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Singular solutions of q-difference-differential equations of Briot-Bouquet type

By

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Abstract

In 1990, Gérard-Tahara [3] introduced the Briot-Bouquet type partial differential equation $t\partial_t u = F(t, x, u, \partial_x u)$. In [17] the author showed existences of holomorphic and singular solutions of the following type of difference-differential equations $tD_q u = F(t, x, u, \partial_x u)$ when the characteristic exponent $\rho(0) \neq (q^N - 1)/(q - 1)$ holds. In this paper the author shows existences of singular solutions with $\rho(0) = (q^N - 1)/(q - 1)$

§ 1. Introduction

In this paper let $q > 1$: for a function $f(t, x)$ we define the q-difference operator D_q by

$$D_q f(t,x) = \frac{\sigma_q f(t,x) - f(t,x)}{qt - t} = \frac{f(qt,x) - f(t,x)}{qt - t}.$$

In [15] Ramis introduced the q-difference operator D_q. We will study the following type of nonlinear q-difference-differential equations:

$$tD_q u = F(t, x, \{\partial_x^\alpha u\}_{|\alpha| \leq m}) \tag{1.1}$$

where $(t, x) = (t, x_1, \ldots, x_n) \in \mathbb{C}_t \times \mathbb{C}_x^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\partial_x^\alpha = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}$, $\partial_i = \partial/\partial_{x_i}$ for $i = 1, \ldots, n$, $F(t, x, Z)$ ($Z = \{Z_\alpha\}_{|\alpha| \leq m}$) is a function defined in a polydisk Δ centered at the origin of $\mathbb{C}_t \times \mathbb{C}_x^n \times \mathbb{C}_Z^\delta$ and δ is the cardinal of $\{\alpha \in \mathbb{N}^n : |\alpha| \leq m\}$.

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In the case of q-difference equations, Vizio, Ramis, Sauloy and Zhang [2] studied linear equations and Joshi and Shi [5], Nishioka [11] and Ohyama [12] obtained some results for nonlinear equations. In the case of q-difference-differential equations, Lastra, Malek and Sanz [7] and Tahara and Yamazawa [14] studied the summability of formal solutions of linear equations.

Let us denote $\Delta_0 = \Delta \cap \{t = 0, Z = 0\}$. In this paper we assume the following conditions:

(A1) $F(t, x, Z)$ is holomorphic in Δ,

(A2) $F(0, x, 0) = 0$ in Δ_0,

(A3) $\dfrac{\partial F}{\partial Z_\alpha}(0, x, 0) = 0$ in Δ_0 for all $1 \leq |\alpha| \leq m$.

Definition 1.1. If the equation (1.1) satisfies $(A1)$, $(A2)$ and $(A3)$ we say that (1.1) is a q-analogue of the Briot-Bouquet type with respect to t (simply the q-Briot-Bouquet type with respect to t).

Definition 1.2. ([3]) Let us define

$$\rho(x) = \frac{\partial F}{\partial u}(0, x, 0),$$

then the holomorphic function $\rho(x)$ is called the characteristic exponent of the equation (1.1).

Let us denote by

1. $\mathcal{R}(\mathbb{C}\backslash\{0\})$ the universal covering space of $\mathbb{C}\backslash\{0\}$,
2. $S_\theta = \{t \in \mathcal{R}(\mathbb{C}\backslash\{0\});\ |\arg t| < \theta\}$,
3. $S(\epsilon(s)) = \{t \in \mathcal{R}(\mathbb{C}\backslash\{0\});\ 0 < |t| < \epsilon(\arg t)\}$ for some positive-valued function $\epsilon(s)$ defined and continuous on $\mathbb{R}$,
4. $S_\theta(T) = S_\theta \cap S(T)$,
5. $D_R = \{x \in \mathbb{C}^n;\ |x_i| < R \text{ for } i = 1, \ldots, n\}$,
6. $\mathbb{C}\{x\}$ the ring of germs of holomorphic functions at the origin of $\mathbb{C}^n$,
7. $\mathcal{O}(D)$ the set of all holomorphic functions on a domain $D \subset \mathbb{C}^n_x$,
8. $||f||_R := \sup_{x \in D_R} |f(x)|$.

Definition 1.3. ([3]) We define the set $\widetilde{\mathcal{O}}_+$ of all functions $u(t,x)$ satisfying the following conditions:
1. $u(t,x)$ is holomorphic in $S(\epsilon(s)) \times D_R$ for some $\epsilon(s)$ and $R > 0$,
2. there is an $a > 0$ such that for any $\theta > 0$ and any compact subset K of D_R

$$\max_{x \in K} |u(t,x)| = O(|t|^a) \quad \text{as} \quad t \to 0 \quad \text{in} \quad S_\theta.$$

Set $\rho_q(x) = \log\{1+(q-1)\rho(x)\}/\log q$. Then the author proved the following result:

Theorem 1.4. *([17]) If the equation* (1.1) *is of the q-Briot-Bouquet type and* $\rho(0) \neq [i]_q := (q^i - 1)/(q-1)$ *for* $i = 1, 2, \ldots$ *then we have;*
(1) (Holomorphic solutions) The equation (1.1) *has a unique solution* $u_0(t,x)$ *holomorphic near the origin of* $\mathbb{C}_t \times \mathbb{C}_x^n$ *satisfying* $u_0(0,x) \equiv 0$.
(2) (Singular solutions) When $\Re\rho(0) > 0$ *for any* $\varphi(x) \in \mathbb{C}\{x\}$ *there exists an* $\widetilde{\mathcal{O}}_+$*-solution* $U(\varphi)$ *of* (1.1) *having an expansion of the following form:*

$$(1.2)\quad U(\varphi) = \sum_{i=1}^{\infty} u_i(x)t^i + \sum_{k \le i+2m(j-1), j \ge 1} \varphi_{i,j,k}(x) t^{i+\rho_q(x)j} (\log t)^k, \quad \varphi_{0,1,0}(x) = \varphi(x).$$

The purpose of our paper is to construct $\widetilde{\mathcal{O}}_+$-solutions of (1.1) on the case $\rho(0) = [N]_q$. The main result of this paper is;

Theorem 1.5. *If* (1.1) *is of the q-Briot-Bouquet type and* $\rho(0) \equiv [N]_q$ *for* $N \in \mathbb{N}$ *and* $\rho(x) \not\equiv \rho(0)$ *then we have;*
For any $\varphi(x) \in \mathbb{C}\{x\}$ *there exists an* $\widetilde{\mathcal{O}}_+$*-solution* $U(\varphi)$ *of* (1.1) *having an expansion of the following form:*

$$(1.3)\qquad U(\varphi) = \varphi(x)t^{\rho_q(x)} + \sum_{k \ge 1} \sum_{1 \le l \le m_k} \phi_{k,l}(x) w_{k,l}(t,x).$$

Here m_k *and* $w_{k,l}(t,x)$ *are as follows: (1)* m_k $(k \ge 1)$ *are positive integers determined by the equation* (1.1)*, and (2)* $w_{k,l}(t,x)$ $(k \ge 1$ *and* $1 \le l \le m_k)$ *are functions also determined by the equation* (1.1) *satisfying the following property: there is a* $\sigma > 0$ *such that* $w_{k,l}(t,x) = O(t^{\sigma k}, \widetilde{\mathcal{O}}_+)$ *(as* $t \to 0$*) holds for all* (k,l)*. The coefficients* $\varphi(x)$ *and* $\phi_{k,l}(x)$ *are as follows: (3)* $\varphi(x)$ *are arbitrary holomorphic function, and (4)* $\phi_{k,l}(x)$ $(k \ge 1$ *and* $1 \le l \le m_k)$ *are holomorphic functions determined by* $\varphi(x)$.

In the above condition (2) the notation

$$w(t,x) = O(t^s, \widetilde{\mathcal{O}}_+) \quad (\text{as } t \to 0)$$

means that the condition $t^{-s}w(t,x) \in \widetilde{\mathcal{O}}_+$ holds.

This paper is organized as follows. In section 2 and 3 we give the definition and estimates of $w_{k,l}(t,x)$ in Theorem 1.5. In section 4 and 5 we prepare lemmas to show our theorem. In section 6 and 7 we give a proof of Theorem 1.5.

§ 2. Definition of the System $\{w_{k,l}\}$

In this section we will define functions $\{w_{k,l}\}$, the idea to construct the functions is in [13]. We choose a constant σ such that $0 < \sigma < \min\{1, \rho_q(0)\}$ and $\{\sigma k;\ k = 1, 2, \dots\} \not\ni \rho_q(0)$. Then we have an integer N^* such that

$$\sigma N^* < \rho_q(0) < \sigma(N^* + 1). \tag{2.1}$$

Remark. By the definition of $\rho_q(x)$, $\rho_q(0) > 0$ holds.

Set $\lambda(x) = 1 + (q-1)\rho(x)$ and $a > 0$. Let us define an operator Q by

$$Q[t^N] = (\sigma_q - \lambda(x))^{-1}[t^N] := \frac{t^{\rho_q(x)} - t^{\rho_q(0)}}{\lambda(x) - \lambda(0)} \tag{2.2}$$

for $N = 1, 2, \dots$ and

$$Q[u(t,x)] := \begin{cases} \displaystyle\sum_{j=0}^{\infty} (\lambda(x))^j \sigma_q^{-(j+1)} u(t,x) & \text{if } ||\lambda||_r < q^a \\ \displaystyle -\sum_{j=0}^{\infty} (\lambda(x))^{-j-1} \sigma_q^j u(t,x) & \text{if } ||\lambda||_r > q^a \end{cases} \tag{2.3}$$

for $u(t,x) = O(t^a, \widetilde{\mathcal{O}}_+)$

Lemma 2.1. *([17], Corollary 5.5, p.196) Suppose that $u(t,x)$ belongs to $\widetilde{\mathcal{O}}_+$ and satisfies*

$$||u(t,\cdot)||_r \le M|t|^a \quad \text{for } t \in S_\theta(T)$$

for any $\theta > 0$ and an $a > 0$. Then there exists a positive constant C such that

$$||Qu(t,\cdot)||_r \le \frac{M}{Cq^a}|t|^a \quad \text{for } t \in S_\theta(T).$$

Then we define function classes in order to construct $w_{k,l}$.

Definition 2.2. We define finite sets $\mathcal{F}_k$, $\mathcal{G}_k$ and $\mathcal{H}_k$ $(k = 1, 2, \dots)$ in $\widetilde{\mathcal{O}}_+$ inductively by the following procedure (1) $\sim$ (3) :
(1) We set $\mathcal{F}_1 = \{Q[t]\}$. If $k \ge 2$ and if $\mathcal{H}_1, \dots, \mathcal{H}_{k-1}$ are already defined, we set

$$\mathcal{F}_k = \bigcup_{2 \le \mu + |\nu| \le k} \ \bigcup_{\mu + k_1 + \cdots + k_{|\nu|} = k} \{Q[t^\mu \phi_{k_1} \cdots \phi_{k_{|\nu|}}];\ \phi_{k_j} \in \mathcal{H}_{k_j}\ (j = 1, \dots, |\nu|)\}.$$

(2) If $\mathcal{F}_k$ is already defined, we set

$$\mathcal{G}_k = \begin{cases} \mathcal{F}_k & (k \neq N^*) \\ \mathcal{F}_k \cup \{t^{\rho_q(x)}\} & (k = N^*). \end{cases}$$

(3) If $\mathcal{G}_k$ is already defined, we set

$$\mathcal{H}_k = \bigcup_{|\alpha| \leq m} \bigcup_{0 \leq \beta_\alpha \leq \alpha} \{k^{|\beta_\alpha|} (\frac{\partial}{\partial x})^{\alpha - \beta_\alpha} W;\ W \in \mathcal{G}_k\}$$

where $\beta \leq \alpha$ means $\beta_j \leq \alpha_j$ for $j = 1, \dots, n$.

Definition 2.3. We define the system of functions $\{w_{k,l}(t,x) :\ k \geq 1,\ 1 \leq l \leq m_k\}$ by the following : set $m_k = \sharp \mathcal{F}_k$ and

$$\mathcal{F}_k = \{w_{k,1}(t,x), \dots, w_{k,m_k}(t,x)\} \quad \text{for } k = 1, 2, \dots .$$

Then we have:

Proposition 2.4. *If $R > 0$ is sufficiently small, we have*

(1) $w_{k,l}(t,x)$ belongs to $\widetilde{\mathcal{O}}_+$ for any $k \geq 1$ and $1 \leq l \leq m_k$.

(2) For any $\theta > 0$ there exists a $T > 0$ and $0 < \sigma' < \sigma$ such that

$$\Big| \Big(\frac{\partial}{\partial x} \Big)^\alpha w_{k,l}(t,x) \Big| \leq \frac{k^{|\alpha|}}{q^{\sigma' k}} |t|^{\sigma' k} \quad \text{on } S_\theta(T) \times D_R$$

holds for any $k \geq 1,\ 1 \leq l \leq m_k$ and $|\alpha| \leq m$.

In the next section, we will show the proof of Proposition 2.4.

§ 3. Proof of Proposition 2.4

In this section we will give the proof of Proposition 2.4.

Recall that $\mathcal{F}_1 = \{w_{1,1}(t,x)\}$ with $w_{1,1} = Q[t]$. Then taking $0 < \sigma < 1$ in Section 2 we can assume that $w_{1,1}$ is holomorphic on $S(1) \times D_R$ and for any $\theta > 0$ there exists $K_\theta > 1$ such that

$$\Big\| \Big(\frac{\partial}{\partial x} \Big)^\alpha w_{1,1}(t,\cdot) \Big\|_R \leq K_\theta \frac{1}{q^\sigma} |t|^\sigma \quad \text{on } S_\theta(1) \text{ for any } |\alpha| \leq m. \tag{3.1}$$

By induction on k we have:

Lemma 3.1. *For any $k = 1, 2, \dots$ we have the following properties $(1)_k$ and $(2)_k$, in which the constant $C_\theta > 1$ is independent of α, k and l.*

$(1)_k$ *$w_{k,l}(t,x)$ is holomorphic on $S_\theta(1)\times D_r$ for any $\theta>0$ and $0<r<R$ and $l=1,\dots,m_k$.*

$(2)_k$ *We have the following estimates for any $\theta>0$:*

$$\Big|\Big|\Big(\frac{\partial}{\partial x}\Big)^{\alpha} w_{k,l}(t,\cdot)\Big|\Big|_r \le \frac{k^{|\alpha|}}{q^{\sigma k}}\frac{C_\theta^{2k-2}}{(R-r)^{m(k-1)}}|t|^{\sigma k} \quad \text{on } S_\theta(1) \tag{3.2}$$

for any $0<r<R$, $|\alpha|\le m$ and $l=1,\dots,m_k$.

Proof. For $t^{\rho_q(x)}$ we can assume

$$\Big|\Big|\Big(\frac{\partial}{\partial x}\Big)^{\alpha} t^{\rho_q(x)}\Big|\Big|_r \le \frac{N^{*|\alpha|}}{q^{\sigma N^*}}\frac{C_\theta^{2N^*-2}}{(R-r)^{m(N^*-1)}}|t|^{\sigma N^*} \quad \text{on } S_\theta(1) \tag{3.3}$$

for any $0<r<R$ and $|\alpha|\le m$.

Let $k\ge 2$. Suppose that $(1)_i$ and $(2)_i$ already hold for $i=1,\dots,k-1$. Set $\varphi_{k_\nu}=k_\nu^{|\beta_\alpha|}(\frac{\partial}{\partial x})^{\alpha-\beta_\alpha}W$ for some $|\alpha|\le m$, $0\le\beta_\alpha\le\alpha$ and $W(t,x)\in\mathcal{G}_{k_\nu}$. Since $1\le k_\nu\le k-1$ holds, in the case $W(t,x)\in\mathcal{F}_{k_\nu}$ we have

$$\begin{aligned}||\varphi_{k_\nu}||_r &\le k_\nu^{|\beta_\alpha|}\frac{k_\nu^{|\alpha|-|\beta_\alpha|}}{q^{\sigma k_\nu}}\frac{C_\theta^{2k_\nu-2}}{(R-r)^{m(k_\nu-1)}}|t|^{\sigma k_\nu}\\ &\le K_m\frac{C_\theta^{2k_\nu-2}}{(R-r)^{m(k_\nu-1)}}|t|^{\sigma k_\nu} \quad \text{on } S_\theta(1)\end{aligned} \tag{3.4}$$

for any $0<r<R$. Here we used that there exists positive constant K_m such that

$$\frac{k^{|\alpha|}}{q^{\sigma k}}\le K_m \quad \text{for } |\alpha|\le m \text{ and } k=1,2,\dots.$$

In the case $k_\nu=N^*$ and $W(t,x)=t^{\rho_q(x)}$, by the estimate (3.3) we can obtain the same estimate as (3.4). By the definition of $\mathcal{F}_k$, we see that $w_{k,l}(t,x)$ is expressed the form

$$w_{k,l}(t,x)=Q[t^\mu\varphi_{k_1}\times\cdots\times\varphi_{k_{|\nu|}}]$$

where $2\le\mu+|\nu|\le k$ and $\mu+k_1+\cdots+k_{|\nu|}=k$.

Then by (3.4) we have

$$\begin{aligned}&||t^\mu\varphi_{k_1}(t,\cdot)\times\cdots\times\varphi_{k_{|\nu|}}(t,\cdot)||_r\\ \le&|t|^\mu\times K_m\frac{C_\theta^{2k_1-2}}{(R-r)^{m(k_1-1)}}|t|^{\sigma k_1}\times\cdots\times K_m\frac{C_\theta^{2k_{|\nu|}-2}}{(R-r)^{m(k_{|\nu|}-1)}}|t|^{\sigma k_{|\nu|}}\\ \le&K_m^{|\nu|}\frac{C_\theta^{2(k_1+\cdots+k_{|\nu|}-|\nu|)}}{(R-r)^{m(k_1+\cdots+k_{|\nu|}-|\nu|)}}|t|^{\sigma k}\le\frac{C_\theta^{2k-2\mu-3/2|\nu|}}{(R-r)^{m(k-\mu-|\nu|)}}|t|^{\sigma k} \quad \text{on } S_\theta(1)\end{aligned} \tag{3.5}$$

when $K_m\le C_\theta^{1/2}$.

Since $\theta > 0$ is arbitrary, this implies that $t^\mu \varphi_{k_1} \times \cdots \times \varphi_{k_{|\nu|}}$ is holomorphic on $S(1) \times D_r$ for any $0 < r < R$. Thus we see that $w_{k,l}(t,x)$ is holomorphic on $S(1) \times D_r$, that is, $w_{k,l}(t,x)$ belongs to $\widetilde{\mathcal{O}}_+$. By Lemma 2.1, we have

$$(3.6) \qquad ||w_{k,l}||_r \leq \frac{1}{Cq^{\sigma k}} \frac{C_\theta^{2k-2\mu-3/2|\nu|}}{(R-r)^{m(k-2)}} |t|^{\sigma k} \leq \frac{1}{Cq^{\sigma k}} \frac{C_\theta^{2k-3}}{(R-r)^{m(k-2)}} |t|^{\sigma k} \quad \text{on } S_\theta(1).$$

We need the following lemma in order to estimate the derivative of $w_{k,l}$.

Lemma 3.2 (Nagumo's lemma). *If a holomorphic function $u(x)$ in D_R satisfies*

$$||u||_r \leq \frac{C}{(R-r)^p} \quad \textit{for } 0 < r < R$$

then we have

$$||\frac{\partial}{\partial x} u||_r \leq \frac{Ce(p+1)}{(R-r)^{p+1}} \quad \textit{for } 0 < r < R.$$

For the proof, see Hörmander ([4], lemma 5.1.3).

By Lemma 3.2, we obtain

(3.7)

$$\begin{aligned}
\Big|\Big| \Big(\frac{\partial}{\partial x}\Big)^\alpha w_{k,l} \Big|\Big|_r &\leq \frac{C_\theta^{2k-3}}{Cq^{\sigma k}} \frac{(m(k-2)+1)(m(k-2)+2)\cdots(m(k-2)+|\alpha|)e^{|\alpha|}}{(R-r)^{m(k-2)+|\alpha|}} |t|^{\sigma k} \\
&\leq \frac{C_\theta^{2k-3}}{Cq^{\sigma k}} \frac{(m(k-1))^{|\alpha|} e^{|\alpha|}}{(R-r)^{m(k-2)+|\alpha|}} |t|^{\sigma k} \leq \frac{C_\theta^{2k-3}(k-1)^{|\alpha|}}{Cq^{\sigma k}} \frac{(me)^m}{(R-r)^{m(k-1)}} |t|^{\sigma k} \\
&\leq \frac{k^{|\alpha|}}{q^{\sigma k}} \frac{C_\theta^{2k-2}}{(R-r)^{m(k-1)}} |t|^{\sigma k} \quad \text{on } S_\theta(1)
\end{aligned}$$

for any $|\alpha| \leq m$ and $(me)^m/C \leq C_\theta$. Q.E.D.

Completion of the proof of Proposition 2.4.

Set $r = R/2$ and take $\sigma' > 0$ with $\sigma' < \sigma$ and (2.1). Then we see that $w_{k,l}(t,x)$ is holomorphic on $S(1) \times D_{R/2}$ for all (k,l) and

$$(3.8) \qquad \begin{aligned}
\Big|\Big| \Big(\frac{\partial}{\partial x}\Big)^\alpha w_{k,l}(t,\cdot) \Big|\Big|_{R/2} &\leq \frac{k^{|\alpha|}}{q^{\sigma k}} \frac{C_\theta^{2k-2}}{(R/2)^{m(k-1)}} |t|^{\sigma k} \\
&\leq \frac{k^{|\alpha|}}{q^{\sigma' k}} \Big(\frac{C_\theta^2}{(R/2)^m} \Big(\frac{|t|}{q}\Big)^{\sigma-\sigma'} \Big)^k |t|^{\sigma' k} \quad \text{on } S_\theta(1)
\end{aligned}$$

holds for any $|\alpha| \leq m$ and (k,l). Then if we take $T > 0$ so that

$$\Big(\frac{C_\theta^2}{(R/2)^m} \Big(\frac{|t|}{q}\Big)^{\sigma-\sigma'} \Big) \leq 1 \quad \text{for } |t| \leq T,$$

we obtain the estimate in Proposition 2.4-(2) with R and σ replaced $R/2$ and σ' respectively.

§ 4. Reduction equations

In this section we will reduct (1.1) into the following equation in order to prove the main theorem:

$$(\sigma_q - \lambda(x))u(t,x) = a(x)t + G_2(x)(t, \{\partial_x^\alpha u(t,x)\}_{|\alpha|\le m}) \tag{4.1}$$

where $a(x) \in \mathcal{O}(D_R)$.

Set $Z^\nu = \prod_{|\alpha|\le m}\{Z_\alpha\}^{\nu_\alpha}$. We assume that the functions $G_2(x)(t,Z)$ have the following expansion:

$$G_2(x)(t,Z) = \sum_{\mu+|\nu|\ge 2} g_{\mu,\nu}(x)t^\mu Z^\nu \tag{4.2}$$

where $g_{\mu,\nu}(x) \in \mathcal{O}(D_R)$.

Lemma 4.1. *If the equation* (1.1) *is of the q-Briot-Bouquet type, then we can reduct* (1.1) *into the equation* (4.1) *with* (4.2).

Proof. We multiply the both side of (1.1) by $q-1$. Then we get (4.1) and $\lambda(x) = 1 + (q-1)\rho(x)$. Q.E.D.

Remark. If $\rho(0) \equiv [N]_q$ then $\lambda(0) \equiv q^N$.

§ 5. Function equation

Let us consider a functional equation in order to estimate the coefficients $w_{l,k}(x)$. Set $A \ge 0$, $G_{\mu,\nu} \ge 0$ $(\mu + |\nu| \ge 2)$ and $0 < r < R$. Then we consider the following functional equation:

$$Y = At + \frac{1}{(R-r)^m} \sum_{\mu+|\nu|\ge 2} \frac{G_{\mu,\nu}}{(R-r)^{m(\mu+|\nu|-2)}} t^\mu ((2me)^m Y)^{|\nu|} \tag{5.1}$$

where a series $\sum_{\mu+|\nu|\ge 2} G_{\mu,\nu}t^\mu Y^{|\nu|}$ converges in a neigbourhood of $(t,Y) = (0,0)$.

For the equation (5.1) we have the following lemma.

Lemma 5.1. *The functional equation* (5.1) *has the holomorphic solution* $Y(t) = \sum_{k=1} Y_k(r)t^k$ *that is expressed in the form*

$$Y_k(r) = \frac{C_k}{(R-r)^{m(k-1)}} \quad \textit{for } k \ge 1$$

with constants $C_1 = A$ *and* $C_k \ge 0$ *for* $k \ge 2$.

Proof. Let us show that the equation (5.1) have formal power series solutions $Y(t) = \sum_{k=1}^{\infty} Y_k(r)t^k$. By substituting $\sum_{k=1}^{\infty} Y_k t^k$ into (5.1), we have $Y_1 = A$ and

$$Y_k = \frac{1}{(R-r)^m} \sum_{2 \le \mu+|\nu| \le k} \frac{G_{\mu,\nu}}{(R-r)^{m(\mu+|\nu|-2)}} \prod_{\mu+|k(\nu)|=k} \prod_{|\alpha| \le m} \prod_{i=1}^{\nu_\alpha} ((2em)^m) Y_{k_\alpha(i)} \tag{5.2}$$

for $k \ge 2$. Let us show that the formal power series solutions $Y(t)$ are holomorphic in a neighborhood of the origin $t = 0$. Set

$$\begin{aligned} F(t,Y) &= Y - At \\ &\quad - \frac{1}{(R-r)^m} \sum_{\mu+|\nu| \ge 2} \frac{G_{\mu,\nu}}{(R-r)^{m(\mu+|\nu|-2)}} t^\mu ((2me)^m Y)^{|\nu|}. \end{aligned} \tag{5.3}$$

We have $F(0,0) = 0$ and $\partial_Y F(0,0) = 1$. Then we get holomorphic solutions $Y(t)$ by the implicit function's theorem. Further we have the coefficients $Y_k(r) = C_k/(R-r)^{m(k-1)}$ by the recurrence formula (5.2) and induction on k. Q.E.D.

§ 6. Proof of Theorem 1.5 of the case $N = 1$

In this section we will give the proof of theorem 1.5. We will show that the equation (4.1) has formal solutions of the form (1.3) and the formal solution converges in $S_\theta(T) \times D_r$ by the implicit function's theorem and the method of majorant functions.

§ 6.1. Construction of a formal solution

We note that we have $N^* = 1$ when $N = 1$. Let us construct a formal solution $u(t,x)$ of the equation (4.1) in the form

$$u(t,x) = \sum_{k \ge 1} u_k(t,x) \tag{6.1}$$

with

$$u_k(t,x) = \begin{cases} \phi_{1,1}(x) w_{1,1}(t,x) + \varphi(x) t^{\rho_q(x)} & k = 1 \\ \displaystyle\sum_{l=1}^{m_k} \phi_{k,l}(x) w_{k,l}(t,x) & k \ge 2 \end{cases} \tag{6.2}$$

where $\varphi(x)$ and $\phi_{k,l}(x)$ are suitable holomorphic functions in a common neighbourhood of $x = 0$. Here set $W_{1,1} := w_{1,1}$, $W_{1,2} := t^{\rho_q(x)}$, $M_1 := m_1 + 1 = 2$, $W_{k,l} := w_{k,l}$ and $M_k := m_k$ for $k \ge 2$.

Remark. We note

$$w_{1,1}(t,x)=\frac{t^{\rho_q(x)}-t^{\rho_q(0)}}{\lambda(x)-\lambda(0)}.$$

By the proof of Proposition 2.4, we have

$$u_k(t,x)=O(t^{\sigma k},\widetilde{\mathcal{O}}_+)\quad\text{as } t\to 0$$

for all $k\geq 1$. Then we will construct a formal solution as follows;

$$(6.3)\qquad \begin{aligned}&(\sigma_q-\lambda(x))u_1=a(x)t\\ &(\sigma_q-\lambda(x))u_k=\sum_{\mu+|k(\nu)|=k}g_{\mu,\nu}(x)t^{\mu}\prod_{|\alpha|\leq m}\prod_{i\leq\nu_\alpha}\big(\frac{\partial}{\partial x}\big)^{\alpha}u_{k_\alpha(i)}(t,x)\end{aligned}$$

where $|\nu|=\sum_{|\alpha|\leq m}\nu_\alpha$ and $|k(\nu)|=\sum_{|\alpha|\leq m}\sum_{i=1}^{\nu_\alpha}k_\alpha(i)$. By substituting $u_k(t,x)$ to the relation (6.3), we have

$$(6.4)\qquad \begin{aligned}&\sum_{l=1}^{m_k}\phi_{k,l}(x)(\sigma_q-\lambda(x))w_{k,l}(t,x)\\ &=\sum_{2\leq\mu+|\nu|\leq k}g_{\mu,\nu}(x)\sum_{(k(\nu),l(k))\in J_k(\mu,\nu)}\sum_{\beta(\nu)\in\Gamma(\nu)}\prod_{|\alpha|\leq m}\prod_{i=1}^{\nu_\alpha}\\ &\times\binom{\alpha}{\beta_\alpha(i)}\frac{1}{k_\alpha(i)^{|\beta_\alpha(i)|}}\big(\frac{\partial}{\partial x}\big)^{\beta_\alpha(i)}\phi_{k_\alpha(i),l_\alpha(i)}(x)\\ &\times\Big[t^{\mu}\prod_{|\alpha|\leq m}\prod_{i=1}^{\nu_\alpha}k_\alpha(i)^{|\beta_\alpha(i)|}\big(\frac{\partial}{\partial x}\big)^{\alpha-\beta_\alpha(i)}W_{k_\alpha(i),l_\alpha(i)}(t,x)\Big]\end{aligned}$$

where $l(\nu)=\{(l_\alpha(i));|\alpha|\leq m \text{ and } 1\leq i\leq\nu_\alpha\}$, $L(\nu,k(\nu))=\{l(\nu);1\leq l_\alpha(i)\leq m_{k_\alpha}(i)\ (|\alpha|\leq m,1\leq i\leq\nu_\alpha)\}$, $J_k(\mu,\nu)=\{(k(\nu),l(\nu));\mu+|k(\nu)|=k,l(\nu)\in L(\nu,k(\nu))\}$, $\beta(\nu)=\{(\beta_\alpha(i));\beta_\alpha(i)\in\mathbb{N}^n,|\alpha|\leq m \text{ and } 1\leq i\leq\nu_\alpha\}$ and $\Gamma(\nu)=\{\beta(\nu);\beta_\alpha(i)\leq\alpha\ (|\alpha|\leq m,1\leq i\leq\nu_\alpha)\}$.

Here we note the following lemma that is the same as Lemma 8 in [13].

Lemma 6.1. *Let $k\geq 2$ and set $\mathcal{A}_k=\{(\mu,\nu,k(\nu),l(\nu),\beta(\nu));2\leq\mu+|\nu|\leq k,(k(\nu),l(\nu))\in\mathcal{J}_k(\mu,\nu)$ and $\beta(\nu)\in\Gamma(\nu)\}$. Then by a suitable injection $\pi_k\colon\mathcal{A}_k\to\{1,2,\dots,m_k\}$ we have the following equality*

$$(6.5)\qquad Q\Big[t^{\mu}\sum_{|\alpha|\leq m}\prod_{i=1}^{\nu_\alpha}k_\alpha(i)^{|\beta_\alpha(i)|}\big(\frac{\partial}{\partial x}\big)^{\alpha-\beta_\alpha(i)}W_{k_\alpha(i),l_\alpha(i)}(t,x)\Big]=w_{k,l}(t,x)$$

and

$$(6.6)\qquad g_{\mu,\nu}(x)\prod_{|\alpha|\leq m}\prod_{i=1}^{\nu_\alpha}\binom{\alpha}{\beta_\alpha(i)}\frac{1}{k_\alpha(i)^{|\beta_\alpha(i)|}}\big(\frac{\partial}{\partial x}\big)^{\beta_\alpha(i)}\phi_{k_\alpha(i),l_\alpha(i)}(x)=\phi_{k,l}(x)$$

under the correspondence $\pi_k(\mu,\nu,k(\nu),l(\nu),\beta(\nu))=l$.

Thus we have;

Proposition 6.2. *We can construct a formal solution* $\hat{u}(t,x)$ *of the form (6.1) and (6.2). Moreover we see the following: (i) the coefficient* $\varphi(x)\in\mathbb{C}\{x\}$ *can be chosen arbitrary, (ii)* $\phi_{1,1}(x)=a(x)$*, and (iii) all the other coefficients* $\phi_{k,l}(x)\in\mathbb{C}\{x\}$ *are determined by (6.6) and therefore they are all holomorphic in a common neighborhood of* $x=0$.

§ 6.2. Proof of the convergence of a formal solution

We will prove the convergence of the formal solution $\hat{u}(t,x)$ in Proposition 6.2. Let $R>0$ be sufficiently small. Set

$$\begin{aligned}&||(\frac{\partial}{\partial x})^{\alpha}\varphi||_R+||(\frac{\partial}{\partial x})^{\alpha}\phi_{1,1}||_R\le A \quad \text{for } |\alpha|\le m \text{ and}\\ &||g_{\mu,\nu}||_R\le G_{\mu,\nu} \quad \text{for } \mu+|\nu|\ge 2.\end{aligned} \tag{6.7}$$

We will assume that

$$At+\sum_{\mu+|\nu|\ge 2}G_{\mu,\nu}t^{\mu}Z^{\nu}$$

is convergent in a neighborhood of $(t,Z)=(0,0)$.

Let $\hat{u}(t,x)=\sum_{k\ge 1}u_k(t,x)$ with

$$u_k(t,x)=\sum_{l=1}^{M_k}\phi_{k,l}(x)W_{k,l}(t,x) \tag{6.8}$$

be the formal solution. We know $M_1=2$, $\phi_{1,1}(x)=a(x)$, $\phi_{1,2}(x)=\varphi(x)$ and

$$||u_k||_r\le\sum_{l=1}^{M_k}||\phi_{k,l}||_r|t|^{\sigma k} \quad \text{for } t\in S_{\theta}(T).$$

Then set

$$||u_k||_r^*:=\sum_{l=1}^{M_k}||\phi_{k,l}||_r \quad \text{and } ||D_x^{\alpha}u_k||_r^*=\sum_{l=1}^{M_k}||(\frac{\partial}{\partial x})^{\alpha}\phi_{k,l}||_r \tag{6.9}$$

in order to estimate the term $u_k(t,x)$ in (6.8).

We have $||u_k||_r^*=\delta_{k,1}||\varphi||_r+\sum_{l=1}^{m_k}||\phi_{k,l}||_r$ where $\delta_{k,1}=1$ if $k=1$, $\delta_{k,1}=0$ if $k\ne 1$. By the relation (6.6), we obtain

$$
(6.10)\quad \begin{aligned} ||u_k||_r^* \le \delta_{k,1}||\varphi||_r + \sum_{2\le\mu+|\nu|\le k} ||g_{\mu,\nu}||_r \sum_{\mu+|k(\nu)|=k,\beta(\nu)\in\Gamma(\nu)} \prod_{|\alpha|\le m} \prod_{i=1}^{\nu_\alpha} \binom{\alpha}{\beta_\alpha(i)} \\ \times \frac{1}{k_\alpha(i)^{|\beta_\alpha(i)|}} ||D_x^{\beta_\alpha(i)} u_{k_\alpha(i)}||_r^*. \end{aligned}
$$

Let us consider the following equation:

$$
(6.11)\qquad Y = At + \frac{1}{(R-r)^m} \sum_{\mu+|\nu|\ge 2} \frac{G_{\mu,\nu}}{(R-r)^{m(\mu+|\nu|-2)}} t^\mu ((2me)^m Y)^{|\nu|}.
$$

Then we have;

Proposition 6.3.

$$
(6.12)\qquad \frac{1}{k^{|\alpha|}} ||D_x^\alpha u_k||_r^* \le (me)^m Y_k \quad \textit{for any } 0<r<R \textit{ and } |\alpha|\le m,
$$

for $k=1,2,\dots.$

Proof.

For $k=1$ by (6.7), we have

$$
(6.13)\quad \begin{aligned} ||D_x^\alpha u_1||_r^* &= ||(\frac{\partial}{\partial x})^\alpha \varphi||_r + ||(\frac{\partial}{\partial x})^\alpha \phi_{1,1}||_r \\ &\le A = Y_1 \le (me)^m Y_1 \quad \text{for any } |\alpha|\le m. \end{aligned}
$$

When $k=1$, the inequality (6.12) holds.

For $k\ge 2$, suppose that (6.12) holds for $i=1,2,\dots,k-1$. Then we have

(6.14)

$$
\begin{aligned} ||u_k||_r^* &\le \sum_{2\le\mu+|\nu|\le k} \frac{G_{\mu,\nu}}{(R-r)^{m(\mu+|\nu|-2)}} \sum_{\mu+|k(\nu)|=k,\beta(\nu)\in\Gamma(\nu)} \prod_{|\alpha|\le m} \prod_{i=1}^{\nu_\alpha} \binom{\alpha}{\beta_\alpha(i)} (me)^m Y_{k_{\alpha(i)}} \\ &\le \sum_{2\le\mu+|\nu|\le k} \frac{G_{\mu,\nu}}{(R-r)^{m(\mu+|\nu|-2)}} \sum_{\mu+|k(\nu)|=k} \prod_{|\alpha|\le m} \prod_{i=1}^{\nu_\alpha} (2me)^m Y_{k_{\alpha(i)}} \\ &= (R-r)^m Y_k = \frac{C_k}{(R-r)^{m(k-2)}}. \end{aligned}
$$

Here we used

$$
\sum_{0\le\beta\le\alpha} \binom{\alpha}{\beta} = 2^{|\alpha|} \le 2^m.
$$

By applying Lemma 3.2 to the inequality (6.14), we obtain

(6.15)
$$\begin{aligned}\frac{1}{k^{|\alpha|}}||D_x^\alpha u_k||_r^* &\le \frac{1}{k^{|\alpha|}}\frac{(m(k-2)+1)\times\cdots\times(m(k-2)+|\alpha|)e^{|\alpha|}C_k}{(R-r)^{m(k-2)+|\alpha|}}\\ &\le \frac{(k-1)^{|\alpha|}}{k^{|\alpha|}}\frac{(me)^{|\alpha|}C_k}{(R-r)^{m(k-2)+|\alpha|}} \le (me)^m\frac{C_k}{(R-r)^{m(k-1)}} = (me)^m Y_k.\end{aligned}$$

Q.E.D.

Proof of the convergence of the formal solution $\hat{u}(t,x)$. Set $r = R/2$. By Proposition 2.4-(2), 6.3, (6.8) and (6.9), we have

$$|u(t,x)| \le \sum_{k\ge 1}||u_k(t,\cdot)||_r \le \sum_{k\ge 1}||u_k||_r^*|t|^{\sigma k} \le (me)^m\sum_{k\ge 1}Y_k|t|^{\sigma k} \quad \text{on } S_\theta(T).$$

$u(t,x)$ converges on $S_\theta(T)\times D_r$. Since $\theta > 0$ is arbitrary, we can conclude that $u(t,x)$ converges in $\widetilde{\mathcal{O}}_+$.

§ 7. Proof of Theorem 1.5 of the case $N \ge 2$

We set

$$u(t,x) = \sum_{i=1}^{N-1} u_i(x)t^i + t^{N-1}w(t,x) \tag{7.1}$$

where $u_i(x) \in \mathbb{C}\{x\}$ $(1 \le i \le N-1)$ and $w(t,x) \in \widetilde{\mathcal{O}}_+$. Then by an easy calculation we see;

Lemma 7.1. *If the function (7.1) is a solution of the equation (4.1), then the functions $u_1(x),\dots,u_{N-1}(x)$ are uniquely determined and $w(t,x)$ satisfies the following equation:*

$$\begin{aligned}(\sigma_q - q^{-(N-1)}\lambda(x))w &= ta(t,x) + t\sum_{|\alpha|\le m}A_\alpha(t,x)\partial_x^\alpha w\\ &\quad + \sum_{|\nu|\ge 2}t^{(N-1)(|\nu|-1)}B_\nu(t,x)\prod_{|\alpha|\le m}\{\partial_x^\alpha w\}^{\nu_\alpha},\end{aligned} \tag{7.2}$$

where

$$a(t,x) = \frac{1}{q^{N-1}t^N}\left(G_2(x)(t,\{\partial_x^\alpha w_0\}_{|\alpha|\le m}) + ta(x) - (\sigma_q - \lambda(x))w_0\right)$$

with $w_0 = \sum_{i=1}^{N-1} u_i(x)t^i$ *and*

$$A_\alpha(t,x) = \frac{1}{q^{N-1}t}\frac{\partial G_2}{\partial Z_\alpha}(x)(t,\{\partial_x^\alpha w_0\}_{|\alpha|\le m}), \quad |\alpha| \le m$$

$$B_\nu(t,x) = \frac{1}{q^{N-1}}\frac{1}{\nu!}\frac{\partial^{|\nu|}G_2}{\partial Z^\nu}(x)(t,\{\partial_x^\alpha w_0\}_{|\alpha|\le m}), \quad |\nu| \ge 2.$$

We can apply the results in Section 6 to the equation (7.2). Hence this completes the proof of Theorem 1.5.

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