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Kyoto University
Local cohomology solutions of holonomic D-modules associated with a non-isolated hypersurface singularity

By

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Abstract

A hypersurface with a smooth 2-dimensional singular locus is considered in the context of Computational Algebraic Analysis. The holonomic D-module associated with each root of the reduced b-function is computed. Local cohomology solutions to the holonomic D-module are explicitly computed.

§1. Introduction

In this paper, we consider a hypersurface with a smooth 2-dimensional singular locus in the context of Computational Algebraic Analysis. We explicitly compute holonomic D-modules associated with the reduced b-function of the hypersurface by using a computer algebra system and we study holonomic D-modules by computing its local cohomology solutions.

In 1970's, M. Kashiwara studied b-functions by using D-modules. It turned out that holonomic D-modules that he introduced to study b-functions contain a wealth of information on singularity. It is important therefore to analyze the structures of the holonomic D-module associated with a root of b-functions.

In a previous paper [15], Y. Umeta and the author of the present paper studied certain kinds of hypersurfaces with a smooth 1-dimensional singular locus. We considered the holonomic D-module associated with a root of the reduced b-function of hypersurfaces. We described in particular a method for computing structures of relevant holonomic D-modules. As a sequel of the previous paper [15], we address the
case of hypersurface with a 2-dimensional singular locus. We consider one example of hypersurface with non-isolated singularity which is given by A. Zaharia in [17]. In this paper, we present in particular an effective method for studying holonomic D-modules associated with roots of b-functions. The key of our approach is a use of the concept of local cohomology.

In section two, we compute the b-function, local b-functions of the hypersurface. We compute the holonomic D-modules associated with roots of the reduced b-function. In section three, we analyze structures of the holonomic D-modules by computing local cohomology solutions.

§ 2. An example of non-isolated hypersurface singularity

Let \( S = \{(x_1, x_2, y_1, y_2) \in X \mid g(x_1, x_2, y_1, y_2) = 0\} \) where \( g = y_1^2(y_1 + x_1^3 + x_2^2) + y_2^2 \). The singular locus \( \Sigma \) of \( S \) is the 2-dimensional plane \( \{(x_1, x_2, 0, 0) \mid x_1, x_2 \in \mathbb{C}\} \). The hypersurface above is taken from Table 4 (page 51) in a paper [17]. The table 4 is obtained by A. Zaharia as one of results of classification of simple germs. The defining function \( g \) is a weighted homogeneous polynomial with respect to the weight vector \( w = \frac{1}{18}(2, 3, 6, 9) \). Let \( E \) denote the Euler operator defined by

\[
E = \frac{1}{9}x_1 \frac{\partial}{\partial x_1} + \frac{1}{6}x_2 \frac{\partial}{\partial x_2} + \frac{1}{3}y_1 \frac{\partial}{\partial y_1} + \frac{1}{2}y_2 \frac{\partial}{\partial y_2}.
\]

Then, \( g \) satisfies, \( E(g) = g \).

In this section, we compute the b-function, local b-functions of the defining function \( g \) and compute the holonomic D-module associated with each root of the reduced b-function of \( g \). For this purpose, we use a computer algebra system Risa/Asir [11] and four algorithms bfct, ndbf.bf-strat, ann and cgsw_dx implemented in Risa/Asir.

By using bfct, we compute the b-function of \( g \).

\[
G = y_1^2*(y_1 + x_1^3 + x_2^2) + y_2^2 \]
\[
y_1^2*(x_1^3 + x_2^2)*y_1^2 + y_2^2
\]

\[
[250] \text{fctr(bfct(G));}
\]
\[
[[1,1],[s+1,2],[2*s+3,1],
[9*s+10,1],[9*s+11,1],[9*s+13,1],[9*s+14,1],[9*s+16,1],[9*s+17,1]]
\]

The output means

\[(s + 1)^2(s + \frac{3}{2})(s + \frac{10}{9})(s + \frac{11}{9})(s + \frac{13}{9})(s + \frac{14}{9})(s + \frac{16}{9})(s + \frac{17}{9})\]

is the b-function of the polynomial \( g \).
§ 2.1. a stratification and local b-functions

First, we briefly recall the concept of local b-functions. Let $X$ be a complex manifold, $\mathcal{O}_X$ the sheaf on $X$ of holomorphic functions, $f$ a germ of holomorphic function at a point $x \in X$. Let $\mathcal{D}_X$ be the sheaf on $X$ of linear partial differential operators with holomorphic coefficients. There exists a polynomial $b(s)$ in $s$ and partial differential operator $P(s) \in \mathcal{D}_X[s]$ in a neighborhood of $x$ such that

$$P(s)f(x)^{s+1} = b(s)f(x)^s.$$  

The monic generator $b_{f,x}(s)$ of the ideal consisting of such polynomials $b(s)$ is the local b-function of $f$ at $x \in X$. This is defined in the context of analytic functions and described in terms of holomorphic linear partial differential operators.

In 1997, T. Oaku showed that local b-functions of a polynomial $f$ in $\mathbb{C}[x_1, x_2, ..., x_n]$ can be computed by applying computations in Weyl algebra and gave an algorithm for computing local b-functions at all points simultaneously by using a primary decomposition algorithm. The algorithm outputs a stratification of $\mathbb{C}^n$ giving the local b-function $b_{f,x}$ which is constant on each stratum. More recently in 2010, K. Nishiyama and M. Noro proposed alternative method for computing local b-functions([10]). The resulting algorithm ndbf.bf_strat is implemented in Risa/Asir([11]).

We compute local b-functions of $g$ by executing ndbf.bf_strat on Risa/Asir. By analyzing the result of computation, we see that the stratification of the space $X = \mathbb{C}^4$ associated with the local b-functions of $g$ consists of 5 strata:

$$\Sigma_0, \Sigma_1, \Sigma_2, S - \Sigma, X - S,$$

where

$$\Sigma_0 = \{(0,0,0,0)\}, \Sigma_1 = \{(x_1, x_2, 0, 0) | x_1^3 + x_2^2 = 0\} - \Sigma_0, \Sigma_2 = \Sigma - \Sigma_1 \cup \Sigma_0.$$

We have $b_{g,x}(s) = (s + 1)$ on the non-singular locus $S - \Sigma$, and

$$b_{g,x}(s) = (s + 1)^2 \text{ on } \Sigma_2, \quad b_{g,x}(s) = (s + 1)^2(2s + 3) \text{ on } \Sigma_1,$$

$$b_{g,x}(s) = (s + 1)^2(s + \frac{3}{2})(s + \frac{10}{9})(s + \frac{11}{9})(s + \frac{13}{9})(s + \frac{14}{9})(s + \frac{16}{9})(s + \frac{17}{9}) \text{ on } \Sigma_0.$$

§ 2.2. holonomic D-modules

First, we recall some basics. Let $\text{Ann}_{\mathcal{D}_X[s]}(f^s)$ denote the annihilating ideal of $f^s$ in the ring $\mathcal{D}_X[s]$

$$\text{Ann}_{\mathcal{D}_X[s]}(f^s) = \{P \in \mathcal{D}_X[s] | Pf^s = 0\}.$$  

Let

$$I = \text{Ann}_{\mathcal{D}_X[s]}f^s + \mathcal{D}_X[s]f + \mathcal{D}_X[s]J_f,$$
where \( D_X[s]J_f \) is the ideal generated by \( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \).

For each root \( \beta \) of the reduced b-function of \( f \), the \( D_X \)-module \( M_\beta \) defined by

\[
D_X[s]/(I + D_X[s](s - \beta))
\]

is holonomic and the support of \( M_\beta \) is a subset of the singular locus of the hypersurface \( S \), in general.

In this subsection we compute holonomic \( D \)-modules associated with roots of the reduced b-function of \( g \) in question.

We define a list of variables, a matrix for term ordering and a list of generators of the ideal \( I \) for preparations. For this purpose, we use the algorithm \textit{ann}.

\[
G = y_1^2(y_1 + x_1^3 + x_2^2) + y_2^2$
\]

\[
W = [[x_1, x_2, y_1, y_2], [dx_1, dx_2, dy_1, dy_2]]$
\]

\[
\text{Mat} = \text{newmat}(11, 10, [[0, 0, 0, 0, 0, 1, 1, 1, 1, 0], [0, 0, 0, 0, 0, 0, 1, 0, 0, 0],
[0, 0, 0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 1, 0],
[0, 0, 1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0],
[0, 0, 0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 1]])$
\]

\[
L = \text{append}([[G, \text{diff}(G, x_1), \text{diff}(G, x_2), \text{diff}(G, y_1), \text{diff}(G, y_2)], \text{ann}(G))])$
\]

Note that in the list \( L \) above \text{ann}(G) is a list of generators of the annihilating ideal \( \text{Ann}_{D_X[s]}g^s \). The contents are as follows.

\[
\text{ann}(G);
\]

\[
[-3*dy_2*y_1^2 + (-2*dy_2*x_1^3 - 2*dy_2*x_2^2)*y_1 + 2*y_2*dy_1,
-3*dy_2*x_1*y_1^2 + 2*dx_1*y_2,
-dy_2*x_2*y_1^2 + dx_2*y_2,
9*dy_2*y_1^3 - 6*y_2*dy_1*y_1 + (4*dx_1*x_1 + 6*dx_2*x_2)*y_2,
(-3*x_1^2*dy_1 + 3*dx_1)*y_1 + 2*dx_1*x_1^3 + 2*dx_1*x_2^2,
(6*x_2*dy_1 - 9*dx_2)*y_1 + 4*dx_1*x_2*x_1 - 6*dx_2*x_2^2,
-3*dx_2*x_1^2 + 2*dx_1*x_2,
18*s - 6*dy_1*y_1 - 9*dy_2*y_2 - 2*dx_1*x_1 - 3*dx_2*x_2]
\]

Now we compute holonomic \( D \)-modules by using the algorithm \text{cgsw..dx} constructed recently in [8] by K. Nabeshima, K. Ohara and the author of the present paper.
local cohomology solutions

[[s+1], [1]]
[y2, y1, -2*dx1*x1-3*dx2*x2-3, -3*dx2*x1^2+2*dx1*x2]

[[2*s+3], [1]]
y2, y1^2, -6*dy1*y1-2*dx1*x1-3*dx2*x2-18, (-2*x2*dy1+dx2)*y1-4*x2,
-3*dx2*x1^2+2*dx1*x2, y1+x1^3+x2^2, (3*x1^2*dy1-dx1)*y1+6*x1^2]

[[9*s+10], [1]]
y2, y1, x2, x1

[[9*s+11], [1]]
y2, y1, x2, -dx1*x1-2, x1^2

[[9*s+13], [1]]
y2, 9*dx2*y1-16*x2, y1^2, x2*y1, 9*y1+16*x2^2,
-6*dy1*y1-2*dx1*x1-3*dx2*x2-17, x1*y1, x2*x1, -3*dx1*y1+8*x1^2]

[[9*s+14], [1]]
y2, 9*dx2*y1-20*x2, y1^2, x2*y1, -9*y1-20*x2^2,
-6*dy1*y1-2*dx1*x1-3*dx2*x2-19, -3*dx1*y1+10*x1^2]

[[9*s+16], [1]]
y2, -6*dy1*y1-2*dx1*x1-3*dx2*x2-23, y1^3,
(-18*x2*dy1+9*dx2)*y1-46*x2, x2*y1^2, 9*y1^2+10*x2+2*y1,
27*x2*y1+28*x2^2, x1*y1^2, x2*x1*y1, 9*x1*y1+28*x2^2*x1,
-3*dx2*x1^2+2*dx1*x2, -9*dy1*y1^2+15*y1+28*x1^3+28*x2^2,
(9*x1^2*dy1-3*dx1)*y1+23*x1^2]

[[9*s+17], [1]]
y2, -6*dy1*y1-2*dx1*x1-3*dx2*x2-25, y1^3,
(-18*x2*dy1+9*dx2)*y1-50*x2, x2*y1^2, 9*y1^2+14*x2^2*y1, 27*x2*y1+32*x2^2-3,
-3*dx2*x1^2+2*dx1*x2, -9*dy1*y1^2+21*y1+32*x1^3+32*x2^2, (9*x1^2*dy1-3*dx1)*y1+25*x1^2,
-9*dy1*y1^2+21*y1+32*x1^3+32*x2^2, (9*x1^2*dy1-3*dx1)*y1+25*x1^2,
x1^2*y1^2, x2*x1^2*y1, 9*x1^2*y1+32*x2^2*x1^2]

[[0], [1062882*s^8+12223143*s^7+61135398*s^6+173682792*s^5]
The output consists of each factor of the reduced b-function, the Gröbner basis of the holonomic D-module associated with each root of the reduced b-function.

§ 3. Local systems and local cohomology solutions

In this section we study structures of the holonomic D-modules presented in the previous section. We explicitly compute algebraic local cohomology solutions.

§ 3.1. Algebraic local cohomology solutions supported on $\Sigma_2$

Let $s + 1 = 0$. Then a Gröbner basis of the holonomic ideal associated with the root $\beta = -1$ is given by

$$y_1, y_2, 2x_1 \frac{\partial}{\partial x_1} + 3x_2 \frac{\partial}{\partial x_2} + 3, 2x_2 \frac{\partial}{\partial x_1} - 3x_1^2 \frac{\partial}{\partial x_2}.$$  

Set

$$H_{\Sigma_2} = \left\{ \psi \in \mathcal{H}^2_{[\Sigma_2]}(\mathcal{O}_X) \mid y_1 \psi = y_2 \psi = 0 \right\}.$$  

Then any germ at a point $Q \in \Sigma_2$ of the sheaf $H_{\Sigma_2}$ can be represented in a form

$$h(x_1, x_2) \begin{bmatrix} 1 \\ y_1 \\ y_2 \end{bmatrix},$$

where $\begin{bmatrix} \cdots \end{bmatrix}$ denotes the Grothendieck symbol and $h(x_1, x_2)$ is a germ at $Q$ of holomorphic functions on $\Sigma_2$.

Algebraic local cohomology solution $\psi$ of the holonomic D-module $M_{-1}$ satisfies the following system of linear partial differential equations

$$(2x_1 \frac{\partial}{\partial x_1} + 3x_2 \frac{\partial}{\partial x_2} + 3)h = 0, (2x_2 \frac{\partial}{\partial x_1} - 3x_1^2 \frac{\partial}{\partial x_2})h = 0.$$
Since \[
\begin{vmatrix}
2x_1 & 3x_2 \\
2x_2 & -3x_1^2
\end{vmatrix} = -6(x_1^2 + x_2^3),
\]
the singular locus of the holonomic D-module \(M_{-1}\) is
\[\{(x_1, x_2, 0, 0) | x_1^3 + x_2^2 = 0\} = \Sigma_1 \cup \Sigma_0.\]

We set \(h(x_1, x_2) = (x_1^3 + x_2^2)^\alpha\). Then we have
\[\left(2x_1 \frac{\partial}{\partial x_1} + 3x_2 \frac{\partial}{\partial x_2} + 3\right)h = (6\alpha + 3)h, \left(2x_2 \frac{\partial}{\partial x_1} - 3x_1^2 \frac{\partial}{\partial x_2}\right)h = 0.\]

Hence,
\[\left(x_1^3 + x_2^2\right)^{-\frac{1}{2}} \begin{bmatrix} 1 \\ y_1 \\ y_2 \end{bmatrix}\]
is a local cohomology solution.

The monodromy structure on \(\Sigma_2\) of the holonomic D-module \(M_{-1}\) is non trivial. Since the local cohomology solution \(\psi\) can not be analytically continued to \(\Sigma_2 \cup \Sigma_0\), there exists no non-trivial algebraic local cohomology solution in \(\mathcal{H}^{2}_{[\Sigma]}(\mathcal{O}_X)\) at \(\Sigma_0\).

§ 3.2. Algebraic local cohomology solutions supported on \(\Sigma_1\)

Since the local b-function is \((s+1)(s + \frac{3}{2})\) on the strutum \(\Sigma_1\), we consider two cases.

(i) Let \(2s + 3 = 0\). Then a Gröbner basis of the holonomic ideal associated with the root \(\beta = -\frac{3}{2}\) is given by
\[
\begin{cases}
x_1^3 + x_2^2 + y_1, \quad y_1^2, \quad y_2, \quad 2x_1 \frac{\partial}{\partial x_1} + 3x_2 \frac{\partial}{\partial x_2} + 6y_1 \frac{\partial}{\partial y_1} + 18, \\
x_2 \frac{\partial}{\partial x_1} - 3x_1^2 \frac{\partial}{\partial x_2}, \quad y_1 \frac{\partial}{\partial x_1} - 3x_2^2 y_1 \frac{\partial}{\partial y_1} - 6x_2^2, \quad y_1 \frac{\partial}{\partial y_1} - 2x_2 y_1 \frac{\partial}{\partial y_1} - 4x_2.
\end{cases}
\]

From \((x_1^3 + x_2^2)^2 = (x_1^3 + x_2^2 - y_1)(x_1^3 + x_2^2 + y_1) + y_1^2\), we see that \((x_1^3 + x_2^2)^2\) is an annihilator. Therefore the holonomic D-module \(M_{-\frac{3}{2}}\) is supported on \(\Sigma_1 \cup \Sigma_0\), where
\[\Sigma_1 \cup \Sigma_0 = \{(x_1, x_2, 0, 0) | x_1^3 + x_2^2 = 0\}.
\]

Now consider algebraic local cohomology class \(\psi\) in \(\mathcal{H}^{3}_{\Sigma_1}(\mathcal{O}_X)\) supported on \(\Sigma_1\) of the form
\[
\psi = a \begin{bmatrix} 1 \\
(x_1^3 + x_2^2) y_1^2 \\
y_2 \end{bmatrix} + b \begin{bmatrix} 1 \\
(x_1^3 + x_2^2)^2 y_1 y_2 \end{bmatrix}.
\]
Note that the weighted degree of \(\psi\) is equal to \(-\frac{27}{18} = -\frac{3}{2}\).

Then, \(\psi\) satisfies, \(y_1^2 \psi = y_2 \psi = 0\) and
\[
(2x_1 \frac{\partial}{\partial x_1} + 3x_2 \frac{\partial}{\partial x_2} + 6y_1 \frac{\partial}{\partial y_1} + 18)\psi = (2x_2 \frac{\partial}{\partial x_1} - 3x_1^2 \frac{\partial}{\partial x_2})\psi = 0.
\]
From
\[(x_1^3 + x_2^2 + y_1)\psi = (a + b) \begin{bmatrix} 1 \\ (x_1^3 + x_2^2) y_1 y_2 \end{bmatrix}\]
we have \(a + b = 0\). Therefore, we set
\[
\psi = \begin{bmatrix} 1 \\ (x_1^3 + x_2^2) y_1^2 y_2 \end{bmatrix} - \begin{bmatrix} 1 \\ (x_1^3 + x_2^2)^2 y_1 y_2 \end{bmatrix}.
\]
It is easy to verify that the local cohomology class \(\psi\) above satisfies
\[
(y_1 \frac{\partial}{\partial x_1} - 3x_1^2 y_1 \frac{\partial}{\partial y_1} - 6x_1^2)\psi = (y_1 \frac{\partial}{\partial x_2} - 2x_2 y_1 \frac{\partial}{\partial y_1} - 4x_2)\psi = 0.
\]
Since the solution \(\psi\) can be analytically continued to \(\Sigma_0\), \(\psi\) belongs to \(\mathcal{H}_{\Sigma_1 \cup \Sigma_0}^3(\mathcal{O}_X)\) and the dimension of the algebraic local cohomology solutions of the holonomic \(D\)-module \(M_{-\frac{3}{2}}\) is equal to one at the origin \(\Sigma_0\).

(ii) Let \(s + 1 = 0\). It is easy to see that the holonomic \(D\)-module \(M_{-1}\) has no non-trivial local cohomology solution in \(\mathcal{H}_{\Sigma_1}^3(\mathcal{O}_X)\).

§ 3.3. Local cohomology solutions supported at \(\Sigma_0\)

Since the reduced local \(b\)-function at \(\Sigma_0\) of \(g\) is equal to
\[
\bar{b}_{g, \Sigma_0} = (s + 1)(s + \frac{3}{2})(s + \frac{10}{9})(s + \frac{11}{9})(s + \frac{13}{9})(s + \frac{14}{9})(s + \frac{16}{9})(s + \frac{17}{9}),
\]
we compute algebraic local cohomology solutions for the holonomic \(D\)-module associated with each root of the reduced local \(b\)-function.

Let \(\mathcal{H}_{\Sigma_0}^4(\mathcal{O}_X)\) be the algebraic local cohomology supported on \(\Sigma_0\).

(i) \(9s + 10 = 0\). Then, a Gröbner basis of the holonomic ideal associated with the root \(\beta = -\frac{10}{9}\) is given by \(\{x_1, x_2, y_1, y_2\}\). We see that the local cohomology class
\[
\begin{bmatrix} 1 \\ x_1 x_2 y_1 y_2 \end{bmatrix}
\]
is a solution of the holonomic \(D\)-module \(M_{-\frac{10}{9}}\).

(ii) \(9s + 11 = 0\). Then, a Gröbner basis of the holonomic ideal associated with the root \(\beta = -\frac{11}{9}\) is given by \(\{x_1^2, x_2, y_1, y_2, x_1 \frac{\partial}{\partial x_1} + 2\}\). The local cohomology class
\[
\begin{bmatrix} 1 \\ x_1^2 x_2 y_1 y_2 \end{bmatrix}
\]
is a solution of the holonomic D-module $M_{-\frac{13}{9}}$.

(iii) $9s + 13 = 0$. Then, a Gröbner basis of the holonomic ideal associated with the root $\beta = -\frac{13}{9}$ is given by

$$\left\{ \begin{array}{l}
x_1 x_2, \; x_1 y_1, \; x_2 y_1, \; y_1^2, \; y_2, \; 16x_2^2 + 9y_1, \\
2x_1 \frac{\partial}{\partial x_1} + 3x_2 \frac{\partial}{\partial x_2} + 6y_1 \frac{\partial}{\partial y_1} + 17, \; 3y_1 \frac{\partial}{\partial x_1} - 8x_1^2, \; 9y_1 \frac{\partial}{\partial x_2} - 16x_2
\end{array} \right.$$

Note that since

$$16x_2^3 = x_2(16x_2^2 + 9y_1) - 9x_2y_1$$

and

$$8x_1^4 = -x_1^2(3y_1 \frac{\partial}{\partial x_1} - 8x_1^2) + 3 \frac{\partial}{\partial x_1}(x_1^2y_1) - 6x_1y_1,$$

partial differential operators $x_1^4, x_2^3$ belong to the annihilating ideals. It follows in particular from this fact that the support of the holonomic D-module $M_{-\frac{13}{9}}$ is the origin $\Sigma_0$.

Since the weighted degree of the solution is equal to $-\frac{13}{9}$, we consider algebraic local cohomology class $\psi$ of the form

$$\psi = a \left[ x_1^4 \; x_2 \; y_1 \; y_2 \right] + b \left[ x_1 \; x_2^3 \; y_1 \; y_2 \right] + c \left[ x_1 \; x_2 \; 11 \; y_1^2 \; y_2 \right].$$

From

$$(3y_1 \frac{\partial}{\partial x_1} - 8x_1^2)\psi = -(8a + 3c) \left[ x_1^2 \; x_2 \; y_1 \; y_2 \right]$$

and

$$(9y_1 \frac{\partial}{\partial x_2} - 16x_2)\psi = -(16b + 9c) \left[ x_1 \; x_2^2 \; y_1 \; y_2 \right],$$

we have

$$8a + 3c = 0, \; 16b + 9c = 0.$$ 

We thus have

$$\psi = \left[ x_1^4 \; x_2 \; y_1 \; y_2 \right] + \frac{3}{2} \left[ x_1 \; x_2^3 \; y_1 \; y_2 \right] - \frac{8}{3} \left[ x_1 \; x_2 \; y_1^2 \; y_2 \right]$$

as local cohomology solution.

(iv) $9s + 14 = 0$. Then, a Gröbner basis of the holonomic ideal associated with the root $\beta = -\frac{14}{9}$ is given by
\[
\begin{cases}
x_2y_1, \ y_1^2, \ y_2, \ 20x_2^2 + 9y_1, \ 2x_1 \frac{\partial}{\partial x_1} + 3x_2 \frac{\partial}{\partial x_2} + 6y_1 \frac{\partial}{\partial y_1} + 19, \\
3y_1 \frac{\partial}{\partial x_1} - 10x_1^2, \ 9y_1 \frac{\partial}{\partial x_2} - 20x_2.
\end{cases}
\]

A direct computation yields the following local cohomology solution:

\[
\psi = \left[ \begin{array}{llll}
1 \\
x_1^5 \\
x_2 \\
y_1 \\
y_2
\end{array} \right] + \frac{3}{4} \left[ \begin{array}{llll}
1 \\
x_1^2 \\
x_2^3 \\
y_1 \\
y_2
\end{array} \right] - \frac{5}{3} \left[ \begin{array}{llll}
1 \\
x_1^2 \\
x_2 \\
y_1^2 \\
y_2
\end{array} \right].
\]

(v) \(9s + 16 = 0\). Then, a Gröbner basis of the holonomic ideal associated with the root \(\beta = -\frac{16}{9}\) is given by

\[
\begin{cases}
28x_1^2x_2 + 9x_1y_1, \ x_1x_2y_1, \ x_1^2y_1^2, \ x_2y_1^2, \ 27x_2y_1 + 28x_2^3, \\
10x_2^2y_1 + 9y_1^2, \ y_1^3, \ y_2, \ P_1, \ P_2, \ Q_1, \ Q_2, \ Q_3,
\end{cases}
\]

where

\[
P_1 = 2x_1 \frac{\partial}{\partial x_1} + 3x_2 \frac{\partial}{\partial x_2} + 6y_1 \frac{\partial}{\partial y_1} + 23, \ P_2 = 2x_2 \frac{\partial}{\partial x_1} - 3x_1^2 \frac{\partial}{\partial x_2},
\]

\[
Q_1 = 3y_1 \frac{\partial}{\partial x_1} - 9x_1^2y_1 \frac{\partial}{\partial y_1} - 23x_1^2, \ Q_2 = 9y_1 \frac{\partial}{\partial x_2} - 18x_2y_1 \frac{\partial}{\partial y_1} - 46x_2,
\]

\[
Q_3 = 9y_1^2 \frac{\partial}{\partial y_1} - 28x_1^3 - 28x_2^2 - 15y_1.
\]

Let

\[
\psi_1 = \left[ \begin{array}{llll}
1 \\
x_1^7 \\
x_2 \\
y_1 \\
y_2
\end{array} \right], \ \psi_2 = \left[ \begin{array}{llll}
1 \\
x_1^4 \\
x_2^3 \\
y_1 \\
y_2
\end{array} \right], \ \psi_3 = \left[ \begin{array}{llll}
1 \\
x_1^4 \\
x_2 \\
y_1^2 \\
y_2
\end{array} \right],
\]

\[
\psi_4 = \left[ \begin{array}{llll}
1 \\
x_1 \\
x_2^5 \\
y_1 \\
y_2
\end{array} \right], \ \psi_5 = \left[ \begin{array}{llll}
1 \\
x_1 \\
x_2^3 \\
y_1^2 \\
y_2
\end{array} \right], \ \psi_6 = \left[ \begin{array}{llll}
1 \\
x_1 \\
x_2 \\
y_1^3 \\
y_2
\end{array} \right].
\]

Then,

\[
\psi = \psi_1 + \frac{3}{8} \psi_2 - \frac{7}{6} \psi_3 + \frac{27}{16} \psi_4 - \frac{7}{4} \psi_5 + \frac{35}{18} \psi_6
\]

is a local cohomology solution.

(vi) \(9s + 17 = 0\). Then, a Gröbner basis of the holonomic ideal associated with the root \(\beta = -\frac{17}{9}\) is given by

\[
\begin{cases}
32x_1^2x_2^2 + 9x_1^2y_1, \ x_1^2x_2y_1, \ x_1^2y_1^2, \ x_2y_1^2, \ 27x_2y_1 + 32x_2^3, \\
14x_2^2y_1 + 9y_1^2, \ y_1^3, \ y_2, \ P_1, \ P_2, \ Q_1, \ Q_2, \ Q_3,
\end{cases}
\]

where

\[
P_1 = 2x_1 \frac{\partial}{\partial x_1} + 3x_2 \frac{\partial}{\partial x_2} + 6y_1 \frac{\partial}{\partial y_1} + 25, \ P_2 = 2x_2 \frac{\partial}{\partial x_1} - 3x_1^2 \frac{\partial}{\partial x_2},
\]
Let
\[ Q_1 = 3y_1 \frac{\partial}{\partial x_1} - 9x_1^2 y_1 \frac{\partial}{\partial y_1} - 25x_1^2, \quad Q_2 = 9y_1 \frac{\partial}{\partial x_2} - 18x_2 y_1 \frac{\partial}{\partial y_1} - 50x_2, \]
\[ Q_3 = 9y_1^2 \frac{\partial}{\partial y_1} - 32x_1^3 - 32x_2^2 - 21y_1. \]

Let
\[ \psi_1 = \begin{bmatrix} 1 \\ x_1^8 x_2 y_1 y_2 \end{bmatrix}, \quad \psi_2 = \begin{bmatrix} 1 \\ x_1^5 x_2^3 y_1 y_2 \end{bmatrix}, \quad \psi_3 = \begin{bmatrix} 1 \\ x_1^5 x_2 y_1^2 y_2 \end{bmatrix}, \]
\[ \psi_4 = \begin{bmatrix} 1 \\ x_1^2 x_2^5 y_2 \end{bmatrix}, \quad \psi_5 = \begin{bmatrix} 1 \\ x_1^2 x_2^3 y_1^2 y_2 \end{bmatrix}, \quad \psi_6 = \begin{bmatrix} 1 \\ x_1^2 x_2 y_1^3 y_2 \end{bmatrix}. \]

Then,
\[ \psi = \psi_1 + \frac{3}{10} \psi_2 - \frac{16}{15} \psi_3 + \frac{27}{40} \psi_4 - \frac{4}{5} \psi_5 + \frac{56}{45} \psi_6 \]
is a local cohomology solution.

(vii) \( s + 1 = 0 \). Gröbner basis of the holonomic ideal associated with the root \( \beta = -1 \)
is \( \{y_1, y_2, 2x_1 \frac{\partial}{\partial x_1} + 3x_2 \frac{\partial}{\partial x_2} + 3, 2x_2 \frac{\partial}{\partial x_1} - 3x_1^2 \frac{\partial}{\partial x_2} \} \)

Let \( \psi = \begin{bmatrix} 1 \\ x_1^i x_2^j y_1 y_2 \end{bmatrix} \). Then,
\[ (2x_1 \frac{\partial}{\partial x_1} + 3x_2 \frac{\partial}{\partial x_2} + 3)\psi = (-2i - 3j + 3)\psi. \]

Therefore there is no non-trivial local cohomology solution supported at the origin \( \Sigma_0 \).

(viii) \( 2s + 3 = 0 \). Recall that
\[
\begin{cases}
x_1^3 + x_2^2 + y_1, & y_1^2, & y_2, & 2x_1 \frac{\partial}{\partial x_1} + 3x_2 \frac{\partial}{\partial x_2} + 6y_1 \frac{\partial}{\partial y_1} + 18, \\
2x_2 \frac{\partial}{\partial x_1} - 3x_1^2 \frac{\partial}{\partial x_2}, & y_1 \frac{\partial}{\partial x_1} - 3x_1^2 y_1 \frac{\partial}{\partial y_1} - 6x_1^2, & y_1 \frac{\partial}{\partial x_2} - 2x_2 y_1 \frac{\partial}{\partial y_1} - 4x_2
\end{cases}
\]
is a Gröbner basis of the holonomic ideal associated with the root \( \beta = -\frac{3}{2} \)

Let \( \psi = \begin{bmatrix} 1 \\ x_1^i x_2^j y_1^k y_2 \end{bmatrix} \). Then,
\[ (2x_1 \frac{\partial}{\partial x_1} + 3x_2 \frac{\partial}{\partial x_2} + 6y_1 \frac{\partial}{\partial y_1} + 18)\psi = (-2i - 3j - 6k + 18)\psi. \]

Therefore, we have \((i, j, k) = (3, 2, 1)\).

Since
\[ (2x_2 \frac{\partial}{\partial x_1} - 3x_1^2 \frac{\partial}{\partial x_2}) \begin{bmatrix} 1 \\ x_1^3 x_2 y_1 y_2 \end{bmatrix} \neq 0. \]
Therefore there is no non-trivial local cohomology solution supported at the origin $\Sigma_0$.

We have verified in this section that

$$\dim_{\mathbb{C}}(\text{Hom}_{D_{X}}(M_{\beta}, \mathcal{H}_{\Sigma_2}^{4}(\mathcal{O}_{X}))) = 1,$$

for $\beta = -\frac{10}{9}, -\frac{11}{9}, -\frac{12}{9}, -\frac{13}{9}, -\frac{14}{9}, -\frac{15}{9}, -\frac{16}{9}, -\frac{17}{9}$ and

$$\text{Hom}_{D_{X}}(M_{\beta}, \mathcal{H}_{\Sigma_2}^{4}(\mathcal{O}_{X})) = \{0\},$$

for $\beta = -1, -\frac{3}{2}$.

References