support of $u(x,y)$ is contained in $[-R/2, R/2] \times U$ with $R > 0$. Then there exists a hyperfunction $v(x,y)$ on $\mathbb{R} \times U$ such that $\partial_x v(x,y) = u(x,y)$ whose singular spectrum S.S. $v(x,y)$ does not contain the points $(\pm R, y; \pm\sqrt{-1}dx)$ with $y \in U$ in the purely imaginary cotangent bundle $\sqrt{-1}T^*(\mathbb{R} \times \mathbb{R}^d)$. See Proposision 3.2.1 and subsequent arguments in Kashiwara-Kawai-Kimura [3] on integration of hyperfunctions. This implies that x is a real analytic parameter of $v(x,y)$ on a neighborhood of $\{\pm R\} \times U$. Hence $v(\pm R, y)$ are well-defined as hyperfunctions on U and one has

$$\int_{-\infty}^{\infty} u(x,y)\,dx = v(R,y) - v(-R,y)$$

by the definition. One also has

$$\int_{-\infty}^{\infty} P(y,\partial_y) u(x,y)\,dx = P(y,\partial_y)v(R,y) - P(y,\partial_y)v(-R,y)$$

for any $P \in D_d$ since $\partial_x P(y,\partial_y)v(x,y) = P(y,\partial_y)u(x,y)$. Thus we get

$$\begin{aligned}
P(y,\partial_y)\int_{-\infty}^{\infty} u(x,y)\,dx &= P(y,\partial_y)v(R,y) - P(y,\partial_y)v(-R,y) \\
&= \int_{-\infty}^{\infty} P(y,\partial_y)u(x,y)\,dx.
\end{aligned}$$

□

Proposition 2.3. Suppose that $u(x,y)$ belongs to $\Gamma(U, \varpi_! \mathcal{B}_{\mathbb{R}^{n+d}})$ with an open set U of $\mathbb{R}^d$, or else to $\mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d)$. Then one has

$$\int_{\mathbb{R}^n} \partial_{x_j} u(x,y)\, dx = 0 \qquad (j = 1, \dots, n).$$

Proof. First suppose that $u(x,y)$ belongs to $\mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d)$. We may assume, without loss of generality, that $u(x,y) = v(x)w(x,y)$ with $v \in \mathcal{S}(\mathbb{R}^n)$ and $w \in \mathcal{S}'(\mathbb{R}^{n+d})$. Then for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\begin{aligned}
&\left\langle \int_{\mathbb{R}^n} \partial_{x_j}(v(x)w(x,y))\, dx,\ \varphi(y) \right\rangle \\
&= \left\langle \int_{\mathbb{R}^n} (\partial_{x_j} v(x)) w(x,y)\, dx,\ \varphi(y) \right\rangle + \left\langle \int_{\mathbb{R}^n} v(x)(\partial_{x_j} w(x,y))\, dx,\ \varphi(y) \right\rangle \\
&= \langle w(x,y),\, (\partial_{x_j} v(x))\varphi(y)\rangle + \langle \partial_{x_j} w(x,y),\, v(x)\varphi(y)\rangle \\
&= \langle w(x,y),\, (\partial_{x_j} v(x))\varphi(y)\rangle - \langle w(x,y),\, \partial_{x_j}(v(x)\varphi(y))\rangle = 0.
\end{aligned}$$

Next, let us assume that $u(x,y)$ belongs to $\Gamma(U, \varpi_! \mathcal{B}_{\mathbb{R}^{n+d}})$. We may also assume that U is convex, $n = 1$, and the support of $u(x,y)$ is contained in $[-R/2, R/2] \times U$ with $R > 0$. Then by the definition of the integration, we have

$$\int_{-\infty}^{\infty} \partial_x u(x,y)\, dx = u(R,y) - u(-R,y) = 0.$$

$\square$

Hence the pairs $(\Gamma(U, \varpi_! \mathcal{B}_{\mathbb{R}^{n+d}}), \Gamma(U, \mathcal{B}_{\mathbb{R}^d}))$ and $(\mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d),\ \mathcal{S}'(\mathbb{R}^d))$ are adapted to the projection ϖ of $\mathbb{R}^{n+d}$ to $\mathbb{R}^d$.

§ 3. Integration of D-modules

Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_d)$ be (complex or real) variables. We set $X = \mathbb{C}^{n+d}$ and $Y = \mathbb{C}^d$ and let $\varpi_{\mathbb{C}} : X \ni (x,y) \longmapsto y \in Y$ be the projection. We denote by $D_X = D_{n+d}$ the ring of differential operators in the variables (x,y), and by $D_Y = D_d$ that in the variables y. The module

$$D_{Y \leftarrow X} := D_X / (\partial_{x_1} D_X + \cdots + \partial_{x_n} D_X)$$

has a structure of (D_Y, D_X)-bimodule. The *integral* of a left D_X-module M along the fibers of $\varpi_{\mathbb{C}}$, or the *direct image* by $\varpi_{\mathbb{C}}$ is defined to be the left D_Y-module

$$(\varpi_{\mathbb{C}})_* M := D_{Y \leftarrow X} \otimes_{D_X} M = M / (\partial_{x_1} M + \cdots + \partial_{x_n} M).$$

For an element u of M, let $[u]$ be its residue class in $(\varpi_{\mathbb{C}})_*M$. If M is generated by $u_1, \ldots, u_r$ over D_X, then $(\varpi_{\mathbb{C}})_*M$ is generated by the set $\{x^\alpha[u_j] \mid 1 \le j \le r,\ \alpha \in \mathbb{N}^n\}$ over D_Y.

Let $(\mathcal{F}_{n,d}, \mathcal{F}_n)$ be a pair adapted to ϖ. Let h be a D_X-homomorphism from M to $\mathcal{F}_{n,d}$. Let us define a $\mathbb{C}$-linear map h' from M to $\mathcal{F}_d$ by

$$h'(u) = \varpi_*(h(u)) \qquad (\forall u \in M),$$

which is D_Y-linear and we have

$$\partial_{x_1}M + \cdots + \partial_{x_n}M \subset \ker h'.$$

Hence h' induces a D_Y-homomorphism

$$\varpi_*(h) : (\varpi_{\mathbb{C}})_*M \longrightarrow \mathcal{F}_d.$$

In conclusion, we have defined a $\mathbb{C}$-linear map

$$\varpi_* \ :\ \mathrm{Hom}_{D_X}(M, \mathcal{F}_{n,d}) \longrightarrow \mathrm{Hom}_{D_Y}((\varpi_{\mathbb{C}})_*M, \mathcal{F}_d).$$

If M is a holonomic D_X-module, then $(\varpi_{\mathbb{C}})_*M$ is a holonomic D_Y-module. An algorithm to compute $(\varpi_{\mathbb{C}})_*M$, which works at least if M is holonomic, was given in [9], [10]; see also [8]. For practical computation, we use a library file `nk_restriction.rr` by Hiromasa Nakayama for the computer algebra system Risa/Asir [7].

§ 4. Oscillatory integrals

Let $f(x)$ be a real polynomial in the real variables $x = (x_1, \ldots, x_n)$. Suppose that $\varphi(x)$ belongs to $\mathcal{SS}'(\mathbb{R}^n)$. Let t and τ be real variables. Since both $\delta(t - f(x))\varphi(x)$ and $e^{itf(x)}\varphi(x)$ belong to $\mathcal{SS}'(\mathbb{R}^n_x \times \mathbb{R}_t)$, the integrals

$$F(t) = I(f, \varphi)(t) = \int_{\mathbb{R}^n} \delta(t - f(x))\varphi(x)\,dx, \qquad G(t) = \hat{I}(f, \varphi)(t) = \int_{\mathbb{R}^n} e^{itf(x)}\varphi(x)\,dx$$

are well-defined as elements of $\mathcal{S}'(\mathbb{R})$. The integral $\hat{I}(f, \varpi)(t)$ is called the oscillatory integral with the phase function $f(x)$ and the amplitude function $\varphi(x)$, which is usually assumed to belong to $C_0^\infty(\mathbb{R}^n)$ in the literature (see e.g., [5]).

Proposition 4.1. *Define $F(t)$ and $G(\tau)$ as above with $\varphi \in \mathcal{SS}'(\mathbb{R}^n)$ and $f \in \mathbb{R}[x]$. Then $F(t)$ and $G(\tau)$ are related by*

$$G(\tau) = \hat{F}(\tau) := \int_{-\infty}^{\infty} e^{it\tau}F(t)\,dt, \qquad F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\tau}G(\tau)\,d\tau,$$

where the integrals make sense as Fourier transformation in $\mathcal{S}'(\mathbb{R})$. Moreover $G(\tau)$ belongs to $C^\infty(\mathbb{R})$.

Proof. We may assume that $\varphi(x) = \psi(x)u(x)$ with $\psi \in \mathcal{S}(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. Then by the definition of the integral of an element of $\mathcal{SS}'(\mathbb{R}^n_x \times \mathbb{R}_t)$, we get, for any $\chi \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned}\langle \hat{F}, \chi\rangle &= \left\langle \int_{\mathbb{R}^n} \psi(x)\delta(t-f(x))u(x)\,dx,\ \hat{\chi}\right\rangle = \langle \delta(t-f(x))u(x),\ \psi(x)\hat{\chi}(t)\rangle \\ &= \langle \delta(t)u(x),\ \psi(x)\hat{\chi}(t+f(x))\rangle = \langle u(x),\ \psi(x)\hat{\chi}(f(x))\rangle_x \\ &= \left\langle u(x),\ \psi(x)\int_{-\infty}^{\infty} e^{itf(x)}\chi(t)\,dt\right\rangle_x \\ &= \int_{-\infty}^{\infty}\left\langle u(x),\ \psi(x)e^{itf(x)}\chi(t)\right\rangle_x dt = \int_{-\infty}^{\infty}\left\langle u(x),\ \psi(x)e^{itf(x)}\right\rangle_x \chi(t)\,dt\end{aligned}$$

in view of the lemma below. This implies

$$\hat{F}(\tau) = \left\langle u(x),\ \psi(x)e^{i\tau f(x)}\right\rangle_x = \int_{\mathbb{R}^n} \psi(x)e^{i\tau f(x)}u(x)\,dx = G(\tau)$$

and that $G(\tau)$ belongs to $C^\infty(\mathbb{R})$. □

Lemma 4.2. *Assume that $f(x)$ is a real polynomial in x, χ belongs to $\mathcal{S}(\mathbb{R})$, and that ψ belongs to $\mathcal{S}(\mathbb{R}^n)$. Then*

$$\psi(x)\int_{-\infty}^{\infty} e^{itf(x)}\chi(t)\,dt = \lim_{R\to\infty}\lim_{N\to\infty}\frac{R}{N}\sum_{k=-N}^{N-1}\psi(x)\exp\Big(i\frac{Rk}{N}f(x)\Big)\chi\Big(\frac{Rk}{N}\Big)$$

holds in the topology of $\mathcal{S}(\mathbb{R}^n)$.

Proof. Since it is easy to see that

$$\psi(x)\int_{-\infty}^{\infty} e^{itf(x)}\chi(t)\,dt = \psi(x)\lim_{R\to\infty}\int_{-R}^{R} e^{itf(x)}\chi(t)\,dt$$

holds in the topology of $\mathcal{S}(\mathbb{R}^n)$, let us show

$$\psi(x)\int_{-R}^{R} e^{itf(x)}\chi(t)\,dt = \lim_{N\to\infty}\frac{R}{N}\sum_{k=-N}^{N-1}\psi(x)\exp\Big(i\frac{Rk}{N}f(x)\Big)\chi\Big(\frac{Rk}{N}\Big)$$

in $\mathcal{S}(\mathbb{R}^n)$. For integers k, ν with $-N \le k \le N-1$ and $\nu \ge 0$, there exist t_k, t'_k in the interval $\left[\frac{R}{N}k,\ \frac{R}{N}(k+1)\right]$, which depend on x, such that

$$\int_{-R}^{R} e^{itf(x)}t^\nu\chi(t)\,dt = \frac{R}{N}\sum_{k=-N}^{N-1}\left\{\cos(t_kf(x))t_k^\nu\chi(t_k) + i\sin(t'_kf(x)){t'_k}^{\nu}\chi(t'_k)\right\}.$$

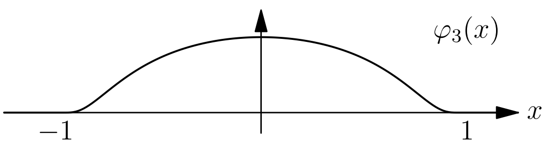
Hence

$$
\begin{aligned}
&\left|\int_{-R}^{R} e^{itf(x)} t^{\nu}\chi(t)\,dt - \frac{R}{N}\sum_{k=-N}^{N-1}\exp\Big(i\frac{Rk}{N}f(x)\Big)\Big(\frac{Rk}{N}\Big)^{\nu}\chi\Big(\frac{Rk}{N}\Big)\right| \\
&\le \frac{R}{N}\sum_{k=-N}^{N-1}\bigg|\big\{\cos(t_k f(x)) t_k^{\nu}\chi(t_k) + i\sin(t_k' f(x)) {t_k'}^{\nu}\chi(t_k')\big\} \\
&\qquad\qquad - \exp\Big(i\frac{Rk}{N}f(x)\Big)\Big(\frac{Rk}{N}\Big)^{\nu}\chi\Big(\frac{Rk}{N}\Big)\bigg| \\
&\le \frac{R}{N}\sum_{k=-N}^{N-1} C(|f(x)|+1)\max\left\{\left|t_k - \frac{Rk}{N}\right|,\ \left|t_k' - \frac{Rk}{N}\right|\right\} \\
&\le \frac{2R^2}{N} C(|f(x)|+1)
\end{aligned}
$$

with some constant C independent of x. This implies the assertion. □

Example 4.3. Set $n = 1$ and $x = x_1$. Let us choose

$$
\varphi_1(x) = \exp\Big(-\frac{x^2}{2}\Big), \quad \varphi_2(x) = Y(1-x^2), \quad \varphi_3(x) = \begin{cases} \exp\Big(-\dfrac{1}{1-x^2}\Big) & \text{if } |x| < 1, \\ 0 & \text{otherwise} \end{cases}
$$



as amplitude functions and set

$$
F_k(t) = \int_{-\infty}^{\infty} \delta(t - x^2)\varphi_k(x)\,dx, \quad G_k(\tau) = \int_{-\infty}^{\infty} e^{i\tau x^2}\varphi_k(x)\,dx \quad (k = 1, 2, 3).
$$

Here $Y(x)$ denotes the Heaviside function. Then $F_k(t)$ satisfy differential equations

$$
(2t\partial_t + t + 1)F_1(t) = 0, \quad (t-1)(2t\partial_t + 1)F_2(t) = 0, \quad (2t(t-1)^2\partial_t + t^2 + 1)F_3(t) = 0
$$

respectively. The point 0 is a regular singular point of the three differential equations with characteristic exponent $-1/2$. The point 1 is a regular singular point of the second equation with characteristic exponent 0, but is an irregular singular point of the last equation. Consequently we get

$$
F_1(t) = t_+^{-1/2} e^{-t/2}, \qquad F_2(t) = t_+^{-1/2} Y(1-t)
$$

in view of

$$
\int_{-\infty}^{\infty} F_k(t)\,dt = \int_{-\infty}^{\infty} \varphi_k(x)\,dx,
$$

and

$$F_3(t) = C_3 t_+^{-1/2} Y(1-t) \exp\Big(-\frac{1}{1-t}\Big)$$

with some constant C_3.

On the other hand, G_k $(k = 1,2,3)$ belong to $\mathcal{S}'(\mathbb{R}) \cap C^\infty(\mathbb{R})$ and satisfy differential equations

$$\begin{aligned}
&((2\tau + i)\partial_\tau + 1)G_1(\tau) = 0,\\
&(2\tau\partial_\tau^2 + (-2i\tau + 3)\partial_\tau - i)G_2(\tau) = 0,\\
&(2\tau\partial_\tau^3 + (-4i\tau + 5)\partial_\tau^2 + (-2\tau - 8i)\partial_\tau - 1)G_3(\tau) = 0
\end{aligned}$$

respectively. In particular, we have $G_1(\tau) = \sqrt{2\pi}(1-2i\tau)^{-1/2}$. The equations for $G_2(\tau)$ and for $G_3(\tau)$ have 0 as a regular singular point and the point at infinity as an irregular singular point. Note that $G_2(\tau)$ and $G_3(\tau)$ are entire, i.e., holomorphic on $\mathbb{C}$.

Example 4.4. Set

$$F(t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \exp\Big(-\frac{|x|^2}{2}\Big)\delta(t - f(x))\,dx$$

with a quadratic form $f(x) = \sum_{i,j=1}^n a_{ij}x_ix_j$. If the absolute values of all the eigenvalues of the matrix (a_{ij}) are the same, then $F(t)$ satisfies a linear differential equation of the second order. We may assume

$$f(x) = a(x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2)$$

with a constant $a > 0$. Then the integrand $u = u(x,t)$ satisfies

$$(t - f(x))u = (\partial_{x_i} + 2ax_i\partial_t + x_i)u = (\partial_{x_j} - 2ax_j\partial_t + x_j)u = 0 \qquad (1 \le i \le p < j \le n).$$

The following operators P and Q annihilate u:

$$\begin{aligned}
P &= \sum_{i=1}^p x_i(\partial_{x_i} + 2ax_i\partial_t + x_i) + \sum_{i=p+1}^n x_i(\partial_{x_i} - 2ax_i\partial_t + x_i)\\
&= \sum_{i=1}^n \partial_{x_i}x_i + 2f\partial_t + |x|^2 - n = \sum_{i=1}^n \partial_{x_i}x_i + 2\partial_t t + |x|^2 - n - 2\partial_t(t-f),\\
Q &= \sum_{i=1}^p x_i(\partial_{x_i} + 2ax_i\partial_t + x_i) - \sum_{i=p+1}^n x_i(\partial_{x_i} - 2ax_i\partial_t + x_i)\\
&= \sum_{i=1}^p \partial_{x_i}x_i - \sum_{i=p+1}^n \partial_{x_i}x_i + 2a|x|^2\partial_t + \frac{1}{a}f + n - 2p.
\end{aligned}$$

Hence

$$2a\partial_t P - Q = 2a\sum_{i=1}^{p}\partial_{x_i}x_i\partial_t - \sum_{i=1}^{p}\partial_{x_i}x_i + \sum_{i=p+1}^{n}\partial_{x_i}x_i + (-4a\partial_t^2 + \frac{1}{a})(t-f)$$
$$+ 4a\partial_t^2 t - 2na\partial_t - \frac{1}{a}t + (2p-n)$$

implies

$$\{4a^2t\partial_t^2 + 2a^2(4-n)\partial_t - t + (2p-n)a\}F(t) = 0.$$

The solutions of this differential equation are expressed as

$$P\left\{\begin{array}{cccc} \multicolumn{2}{c}{\overbrace{\qquad\qquad\qquad\qquad\qquad}^{\infty}} & 0 & \\ \frac{1}{2a} & 1-\frac{p}{2} & 0 & t \\ -\frac{1}{2a} & -\frac{1}{4}(2n-2p-4) & \frac{n-2}{2} & \end{array}\right\}.$$

On the other hand,

$$G(\tau) = \int_{\mathbb{R}^n}\exp\Big(i\tau f(x) - \frac{|x|^2}{2}\Big)$$

satisfies a differential equation

$$\{(4a^2\tau^2+1)\partial_\tau + a(2na\tau + (n-2p)i)\}G(\tau) = 0.$$

It follows that

$$G(\tau) = \exp\left(i\Big(p-\frac{n}{2}\Big)\tan^{-1}(2a\tau)\right)(4a^2\tau^2+1)^{-n/4}.$$

More generally, if $f(x)$ is a general quadratic form with eigenvalues $a_1,\dots,a_n$, then one has

$$G(\tau) = \prod_{k=1}^{n}(1-2ia_k\tau)^{-1/2} = \prod_{k=1}^{n}(4a_k^2\tau^2+1)^{-1/4}\exp\Big(-\frac{i}{2}\tan^{-1}(2a_k\tau)\Big)$$

since $G(\tau) = (1-2i\tau)^{-1/2}$ if $n=1$ and $a=a_1=1$.

Example 4.5. Set $f(x,y) = x^3 - y^2$ and

$$F(t) = (2\pi)^{-1}\int_{\mathbb{R}^2}\exp\Big(-\frac{x^2+y^2}{2}\Big)\delta(t-f(x,y))\,dxdy.$$

Then $F(t)$ satisfies

$$\{108t^2\partial_t^5 + (-108t^2 + 648t)\partial_t^4 + (27t^2 - 486t + 627)\partial_t^3 + (85t - 303)\partial_t^2 + (-4t + 21)\partial_t + t - 3\}F(t) = 0.$$

It has a regular singularity at 0 with the indicial equation $s(s-1)(s-2)(6s+1)(6s-7)$. Note that the b-function of f is $(s+1)(6s+5)(6s+7)$.

Example 4.6. Set $f(x, y, z) = x^2 - y^2 z$ and

$$F(t) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \exp\Big(-\frac{x^2 + y^2 + z^2}{2}\Big)\delta(t - f(x, y, z))\, dxdydz.$$

Then $F(t)$ satisfies

$$\{16t^2\partial_t^5 + (16t^2 + 96t)\partial_t^4 + (4t^2 + 72t + 96)\partial_t^3 + (16t + 48)\partial_t^2 + (4t + 9)\partial_t + t + 3\}F(t) = 0.$$

It has a regular singularity at 0 with the indicial equation $s^2(s-1)^2(s-2)$. Note that the b-function of f is $(s+1)^2(2s+3)$.

Example 4.7. Set $f(x, y, z) = x^3 - y^2 z^2$ and

$$F(t) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \exp\Big(-\frac{x^2 + y^2 + z^2}{2}\Big)\delta(t - f(x, y, z))\, dxdydz.$$

Then $F(t)$ satisfies a linear ordinary differential equation of order 10 which has a regular singularity at 0 with the indicial equation

$$s(s-1)(s-2)(s-3)(s-4)(s-5)(6s+1)^2(6s-7)^2 = 0.$$

Note that the b-function of f is $(s+1)(3s+4)(3s+5)(6s+5)^2(6s+7)^2$.

§ 5. Cutkosky-type phase space integrals associated with Feynman diagrams

The Cutkosky-type phase space integral associated with a Feynman diagram G, which we will call the phase space integral for short, describes the discontinuity of the Feynman integral $F_G(p)$ along its singularity locus. We consider simple Feynman diagrams in two-dimensional space-time for the sake of simplicity in actual computation, inspired by the recent work by Honda and Kawai (see e.g., [1], [2]) on the Landau-Nakanishi surface associated with G.

In general, for a two-dimensional vector $\mathbf{p} = (p_0, p_1)$, we denote $\mathbf{p}^2 = p_0^2 - p_1^2$ for the Lorentz norm and $d\mathbf{p} = dp_0 dp_1$ for the volume element. Let m be a positive constant.

Then the delta function $\delta(\mathbf{p}^2 - m^2)$ is well-defined and its support coincides with the curve $\mathbf{p}^2 - m^2 = 0$ in the 2-dimensional space-time $\mathbb{R}^2$. We set

$$\delta_+(\mathbf{p}^2 - m^2) = Y(p_0)\delta(\mathbf{p}^2 - m^2),$$

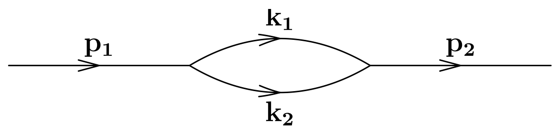
which is well-defined since the line $p_0 = 0$ is disjoint from the curve $\mathbf{p}^2 = m^2$. Its support is contained in $\{\mathbf{p} \mid \mathbf{p}^2 = m^2, p_0 \geq m\}$. Moreover, if $P \in D_2$ annihilates $\delta(\mathbf{p}^2 - m^2)$, then it also annihilates $\delta_+(\mathbf{p}^2 - m^2)$. More precisely it is easy to see that

$$\mathrm{Ann}_{D_2}\delta_+(\mathbf{p}^2 - m^2) = \mathrm{Ann}_{D_2}\delta(\mathbf{p}^2 - m^2)$$

holds, where D_2 is the ring of differential operators with polynomial coefficients with respect to the variables p_0, p_1. On the other hand, we denote by $\delta(\mathbf{p})$ the delta function $\delta(p_0)\delta(p_1)$ supported at the origin of $\mathbb{R}^2$.

We give a precise definition of the Cutkosky-type phase space integral associated with a Feynman diagram in each example instead of presenting a general formulation.

Example 5.1. Let us study the Feynman diagram G with two vertices, two external lines, and two internal lines as below:



Let us assign 2-vectors $\mathbf{p}_1 = (p_{10}, p_{11})$ and $\mathbf{p}_2 = (p_{20}, p_{21})$ to the external lines, $\mathbf{k}_1 = (k_{10}, k_{11})$ and $\mathbf{k}_2 = (k_{20}, k_{21})$ to the internal lines. Then the phase space integral associated with this diagram is defined to be

$$\tilde{I}_G(\mathbf{p}_1, \mathbf{p}_2) = \int \delta(\mathbf{p}_1 - \mathbf{k}_1 - \mathbf{k}_2)\delta(-\mathbf{p}_2 + \mathbf{k}_1 + \mathbf{k}_2)\delta_+(\mathbf{k}_1^2 - m_1^2)\delta_+(\mathbf{k}_2^2 - m_2^2)\, d\mathbf{k}_1 d\mathbf{k}_2.$$

It is easy to see that the product in the integrand makes sense as a hyperfunction by considering the singular spectrum (the analytic wave front set) of each factor. Integration with respect to $\mathbf{k}_2$ yields the expression

$$\tilde{I}_G(\mathbf{p}_1, \mathbf{p}_2) = \delta(\mathbf{p}_1 - \mathbf{p}_2)I_G(\mathbf{p}_1)$$

with

$$I_G(\mathbf{p}_1) = \int \delta_+(\mathbf{k}_1^2 - m_1^2)\delta_+((\mathbf{p}_1 - \mathbf{k}_1)^2 - m_2^2)\, d\mathbf{k}_1.$$

Since $I_G(\mathbf{p}_1)$ is invariant under Lorentz transformations of $\mathbf{p}_1$, we may put $\mathbf{p}_1 = (x, 0)$. Then the support of the integrand of $I_G((x, 0))$ is contained in the set

$$\begin{aligned}
&\{(x, \mathbf{k}_1) \mid \mathbf{k}_1^2 - m_1^2 = (x - k_{10})^2 - k_{11}^2 - m_2^2 = 0,\ k_{10} > 0,\ x - k_{10} > 0\} \\
&\subset \{(x, \mathbf{k}_1) \mid k_{10} \geq m_1,\ x - k_{10} \geq m_2,\ |k_{11}| < k_{10}\} \\
&\subset \{(x, \mathbf{k}_1) \mid x \geq m_1 + m_2,\ m_1 \leq k_{10} \leq x - m_2,\ |k_{11}| < k_{10}\}.
\end{aligned}$$

Hence the support of the integrand is proper with respect to the projection $\varpi : \mathbb{R}^3 \ni (x, \mathbf{k}_1) \mapsto x \in \mathbb{R}$. It follows that $I_G((x,0))$ is well-defined as a hyperfunction on $\mathbb{R}$ and its support is contained in $\{x \in \mathbb{R} \mid x \geq m_1 + m_2\}$.

Let us consider the second local cohomology group $H_I^2(\mathbb{C}[x, \mathbf{k}_1])$ with the ideal I generated by two polynomials $f_1 := \mathbf{k}_1^2 - m_1^2$ and $f_2 := (x - k_{10})^2 - k_{11}^2 - m_2^2$. Then we can identify the integrand with the cohomology class $\delta(f_1, f_2) = [1/(f_1 f_2)]$ in this local cohomology group. Since the variety $f_1 = f_2 = 0$ is non-singular, the annihilator in the ring D_3 of the integrand coincides with that of $\delta(f_1, f_2)$, which consists of first order operators together with f_1, f_2 and can be computed easily.

By virtue of Propositions 2.2 and 2.3, the integration algorithm described in [8] gives us a differential equation $PI_G((x,0)) = 0$ with

$$P = (x - m_1 - m_2)(x - m_1 + m_2)(x + m_1 - m_2)(x + m_1 + m_2)\partial_x + 2(x^2 - m_1^2 - m_2^2)x.$$

If $m_1 \neq m_2$, then we get

$$I_G((x,0)) = C(x - m_1 + m_2)^{-1/2}(x + m_1 - m_2)^{-1/2}(x + m_1 + m_2)^{-1/2}(x - m_1 - m_2)_+^{-1/2}$$

with some constant C by quadrature noting that its support is contained in the interval $[m_1 + m_2, \infty)$ and that there is no hyperfunction solution of the differential equation above whose support is the point $m_1 + m_2$.

On the other hand, if $m_1 = m_2$, then $I_G((x,0))$ satisfies

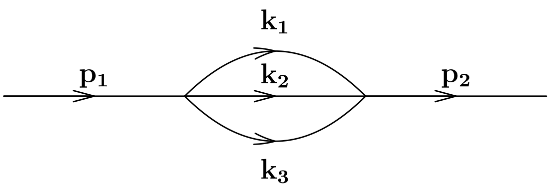
$$\{x(x^2 - 4m_1^2)\partial_x + 2(x^2 - 2m_1^2)\} I_G((x,0)) = 0.$$

It follows that

$$I_G((x,0)) = Cx^{-1}(x + 2m_1)^{-1/2}(x - 2m_1)_+^{-1/2}$$

with some constant C.

Example 5.2. The phase space integral associated with the Feynman diagram G with two vertices, two external lines, and three internal lines as below



is given by

$$\tilde{I}_G(\mathbf{p}_1, \mathbf{p}_2) = \int \delta(\mathbf{p}_1 - \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)\delta(-\mathbf{p}_2 + \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \\ \times \delta_+(\mathbf{k}_1^2 - m_1^2)\delta_+(\mathbf{k}_2^2 - m_2^2)\delta_+(\mathbf{k}_3^2 - m_3^2)\, d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3$$

with variables $\mathbf{p}_1 = (p_{10}, p_{11})$, $\mathbf{p}_2 = (p_{20}, p_{21})$, $\mathbf{k}_1 = (k_{10}, k_{11})$, $\mathbf{k}_2 = (k_{20}, k_{21})$, $\mathbf{k}_3 = (k_{30}, k_{31})$ and positive constants m_1, m_2, m_3. We rewrite this integral as

$$\tilde{I}_G(\mathbf{p}_1, \mathbf{p}_2) = \delta(\mathbf{p}_1 - \mathbf{p}_2) I_G(\mathbf{p}_1)$$

with

$$I_G(\mathbf{p}_1) = \int \delta_+(\mathbf{k}_1^2 - m_1^2)\delta_+(\mathbf{k}_2^2 - m_2^2)\delta_+((\mathbf{p}_1 - \mathbf{k}_1 - \mathbf{k}_2)^2 - m_3^2)\, d\mathbf{k}_1 d\mathbf{k}_2.$$

As in the previous example we can set $\mathbf{p}_1 = (x, 0)$. The support of the integrand is contained in

$$\{(x, \mathbf{k}_1, \mathbf{k}_2) \mid k_{10} \geq m_1,\ k_{20} \geq m_2,\ x - k_{10} - k_{20} \geq m_3,\ |k_{11}| < k_{10},\ |k_{21}| < k_{20}\}.$$

Hence $I_G((x, 0))$ is well-defined as a hyperfunction on $\mathbb{R}$ and its support is contained in the interval $[m_1 + m_2 + m_3, \infty)$.

Since the computation for general m_1, m_2, m_3 is intractable, let us set $m_1 = m_2 = m_3 = 1$. Then by the integration algorithm we obtain

$$\{(x(x-1)(x+1)(x-3)(x+3)\partial_x^2 + (5x^4 - 30x^2 + 9)\partial_x + 4x^3 - 12x\} I_G((x, 0)) = 0.$$

The points $0, \pm 1, \pm 3$ are regular singular and the indicial equations at these points are all s^2. It follows that $I_G((x, 0))$ is a locally integrable function of the form

$$I_G((x, 0)) = a(x) Y(x - 3)$$

with a real analytic function $a(x)$ on the interval $(1, \infty)$ in view of the lemma below and analytic continuation. Here we have $a(3) \neq 0$ unless $I_G((x, 0))$ vanishes everywhere. The point at infinity is also a regular singular point with the indicial equation $(s-2)^2$.

On the other hand, if we set $m_1 = 1$, $m_2 = 2$, $m_3 = 3$, then we obtain

$$\begin{aligned}
&\{56x^2(x-2)(x+2)(x-4)(x+4)(x-6)(x+6)\partial_x^3 \\
&+ (-15x^9 + 1680x^7 - 46256x^5 + 341888x^3 - 387072x)\partial_x^2 \\
&+ (-75x^8 + 5544x^6 - 98000x^4 + 404480x^2 - 96768)\partial_x \\
&- 60x^7 + 3192x^5 - 33712x^3 + 44608x\} I_G((x, 0)) = 0.
\end{aligned}$$

The points $0, \pm 2, \pm 4, \pm 6$ are regular singular points. The indicial equation at 6 is $s^2(s-1)$. It follows that $I_G((x, 0))$ is a locally integrable function of the form

$$I_G((x, 0)) = a(x) Y(x - 6)$$

with a real analytic function $a(x)$ on the interval $(4, \infty)$ such that $a(6) \neq 0$ unless $I_G((x, 0))$ vanishes everywhere, in view of the lemma below. The point at infinity is an irregular singular point.

Lemma 5.3. *Let P be a differential operator of the form*

$$P = x\partial_x^m + a_1(x)\partial_x^{m-1} + \cdots + a_m(x)$$

with a positive integer m and analytic functions $a_1(x), \ldots, a_m(x)$ defined on a neighborhood of $x = 0$. Assume $a_1(0) = m - 1$. Let $u(x)$ be a hyperfunction defined on a neighborhood of 0 whose support is contatined in $\{x \in \mathbb{R} \mid x \geq 0\}$ such that $Pu(x) = 0$. Then $u(x)$ is written in the form

$$u(x) = a(x)Y(x)$$

with a real anayltic function $a(x)$ on a neighborhood of 0 such that $Pa(x) = 0$. Moreover, we have $a(0) \neq 0$ unless $u(x)$ vanishes everywhere.

Proof. Note that 0 is a regular singular point of P and its indicial polynomial at 0 is given by

$$\begin{aligned} b(\lambda) &= \lambda(\lambda-1)\cdots(\lambda-m+1) + a_1(0)\lambda(\lambda-1)\cdots(\lambda-m+2) \\ &= \lambda(\lambda-1)\cdots(\lambda-m+2)(\lambda-m+1+a_1(0)) \\ &= \lambda^2(\lambda-1)\cdots(\lambda-m+2). \end{aligned}$$

Since $b(\lambda) = 0$ has no integer roots greater than $m-2$, it follows that the homomorphism $P : \mathbb{C}\{x\} \to \mathbb{C}\{x\}$ is surjective and the dimension of its kernel is $m - 1$.

By the assumption, there exists a holomorphic function $F(z)$ on the set $\{z \in \mathbb{C} \mid |z| < \varepsilon\} \setminus [0, \infty]$ with $\varepsilon > 0$ such that

$$u(x) = F(x + \sqrt{-1}0) - F(x - \sqrt{-1}0).$$

Since $P : \mathbb{C}\{x\} \to \mathbb{C}\{x\}$ is surjective, we may assume $PF(z) = 0$. Let us rewrite P as

$$P = z\partial_z^m + (m-1)\partial_z^{m-1} + P_1\partial_z^{m-2} + \cdots + P_{m-2}\partial_z + P_{m-1} + P_m + \cdots,$$

where P_k is a differential operator of order at most $\min\{k, m-1\}$ such that

$$P_k z^\lambda = p_k(\lambda) z^{\lambda + \max\{0, k-m+1\}}$$

with a polynomial $p_k(\lambda)$ of λ. Following the Frobenius method, we can construct a series

$$v(z, \lambda) = \sum_{n=0}^{\infty} c_n(\lambda) z^{\lambda+n}$$

with rational functions $c_n(\lambda)$ of λ such that

$$Pv(z, \lambda) = b(\lambda) z^\lambda \tag{5.1}$$

and $c_0(\lambda) = 1$ by the recursion formula

$$\begin{aligned} c_n(\lambda) &= -\sum_{k=1}^{\max\{m-2,n\}} \frac{(\lambda+n-k)\cdots(\lambda+n-m+2)}{b(\lambda+n)} p_k(\lambda+n-m+1)c_{n-k}(\lambda) \\ &\quad - \sum_{k=m-1}^{n} \frac{p_k(\lambda+n-k)}{b(\lambda+n)} c_{n-k}(\lambda) \\ &= -\sum_{k=1}^{\max\{m-2,n\}} \frac{p_k(\lambda+n-m+1)}{(\lambda+n)^2(\lambda+n-1)\cdots(\lambda+n-k+1)} c_{n-k}(\lambda) \\ &\quad - \sum_{k=m-1}^{n} \frac{p_k(\lambda+n-k)}{b(\lambda+n)} c_{n-k}(\lambda) \end{aligned}$$

for $n = 1, 2, 3, \ldots$. This implies that $c_n(\lambda)$ are regular at $\lambda = 0$. Differentiating (5.1) with respect to λ and substituting 0 for λ, we get

$$P\frac{\partial v}{\partial \lambda}(z,0) = \frac{\partial b}{\partial \lambda}(0) + b(0)\log z = 0$$

with

$$\frac{\partial v}{\partial \lambda}(z,0) = \sum_{n=0}^{\infty} \frac{\partial c_n}{\partial \lambda}(0) z^n + \sum_{n=0}^{\infty} c_n(0) z^n \log z.$$

Hence $F(z)$ is written in the form

$$F(z) = G(z) + a\sum_{n=0}^{\infty} c_n(0) z^n \log z$$

with a holomorphic function $G(z)$ on a neighborhood of 0 and $a \in \mathbb{C}$. Hence we get

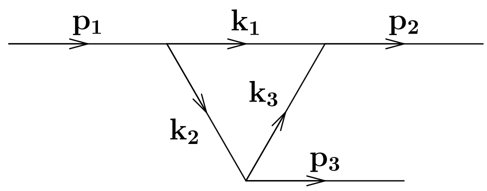
$$u(x) = 2\pi\sqrt{-1}a\sum_{n=0}^{\infty} c_n(0) x^n Y(x) = 2\pi\sqrt{-1}av(x,0)Y(x)$$

with $c_0(0) = 1$ and $Pv(x,0) = b(0) = 0$. □

Example 5.4. Let us consider the phase space integral

$$\begin{aligned} \tilde{I}_G(\mathbf{p}_1, \mathbf{p}_2) = \int & \delta(\mathbf{p}_1 - \mathbf{k}_1 - \mathbf{k}_2)\delta(-\mathbf{p}_2 + \mathbf{k}_1 + \mathbf{k}_3)\delta(-\mathbf{p}_3 + \mathbf{k}_2 - \mathbf{k}_3) \\ & \times \delta_+(\mathbf{k}_1^2 - m_1^2)\delta_+(\mathbf{k}_2^2 - m_2^2)\delta_+(\mathbf{k}_3^2 - m_3^2)\, d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \end{aligned}$$

associated with the diagram G below.



Performing the integration with respect to $\mathbf{k}_2$ and $\mathbf{k}_3$, we obtain

$$\tilde{I}_G(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = \delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) I_G(\mathbf{p}_1, \mathbf{p}_2)$$

with

$$I_G(\mathbf{p}_1, \mathbf{p}_2) = \int \delta_+(\mathbf{k}_1^2 - m_1^2)\delta_+((\mathbf{p}_1 - \mathbf{k}_1)^2 - m_2^2)\delta_+((\mathbf{p}_2 - \mathbf{k}_1)^2 - m_3^2)\, d\mathbf{k}_1.$$

The support of the integrand is contained in

$$\{(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_1) \mid k_{10} \geq m_1,\, p_{10} - k_{10} \geq m_2,\, p_{20} - k_{10} \geq m_3,\ |k_{11}| < k_{10}\}.$$

Hence $I_G(\mathbf{p}_1, \mathbf{p}_2)$ is well-defined as a hyperfunction on $\mathbb{R}^4$ and one has

$$\operatorname{supp} I_G(\mathbf{p}_1, \mathbf{p}_2) \subset \{(\mathbf{p}_1, \mathbf{p}_2) \mid p_{10} \geq m_1 + m_2,\ p_{20} \geq m_1 + m_3\}.$$

Let us set $m_1 = m_2 = m_3 = 1$, $\mathbf{p}_1 = (x, 0)$ and $\mathbf{p}_2 = ((y+z)/2, (y-z)/2)$ following Honda-Kawai-Stapp [2] and set

$$I_G(x, y, z) = I_G((x, 0), ((y+z)/2, (y-z)/2))$$

by abuse of notation. Then the integration algorithm returns a holonomic system $M = D_3/I$ for $I_G(x, y, z)$ with a left ideal I of D_3, which is too complicated to show here. The characteristic variety of M is given by

$$\begin{aligned}\mathrm{Char}(M) = T^*_{\{f=0\}}\mathbb{C}^3 &\cup T^*_{\{x=0\}}\mathbb{C}^3 \cup T^*_{\{x=f_0=0\}}\mathbb{C}^3 \cup T^*_{\{x=yz-4=0\}}\mathbb{C}^3 \\ &\cup T^*_{\{x=y=0\}}\mathbb{C}^3 \cup T^*_{\{x=z=0\}}\mathbb{C}^3 \cup T^*_{\{x=y=z=0\}}\mathbb{C}^3\end{aligned}$$

with

$$f(x, y, z) = yzx^2 - yz(y+z)x + y^2z^2 + (y-z)^2, \qquad f_0(y, z) = f(0, y, z),$$

where we denote by $T^*_Z\mathbb{C}^3$ the closure of the conormal bundle of the regular part of an analytic set Z of $\mathbb{C}^3$. The decomposition of $\mathrm{Char}(M)$ was done by using a library file `noro_pd.rr` of Risa/Asir [7] for prime and primary decomposition of polynomial ideals developed by M. Noro (see e.g., [4] for algorithms); he also computed a primary decomposition of the symbol ideal of I, which enabled us to compute the multiplicity of each component of $\mathrm{Char}(M)$. Thus the characteristic cycle, i.e., the characteristic variety with multiplicity of each component, of M is

$$\begin{aligned}T^*_{\{f=0\}}\mathbb{C}^3 &+ 2\,T^*_{\{x=0\}}\mathbb{C}^3 + T^*_{\{x=f_0=0\}}\mathbb{C}^3 + T^*_{\{x=yz-4=0\}}\mathbb{C}^3 \\ &+ T^*_{\{x=y=0\}}\mathbb{C}^3 + T^*_{\{x=z=0\}}\mathbb{C}^3 + 2\,T^*_{\{x=y=z=0\}}\mathbb{C}^3.\end{aligned}$$

In particular, the support of M as D-module is the hypersuface of $\mathbb{C}^3$ defined by $xf(x,y,z) = 0$. The singular locus of the complex hypersurface $f = 0$ is the union of two complex lines $\{x = y = z\}$ and $\{y = z = 0\}$. There is a stratification of the hypersurface $f = 0$ of $\mathbb{C}^3$ with respect to the (local) b-function $b_{f,p}(s)$ of f at a point p of each stratum as follows:

strata	$b_{f,p}(s)$
$\{(0,0,0)\}$	$(s+1)^3(2s+3)$
$\{(2,0,0),(-2,0,0),(2,2,2),(-2,-2,-2)\}$	$(s+1)^2(2s+3)$
$\{x=y=z\}\cup\{y=z=0\}\setminus\{(0,0,0),(\pm 2,0,0),\pm(2,2,2)\}$	$(s+1)^2$
$\{f=0\}\setminus(\{x=y=z\}\cup\{y=z=0\})$	$s+1$

Note that the b-function of f at the points $(\pm 2, 0, 0)$ and $\pm(2, 2, 2)$ coincides with that of $g := x^2 - y^2 z$ at the origin, which defines what is called the Whitney umbrella. More precisely, the b-function of g at each stratum is as follows:

stratum	$b_{g,p}(s)$
$\{(0,0,0)\}$	$(s+1)^2(2s+3)$
$\{x=y=0\}\setminus\{(0,0,0)\}$	$(s+1)^2$
$\{g=0\}\setminus\{x=y=0\}$	$s+1$

However, the present author does not know whether the germ of analytic function $(f, (2, 2, 2))$, for example, is (real) analytically equivalent to $(g, (0, 0, 0))$. We used a library file `nn_ndbf.rr` of Risa/Asir [7] for the computation of the stratifications above (see e.g., [6] for algorithms).

The fiber of the characteristic variety of M at each zero-dimensional stratum of $f = 0$ is given by

$$\pi^{-1}((0,0,0)) \cap \mathrm{Char}(M) = \{(0,0,0;\xi,\eta,\zeta) \mid \xi,\eta,\zeta \in \mathbb{C}\},$$
$$\pi^{-1}(p) \cap \mathrm{Char}(M) = \{(p;\xi,\eta,\zeta) \mid \xi,\eta,\zeta \in \mathbb{C},\ \eta = \zeta\}$$

with $p = (\pm 2, 0, 0)$ or $p = \pm(2, 2, 2)$, where $\pi : T^*\mathbb{C}^3 \to \mathbb{C}^3$ is the projection of the cotangent bundle to the base space. The fiber of $\mathrm{Char}(M)$ at a point in a one-dimensional stratum, e.g., $x = y = z$, consists of two complex lines while that at a regular point of $f = 0$ consists of one complex line.

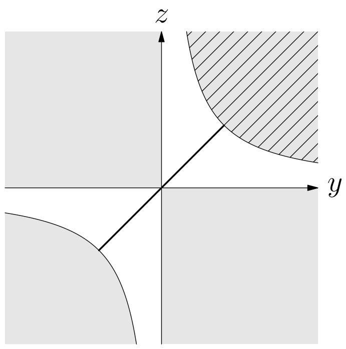
Next let us consider the real hypersurface

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = 0\}$$

of $\mathbb{R}^3$. Since the discriminant of f with respect to x is $yz(y-z)^2(yz-4)$, the projection of S to the yz-plane is given by

$$S_{yz} = \{(y,z) \in \mathbb{R}^2 \mid yz(yz-4) \geq 0\} \cup \{(y,z) \mid y = z\}$$

as shown by the grey regions and the line segment in the figure below:



Except at the origin, the fiber of the projection of S to S_{yz} consists of one or two points. In particular, the fiber at a point in the line $y = z$ consists of only one point. Hence S has a line segment connecting the two points $\pm(2, 2, 2)$ as a one-dimensional component.

Since the support of $I_G(x, y, z)$ is contained in the support of M, i.e., $xf(x, y, z) = 0$, as well as in the set $\{(x, y, z) \in \mathbb{R}^3 \mid x \geq 2,\ y + z \geq 4\}$, we have

$$\operatorname{supp} I_G(x, y, z) \subset \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = 0,\ x \geq 2,\ y + z \geq 4\}.$$

In addition, since $I_G(x, y, z)$ coincides with a finite sum of derivatives of $\delta(f)$ multiplied by real analytic functions on the regular part S_{reg} of S, the intersection of the support of $I_G(x, y, z)$ and S_{reg} is a union of connected components of S_{reg}. Thus we conclude that the support of $I_G(x, y, z)$ satisfies

$$\operatorname{supp} I_G(x, y, z) \subset \{(x, y, z) \mid f(x, y, z) = 0,\ x \geq 2,\ y > 0,\ z > 0,\ yz \geq 4\},$$

the projection of which to the yz-plane is shown as the hatched region in the figure above. More precisely, we can confirm by direct computation that $xf(x, y, z)$ belongs to the left ideal I, which implies $f(x, y, z)I_G(x, y, z) = 0$. It follows that $I_G(x, y, z)$ is the product of a real analytic function and $\delta(f)$ on S_{reg}.

We remark that the hypersurface S of $\mathbb{R}^3$ coincides with the Landau-Nakanishi surface associated with the triangle diagram T_1 studied by Honda-Kawai-Stapp in Appendix A of [2].

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