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Local cohomology solutions of holonomic D-modules associated with a non-isolated hypersurface singularity

By

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Abstract

A hypersurface with a smooth 2-dimensional singular locus is considered in the context of Computational Algebraic Analysis. The holonomic D-module associated with each root of the reduced b-function is computed. Local cohomology solutions to the holonomic D-module are explicitly computed.

§ 1. Introduction

In this paper, we consider a hypersurface with a smooth 2-dimensional singular locus in the context of Computational Algebraic Analysis. We explicitly compute holonomic D-modules associated with the reduced b-function of the hypersurface by using a computer algebra system and we study holonomic D-modules by computing its local cohomology solutions.

In 1970's, M. Kashiwara studied b-functions by using D-modules. It turned out that holonomic D-modules that he introduced to study b-functions contain a wealth information on singularity. It is important therefore to analyze the structures of the holonomic D-module associated with a root of b-functions.

In a previous paper [15], Y. Umeta and the author of the present paper studied certain kinds of hypersurfaces with a smooth 1-dimensional singular locus. We considered the holonomic D-module associated with a root of the reduced b-function of hypersurfaces. We described in particular a method for computing structures of relevant holonomic D-modules. As a sequel of the previous paper [15], we address the

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case of hypersurface with a 2-dimensional singular locus. We consider one example of hypersurface with non-isolated singularity which is given by A. Zaharia in [17]. In this paper, we present in particular an effective method for studying holonomic D-modules associated with roots of b-functions. The key of our approach is a use of the concept of local cohomology.

In section two, we compute the b-function, local b-functions of the hypersurface. We compute the holonomic D-modules associated with roots of the reduced b-function. In section three, we analyze structures of the holonomic D-modules by computing local cohomology solutions.

§ 2. An example of non-isolated hypersurface singularity

Let $S = \{(x_1, x_2, y_1, y_2) \in X \mid g(x_1, x_2, y_1, y_2) = 0\}$ where $g = y_1^2(y_1+x_1^3+x_2^2)+y_2^2$. The singular locus Σ of S is the 2-dimensional plane $\{(x_1, x_2, 0, 0) \mid x_1, x_2 \in \mathbb{C}\}$. The hypersurface above is taken from Table 4 (page 51) in a paper [17]. The table 4 is obtained by A. Zaharia as one of results of classification of simple germs.

The defining function g is a weighted homogeneous polynomial with respect to the weight vector $w = \frac{1}{18}(2, 3, 6, 9)$. Let E denote the Euler operator defined by

$$E = \frac{1}{9}x_1\frac{\partial}{\partial x_1} + \frac{1}{6}x_2\frac{\partial}{\partial x_2} + \frac{1}{3}y_1\frac{\partial}{\partial y_1} + \frac{1}{2}y_2\frac{\partial}{\partial y_2}.$$

Then, g satisfies, $E(g) = g$.

In this section, we compute the b-function, local b-functions of the defining function g and compute the holonomic D-module associated with each root of the reduced b-function of g. For this purpose, we use a computer algebra system Risa/Asir [11] and four algorithms bfct, ndbf.bf_strat, ann and cgsw_dx implemented in Risa/Asir.

By using bfct, we compute the b-function of g.

```
[250] G=y1^2*(y1+x1^3+x2^2)+y2^2;
y1^3+(x1^3+x2^2)*y1^2+y2^2

[251] fctr(bfct(G));
[[1,1],[s+1,2],[2*s+3,1],
[9*s+10,1],[9*s+11,1],[9*s+13,1],[9*s+14,1],[9*s+16,1],[9*s+17,1]]
```

The output means

$$(s+1)^2(s+\frac{3}{2})(s+\frac{10}{9})(s+\frac{11}{9})(s+\frac{13}{9})(s+\frac{14}{9})(s+\frac{16}{9})(s+\frac{17}{9})$$

is the b-function of the polynomial g.

§ 2.1. a stratification and local b-functions

First, we briefly recall the concept of local b-functions. Let X be a complex manifold, $\mathcal{O}_X$ the sheaf on X of holomorphic functions, f a germ of holomorphic function at a point $x \in X$. Let $\mathcal{D}_X$ be the sheaf on X of linear partial differential operators with holomorphic coefficients. There exists a polynomial $b(s)$ in s and partial differential operator $P(s) \in \mathcal{D}_X[s]$ in a neighborhood of x such that

$$P(s)f(x)^{s+1} = b(s)f(x)^s.$$

The monic generator $b_{f,x}(s)$ of the ideal consisting of such polynomials $b(s)$ is the local b-function of f at $x \in X$. This is defined in the context of analytic functions and described in terms of holomorphic linear partial differential operators.

In 1997, T. Oaku showed that local b-functions of a polynomial f in $\mathbb{C}[x_1, x_2, ..., x_n]$ can be computed by applying computations in Weyl algebra and gave an algorithm for computing local b-functions at all points simulateneously by using a primary decomposition algorithm. The algorithm outputs a stratification of $\mathbb{C}^n$ giving the local b-function $b_{f,x}$ which is constant on each stratum. More recently in 2010, K. Nishiyama and M. Noro proposed alternative method for computing local b-functions([10]). The resulting algorithm ndbf.bf_strat is implemented in Risa/Asir([11]).

We compute local b-functions of g by executing ndbf.bf_strat on Risa/Asir. By analyzing the result of computation, we see that the stratification of the space $X = \mathbb{C}^4$ associated with the local b-functions of g consists of 5 strata :

$$\Sigma_0, \Sigma_1, \Sigma_2, S - \Sigma, X - S,$$

where

$$\Sigma_0 = \{(0,0,0,0)\},\ \Sigma_1 = \{(x_1, x_2, 0, 0) \mid x_1^3 + x_2^2 = 0\} - \Sigma_0,\ \Sigma_2 = \Sigma - \Sigma_1 \cup \Sigma_0.$$

We have $b_{g,x}(s) = (s+1)$ on the non-singular locus $S - \Sigma$, and

$$b_{g,x}(s) = (s+1)^2 \text{ on } \Sigma_2, \quad b_{g,x}(s) = (s+1)^2(2s+3) \text{ on } \Sigma_1,$$

$$b_{g,x}(s) = (s+1)^2(s+\tfrac{3}{2})(s+\tfrac{10}{9})(s+\tfrac{11}{9})(s+\tfrac{13}{9})(s+\tfrac{14}{9})(s+\tfrac{16}{9})(s+\tfrac{17}{9}) \text{ on } \Sigma_0.$$

§ 2.2. holonomic D-modules

First, we recall some basics. Let $\mathrm{Ann}_{D_X[s]}(f^s)$ denote the annihilating ideal of f^s in the ring $D_X[s]$:

$$\mathrm{Ann}_{D_X[s]}(f^s) = \{P \in D_X[s] \mid Pf^s = 0\}.$$

Let

$$I = \mathrm{Ann}_{D_X[s]} f^s + D_X[s]f + D_X[s]J_f,$$

where $D_X[s]J_f$ is the ideal generated by $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n}$.

For each root β of the reduced b-function of f, the D_X-module M_β defined by

$$D_X[s]/(I + D_X[s](s - \beta))$$

is holonomic and the support of M_β is a subset of the singular locus of the hypersurface S, in general.

In this subsection we compute holonomic D-modules associated with roots of the reduced b-function of g in question.

We define a list of variables, a matrix for term ordering and a list of generators of the ideal I for preparations. For this purpose, we use the algorithm ann.

```
[300] G=y1^2*(y1+x1^3+x2^2)+y2^2$

[301] W=[[x1,x2,y1,y2],[dx1,dx2,dy1,dy2]]$

[302] Mat=newmat(11,10,[[0,0,0,0,0,1,1,1,1,0],[0,0,0,0,0,1,0,0,0,0],
[0,0,0,0,0,0,1,0,0,0],[0,0,0,0,0,0,0,1,0,0],[0,0,0,0,0,0,0,0,1,0],
[1,1,1,1,0,0,0,0,0,0],[1,0,0,0,0,0,0,0,0,0],[0,1,0,0,0,0,0,0,0,0],
[0,0,1,0,0,0,0,0],[0,0,0,1,0,0,0,0,0,0],[0,0,0,0,0,0,0,0,0,1]])$

[303] L=append([G,diff(G,x1),diff(G,x2),diff(G,y1),diff(G,y2)], ann(G))$
```

Note that in the list L above ann(G) is a list of generators of the annihilating ideal $\mathrm{Ann}_{D_X[s]}g^s$. The contents are as follows.

```
[304] ann(G);

[-3*dy2*y1^2+(-2*dy2*x1^3-2*dy2*x2^2)*y1+2*y2*dy1,
-3*dy2*x1^2*y1^2+2*dx1*y2,
-dy2*x2*y1^2+dx2*y2,
9*dy2*y1^3-6*y2*dy1*y1+(4*dx1*x1+6*dx2*x2)*y2,
(-3*x1^2*dy1+3*dx1)*y1+2*dx1*x1^3+2*dx1*x2^2,
(6*x2*dy1-9*dx2)*y1-4*dx1*x2*x1-6*dx2*x2^2,
-3*dx2*x1^2+2*dx1*x2,
18*s-6*dy1*y1-9*dy2*y2-2*dx1*x1-3*dx2*x2]
```

Now we compute holonomic D-modules by using the algorithm cgsw_dx constructed recently in [8] by K. Nabeshima, K. Ohara and the author of the present paper.

```
[305] cgsw_dx(L,[s],W,1,Mat);

[[s+1],[1]]
[y2,y1,-2*dx1*x1-3*dx2*x2-3,-3*dx2*x1^2+2*dx1*x2]

[[2*s+3],[1]]
[y2,y1^2,-6*dy1*y1-2*dx1*x1-3*dx2*x2-18,(-2*x2*dy1+dx2)*y1-4*x2,
-3*dx2*x1^2+2*dx1*x2,y1+x1^3+x2^2,(3*x1^2*dy1-dx1)*y1+6*x1^2]

[[9*s+10],[1]]
[y2,y1,x2,x1]

[[9*s+11],[1]]
[y2,y1,x2,-dx1*x1-2,x1^2]

[[9*s+13],[1]]
[y2,9*dx2*y1-16*x2,y1^2,x2*y1,9*y1+16*x2^2,
-6*dy1*y1-2*dx1*x1-3*dx2*x2-17,x1*y1,x2*x1,-3*dx1*y1+8*x1^2]

[[9*s+14],[1]]
[y2,9*dx2*y1-20*x2,y1^2,x2*y1,-9*y1-20*x2^2,
-6*dy1*y1-2*dx1*x1-3*dx2*x2-19,-3*dx1*y1+10*x1^2]

[[9*s+16],[1]]
[y2,-6*dy1*y1-2*dx1*x1-3*dx2*x2-23,y1^3,
(-18*x2*dy1+9*dx2)*y1-46*x2,x2*y1^2,9*y1^2+10*x2^2*y1,
27*x2*y1+28*x2^3,x1*y1^2,x2*x1*y1,9*x1*y1+28*x2^2*x1,
-3*dx2*x1^2+2*dx1*x2,-9*dy1*y1^2+15*y1+28*x1^3+28*x2^2,
(9*x1^2*dy1-3*dx1)*y1+23*x1^2]

[[9*s+17],[1]]
[y2,-6*dy1*y1-2*dx1*x1-3*dx2*x2-25,y1^3,
(-18*x2*dy1+9*dx2)*y1-50*x2,x2*y1^2,
9*y1^2+14*x2^2*y1,27*x2*y1+32*x2^3,-3*dx2*x1^2+2*dx1*x2,
-9*dy1*y1^2+21*y1+32*x1^3+32*x2^2,(9*x1^2*dy1-3*dx1)*y1+25*x1^2,
x1^2*y1^2,x2*x1^2*y1,9*x1^2*y1+32*x2^2*x1^2]

[[0],[1062882*s^8+12223143*s^7+61135398*s^6+173682792*s^5
```

```
+306514368*s^4+344058597*s^3+239861032*s^2+94945468*s+16336320]]
[1]

No. of segment is
9
0.0312sec(0.093sec)
```

The output consists of each factor of the reduced b-function, the Gröbner basis of the holonomic D-module associated with each root of the reduced b-function.

§ 3. Local systems and local cohomology solutions

In this section we study structures of the holonomic D-modules presented in the previous section. We explicitly compute algebraic local cohomology solutions.

§ 3.1. Algebraic local cohomology solutions supported on Σ_2

Let $s+1=0$. Then a Gröbner basis of the holonomic ideal associated with the root $\beta=-1$ is given by

$$y_1,\ y_2,\ 2x_1\frac{\partial}{\partial x_1}+3x_2\frac{\partial}{\partial x_2}+3,\ 2x_2\frac{\partial}{\partial x_1}-3x_1^2\frac{\partial}{\partial x_2}.$$

Let $\mathcal{H}^2_{[\Sigma_2]}(\mathcal{O}_X)$ be the sheaf of algebraic local cohomology supported on Σ_2, where

$$\Sigma_2=\{(x_1,x_2,0,0)\mid x_1,x_2\in\mathbb{C}\}-\{(x_1,x_2,0,0)\mid x_1^3+x_2^2=0\}.$$

Set

$$H_{\Sigma_2}=\left\{\psi\in\mathcal{H}^2_{[\Sigma_2]}(\mathcal{O}_X)\ \mid\ y_1\psi=y_2\psi=0\right\}.$$

Then any germ at a point $Q\in\Sigma_2$ of the sheaf H_{Σ_2} can be represented in a form

$$h(x_1,x_2)\begin{bmatrix}1\\ y_1\ y_2\end{bmatrix},$$

where $[\quad]$ denotes the Grothendieck symbol and $h(x_1,x_2)$ ia a germ at Q of holomorphic functions on Σ_2.

Algebraic local cohomology solution ψ of the holonomic D-module M_{-1} satisfies the following system of linear partial differential equations

$$(2x_1\frac{\partial}{\partial x_1}+3x_2\frac{\partial}{\partial x_2}+3)h=0,\ (2x_2\frac{\partial}{\partial x_1}-3x_1^2\frac{\partial}{\partial x_2})h=0.$$

Since $\begin{vmatrix} 2x_1 & 3x_2 \\ 2x_2 & -3x_1^2 \end{vmatrix} = -6(x_1^3 + x_2^2)$, the singular locus of the holonomic D-module M_{-1} is

$$\{(x_1, x_2, 0, 0) \mid x_1^3 + x_2^2 = 0\} = \Sigma_1 \cup \Sigma_0.$$

We set $h(x_1, x_2) = (x_1^3 + x_2^2)^\alpha$. Then we have

$$(2x_1\frac{\partial}{\partial x_1} + 3x_2\frac{\partial}{\partial x_2} + 3)h = (6\alpha + 3)h, \ (2x_2\frac{\partial}{\partial x_1} - 3x_1^2\frac{\partial}{\partial x_2})h = 0.$$

Hence,

$$(x_1^3 + x_2^2)^{-\frac{1}{2}} \begin{bmatrix} 1 \\ y_1 \ y_2 \end{bmatrix}$$

is a local cohomology solution.

The monodromy structure on Σ_2 of the holonomic D-module M_{-1} is non trivial. Since the local cohomology solution ψ can not be analytically continued to $\Sigma_1 \cup \Sigma_0$, there exists no non-trivial algebraic local cohomology solution in $\mathcal{H}^2_{[\Sigma]}(\mathcal{O}_X)$ at Σ_0.

§ 3.2. Algebraic local cohomology solutions supported on Σ_1

Since the local b-function is $(s + 1)(s + \frac{3}{2})$ on the strutum Σ_1, we consider two cases.

(i) Let $2s + 3 = 0$. Then a Gröbner basis of the holonomic ideal associated with the root $\beta = -\frac{3}{2}$ is given by

$$\begin{cases} x_1^3 + x_2^2 + y_1, \quad y_1^2, \quad y_2, \quad 2x_1\frac{\partial}{\partial x_1} + 3x_2\frac{\partial}{\partial x_2} + 6y_1\frac{\partial}{\partial y_1} + 18, \\ 2x_2\frac{\partial}{\partial x_1} - 3x_1^2\frac{\partial}{\partial x_2}, \ y_1\frac{\partial}{\partial x_1} - 3x_1^2 y_1\frac{\partial}{\partial y_1} - 6x_1^2, \ y_1\frac{\partial}{\partial x_2} - 2x_2 y_1\frac{\partial}{\partial y_1} - 4x_2. \end{cases}$$

From $(x_1^3 + x_2^2)^2 = (x_1^3 + x_2^2 - y_1)(x_1^3 + x_2^2 + y_1) + y_1^2$, we see that $(x_1^3 + x_2^2)^2$ is an annihilator. Therefore the holonomic D-module $M_{-\frac{3}{2}}$ is supported on $\Sigma_1 \cup \Sigma_0$, where

$$\Sigma_1 \cup \Sigma_0 = \{(x_1, x_2, 0, 0) \mid x_1^3 + x_2^2 = 0\}.$$

Now consider algebraic local cohomology class ψ in $\mathcal{H}^3_{\Sigma_1}(\mathcal{O}_X)$ supported on Σ_1 of the form

$$\psi = a\begin{bmatrix} 1 \\ (x_1^3 + x_2^2) \ y_1^2 \ y_2 \end{bmatrix} + b\begin{bmatrix} 1 \\ (x_1^3 + x_2^2)^2 \ y_1 y_2 \end{bmatrix}.$$

Note that the weighted degree of ψ is equal to $-\frac{27}{18} = -\frac{3}{2}$.

Then, ψ satisfies, $y_1^2\psi = y_2\psi = 0$ and

$$(2x_1\frac{\partial}{\partial x_1} + 3x_2\frac{\partial}{\partial x_2} + 6y_1\frac{\partial}{\partial y_1} + 18)\psi = (2x_2\frac{\partial}{\partial x_1} - 3x_1^2\frac{\partial}{\partial x_2})\psi = 0.$$

From

$$(x_1^3+x_2^2+y_1)\psi = (a+b)\begin{bmatrix} 1 \\ (x_1^3+x_2^2)\ y_1\ y_2 \end{bmatrix}$$

we have $a+b=0$. Therefore, we set

$$\psi = \begin{bmatrix} 1 \\ (x_1^3+x_2^2)\ y_1^2\ y_2 \end{bmatrix} - \begin{bmatrix} 1 \\ (x_1^3+x_2^2)^2\ y_1 y_2 \end{bmatrix}.$$

It is easy to verify that the local cohomology class ψ above satisfies

$$(y_1\frac{\partial}{\partial x_1} - 3x_1^2 y_1\frac{\partial}{\partial y_1} - 6x_1^2)\psi = (y_1\frac{\partial}{\partial x_2} - 2x_2 y_1\frac{\partial}{\partial y_1} - 4x_2)\psi = 0.$$

Since the solution ψ can be analytically continued to Σ_0, ψ belongs to

$$\mathcal{H}^3_{\Sigma_1\cup\Sigma_0}(\mathcal{O}_X)$$

and the dimension of the algebraic local cohomology solutions of the holonomic D-module $M_{-\frac{3}{2}}$ is equal to one at the origin Σ_0.

(ii) Let $s+1=0$. It is easy to see that the holonomic D-module M_{-1} has no non-trivial local cohomology solution in $\mathcal{H}^3_{\Sigma_1}(\mathcal{O}_X)$.

§ 3.3. Local cohomology solutions supported at Σ_0

Since the reduced local b-function at Σ_0 of g is equal to

$$\tilde{b}_{g,\Sigma_0} = (s+1)(s+\frac{3}{2})(s+\frac{10}{9})(s+\frac{11}{9})(s+\frac{13}{9})(s+\frac{14}{9})(s+\frac{16}{9})(s+\frac{17}{9}),$$

we compute algebraic local cohomology solutions for the holonomic D-module associated with each root of the reduced local b-function.

Let $\mathcal{H}^4_{[\Sigma_2]}(\mathcal{O}_X)$ be the algebraic local cohomology supported on Σ_0.

(i) $9s+10=0$. Then, a Gröbner basis of the holonomic ideal associated with the root $\beta=-\frac{10}{9}$ is given by $\{x_1,\ x_2,\ y_1,\ y_2\}$. We see that the local cohomology class

$$\begin{bmatrix} 1 \\ x_1\ x_2\ y_1\ y_2 \end{bmatrix}$$

is a solution of the holonomic D-module $M_{-\frac{10}{9}}$.

(ii) $9s+11=0$. Then, a Gröbner basis of the holonomic ideal associated with the root $\beta=-\frac{11}{9}$ is given by $\{x_1^2,\ x_2,\ y_1,\ y_2,\ x_1\frac{\partial}{\partial x_1}+2\}$. The local cohomology class

$$\begin{bmatrix} 1 \\ x_1^2\ x_2\ y_1\ y_2 \end{bmatrix}$$

is a solution of the holonomic D-module $M_{-\frac{11}{9}}$.

(iii) $9s+13=0$. Then, a Gröbner basis of the holonomic ideal associated with the root $\beta=-\frac{13}{9}$ is given by

$$\begin{cases} x_1x_2, \quad x_1y_1, \quad x_2y_1, \quad y_1^2, \quad y_2, \quad 16x_2^2+9y_1, \\ 2x_1\frac{\partial}{\partial x_1}+3x_2\frac{\partial}{\partial x_2}+6y_1\frac{\partial}{\partial y_1}+17,\ 3y_1\frac{\partial}{\partial x_1}-8x_1^2,\ 9y_1\frac{\partial}{\partial x_2}-16x_2. \end{cases}$$

Note that since

$$16x_2^3 = x_2(16x_2^2+9y_1)-9x_2y_1$$

and

$$8x_1^4 = -x_1^2(3y_1\frac{\partial}{\partial x_1}-8x_1^2)+3\frac{\partial}{\partial x_1}(x_1^2y_1)-6x_1y_1,$$

partial differntial operators x_1^4, x_2^3 belong to the annihilating ideals. It follows in particular from this fact that the support of the holonomic D-module $M_{-\frac{13}{9}}$ is the origin Σ_0.

Since the weighted degree of the solution is equal to $-\frac{13}{9}$, we consider algebraic local cohomology class ψ of the form

$$\psi = a\begin{bmatrix} 1 \\ x_1^4\ x_2\ y_1\ y_2 \end{bmatrix} + b\begin{bmatrix} 1 \\ x_1\ x_2^3\ y_1\ y_2 \end{bmatrix} + c\begin{bmatrix} 1 \\ x_1\ x_2\ y_1^2\ y_2 \end{bmatrix}.$$

From

$$(3y_1\frac{\partial}{\partial x_1}-8x_1^2)\psi = -(8a+3c)\begin{bmatrix} 1 \\ x_1^2\ x_2\ y_1\ y_2 \end{bmatrix}$$

and

$$(9y_1\frac{\partial}{\partial x_2}-16x_2)\psi = -(16b+9c)\begin{bmatrix} 1 \\ x_1\ x_2^2\ y_1\ y_2 \end{bmatrix},$$

we have

$$8a+3c=0,\ 16b+9c=0.$$

We thus have

$$\psi = \begin{bmatrix} 1 \\ x_1^4\ x_2\ y_1\ y_2 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} 1 \\ x_1\ x_2^3\ y_1\ y_2 \end{bmatrix} - \frac{8}{3}\begin{bmatrix} 1 \\ x_1\ x_2\ y_1^2\ y_2 \end{bmatrix}$$

as local cohomology solution.

(iv) $9s+14=0$. Then, a Gröbner basis of the holonomic ideal associated with the root $\beta=-\frac{14}{9}$ is given by

$$\begin{cases} x_2y_1,\ y_1^2,\ y_2,\ 20x_2^2+9y_1,\ 2x_1\dfrac{\partial}{\partial x_1}+3x_2\dfrac{\partial}{\partial x_2}+6y_1\dfrac{\partial}{\partial y_1}+19, \\ \qquad 3y_1\dfrac{\partial}{\partial x_1}-10x_1^2, \qquad 9y_1\dfrac{\partial}{\partial x_2}-20x_2. \end{cases}$$

A direct computation yields the following local cohomology solution :

$$\psi = \begin{bmatrix} 1 \\ x_1^5\ x_2\ y_1\ y_2 \end{bmatrix} + \frac{3}{4}\begin{bmatrix} 1 \\ x_1^2\ x_2^3\ y_1\ y_2 \end{bmatrix} - \frac{5}{3}\begin{bmatrix} 1 \\ x_1^2\ x_2\ y_1^2\ y_2 \end{bmatrix}.$$

(v) $9s+16=0$. Then, a Gröbner basis of the holonomic ideal associated with the root $\beta=-\frac{16}{9}$ is given by

$$\begin{cases} 28x_1x_2^2+9x_1y_1,\ x_1x_2y_1,\ x_1y_1^2,\ x_2y_1^2,\ 27x_2y_1+28x_2^3, \\ \quad 10x_2^2y_1+9y_1^2,\ y_1^3,\ y_2,\ P_1,\ P_2,\ Q_1,\ Q_2,\ Q_3, \end{cases}$$

where

$P_1=2x_1\frac{\partial}{\partial x_1}+3x_2\frac{\partial}{\partial x_2}+6y_1\frac{\partial}{\partial y_1}+23,\ P_2=2x_2\frac{\partial}{\partial x_1}-3x_1^2\frac{\partial}{\partial x_2},$
$Q_1=3y_1\frac{\partial}{\partial x_1}-9x_1^2y_1\frac{\partial}{\partial y_1}-23x_1^2,\ Q_2=9y_1\frac{\partial}{\partial x_2}-18x_2y_1\frac{\partial}{\partial y_1}-46x_2,$
$Q_3=9y_1^2\frac{\partial}{\partial y_1}-28x_1^3-28x_2^2-15y_1.$

Let

$$\psi_1=\begin{bmatrix} 1 \\ x_1^7\ x_2\ y_1\ y_2 \end{bmatrix},\ \psi_2=\begin{bmatrix} 1 \\ x_1^4\ x_2^3\ y_1\ y_2 \end{bmatrix},\ \psi_3=\begin{bmatrix} 1 \\ x_1^4\ x_2\ y_1^2\ y_2 \end{bmatrix},$$

$$\psi_4=\begin{bmatrix} 1 \\ x_1\ x_2^5\ y_1\ y_2 \end{bmatrix},\ \psi_5=\begin{bmatrix} 1 \\ x_1\ x_2^3\ y_1^2\ y_2 \end{bmatrix},\ \psi_6=\begin{bmatrix} 1 \\ x_1\ x_2\ y_1^3\ y_2 \end{bmatrix}.$$

Then,

$$\psi=\psi_1+\frac{3}{8}\psi_2-\frac{7}{6}\psi_3+\frac{27}{16}\psi_4-\frac{7}{4}\psi_5+\frac{35}{18}\psi_6$$

is a local cohomology solution.

(vi) $9s+17=0$. Then, a Gröbner basis of the holonomic ideal associated with the root $\beta=-\frac{17}{9}$ is given by

$$\begin{cases} 32x_1^2x_2^2+9x_1^2y_1,\ x_1^2x_2y_1,\ x_1^2y_1^2,\ x_2y_1^2,\ 27x_2y_1+32x_2^3, \\ \quad 14x_2^2y_1+9y_1^2,\ y_1^3,\ y_2,\ P_1,\ P_2,\ Q_1,\ Q_2,\ Q_3, \end{cases}$$

where

$P_1=2x_1\frac{\partial}{\partial x_1}+3x_2\frac{\partial}{\partial x_2}+6y_1\frac{\partial}{\partial y_1}+25,\ P_2=2x_2\frac{\partial}{\partial x_1}-3x_1^2\frac{\partial}{\partial x_2},$

$Q_1 = 3y_1\frac{\partial}{\partial x_1} - 9x_1^2y_1\frac{\partial}{\partial y_1} - 25x_1^2,\ Q_2 = 9y_1\frac{\partial}{\partial x_2} - 18x_2y_1\frac{\partial}{\partial y_1} - 50x_2,$
$Q_3 = 9y_1^2\frac{\partial}{\partial y_1} - 32x_1^3 - 32x_2^2 - 21y_1.$

Let

$$\psi_1 = \begin{bmatrix} 1 \\ x_1^8\ x_2\ y_1\ y_2 \end{bmatrix},\ \psi_2 = \begin{bmatrix} 1 \\ x_1^5\ x_2^3\ y_1\ y_2 \end{bmatrix},\ \psi_3 = \begin{bmatrix} 1 \\ x_1^5\ x_2\ y_1^2\ y_2 \end{bmatrix},$$

$$\psi_4 = \begin{bmatrix} 1 \\ x_1^2\ x_2^5\ y_1\ y_2 \end{bmatrix},\ \psi_5 = \begin{bmatrix} 1 \\ x_1^2\ x_2^3\ y_1^2\ y_2 \end{bmatrix},\ \psi_6 = \begin{bmatrix} 1 \\ x_1^2\ x_2\ y_1^3\ y_2 \end{bmatrix}.$$

Then,

$$\psi = \psi_1 + \frac{3}{10}\psi_2 - \frac{16}{15}\psi_3 + \frac{27}{40}\psi_4 - \frac{4}{5}\psi_5 + \frac{56}{45}\psi_6$$

is a local cohomology solution.

(vii) $s+1=0$. Gröbner basis of the holonomic ideal associated with the root $\beta = -1$ is $\{y_1,\ y_2,\ 2x_1\frac{\partial}{\partial x_1} + 3x_2\frac{\partial}{\partial x_2} + 3,\ 2x_2\frac{\partial}{\partial x_1} - 3x_1^2\frac{\partial}{\partial x_2}\}$

Let $\psi = \begin{bmatrix} 1 \\ x_1^i\ x_2^j\ y_1\ y_2 \end{bmatrix}$. Then,

$$(2x_1\frac{\partial}{\partial x_1} + 3x_2\frac{\partial}{\partial x_2} + 3)\psi = (-2i - 3j + 3)\psi.$$

Therefore there is no non-trivial local cohomology solution supported at the origin Σ_0.

(viii) $2s+3=0$. Recall that

$$\begin{cases} x_1^3 + x_2^2 + y_1, \quad y_1^2, \quad y_2, \quad 2x_1\dfrac{\partial}{\partial x_1} + 3x_2\dfrac{\partial}{\partial x_2} + 6y_1\dfrac{\partial}{\partial y_1} + 18, \\ 2x_2\dfrac{\partial}{\partial x_1} - 3x_1^2\dfrac{\partial}{\partial x_2},\ y_1\dfrac{\partial}{\partial x_1} - 3x_1^2y_1\dfrac{\partial}{\partial y_1} - 6x_1^2,\ y_1\dfrac{\partial}{\partial x_2} - 2x_2y_1\dfrac{\partial}{\partial y_1} - 4x_2 \end{cases}$$

is a Gröbner basis of the holonomic ideal associated with the root $\beta = -\frac{3}{2}$

Let $\psi = \begin{bmatrix} 1 \\ x_1^i\ x_2^j\ y_1^k\ y_2 \end{bmatrix}$. Then,

$$(2x_1\frac{\partial}{\partial x_1} + 3x_2\frac{\partial}{\partial x_2} + 6y_1\frac{\partial}{\partial y_1} + 18)\psi = (-2i - 3j - 6k + 18)\psi.$$

Therefore, we have $(i,j,k) = (3,2,1)$.

Since

$$(2x_2\frac{\partial}{\partial x_1} - 3x_1^2\frac{\partial}{\partial x_2})\begin{bmatrix} 1 \\ x_1^3\ x_2^2\ y_1\ y_2 \end{bmatrix} \neq 0.$$

Therefore there is no non-trivial local cohomology solution supported at the origin Σ_0.

We have verified in this section that

$$\dim_{\mathbb{C}}(Hom_{D_X}(M_\beta, \mathcal{H}^4_{[\Sigma_2]}(\mathcal{O}_X)) = 1, \quad \text{for } \beta = -\tfrac{10}{9}, -\tfrac{11}{9}, -\tfrac{13}{9}, -\tfrac{14}{9}, -\tfrac{16}{9}, -\tfrac{17}{9}$$

and

$$Hom_{D_X}(M_\beta, \mathcal{H}^4_{[\Sigma_2]}(\mathcal{O}_X)) = \{0\}, \quad \text{for } \beta = -1, -\tfrac{3}{2}.$$

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