RIMS Kôkyûroku Bessatsu
B75 (2019), 089–099

Movable singularity of generalized Emden equation via Birkhoff reduction

By

Masafumi YOSHINO*

Abstract

This paper studies the movable singularity of the generalized Emden-Fowler equation. For a certain nonlinear term we shall show the existence of infinitely many movable singularities in the complex plane by virtue of the reduction similar to Birkhoff's normal form theory.

§ 1. Motivation and the result

In this paper we shall study the movable singularity of generalized Emden -Fowler equation

$$\frac{d^2u}{dt^2} + \frac{n-1}{t}\frac{du}{dt} + u^\ell = 0, \quad t \in \mathbb{C}, \tag{1.1}$$

where $n \neq 2, \ell \geq 2$ are integers. By the movable singularity we mean the singularity which depends on the respective solution and that does not appear in the coefficients of the equation. There are many works as to the movable singularity of nonlinear ordinary differential equation. We cite [1] and the references therein. The existence of movable branching singularity of (1.1) was shown in [1] by the argument closely related with Painlevé's theorem. Recently, detailed analysis on the singularity was also made in terms of phase space analysis. (cf. [4]). The object of the present paper is to give an alternative proof of the existence of a movable singularity by virtue of the reduction argument similar to the Birkhoff normal form theory of a Hamiltonian system. The advantage of our method is that one can show some structure of the movable singularity

Received April 9, 2017. Revised January 14, 2017.
2010 Mathematics Subject Classification(s): Primary 35C10; Secondary 45E10
Key Words: *movable singularity, Birkhoff reduction, elliptic function.*
Partially supported by Grant-in-Aid for Scientific Research (No. 26400118), JSPS, Japan.
*Hiroshima University, Hiroshima 739-8526, Japan. e-mail: `yoshinom@hiroshima-u.ac.jp`

in terms of Jacobi's elliptic function. Indeed, we write (1.1) in the Hamiltonian system and we transform it to the autonomous form by some analytic transformation in certain domain. By virtue of Jacobi's elliptic function we shall show the existence of infinitely many movable singularities. We shall prove

Theorem 1.1. *Assume that $\ell = 3$ in (1.1). Then (1.1) has an infinitely many movable singularities.*

§ 2. Normalization via Birkhoff reduction

We shall transform (1.1) to a simpler form. For this purpose we write (1.1) in a Hamiltonian system. Set $\ell = 3$, $t = e^z$ and define $v = e^{(2-n)z/2}u$. Then, by (1.1) we have

$$u_{zz} - \tilde{c}_0 u + e^{z\beta}u^3 = 0, \tag{2.1}$$

where $\tilde{c}_0 = (n-2)^2/4$ and $\beta = 4 - n$. Next, by putting $q := u$ and $p := u_z$, we write (2.1) in the system of equations for q and p. Indeed, the Hamiltonian function is given by

$$\frac{1}{2}(p^2 - \tilde{c}_0 q^2) + \frac{e^{z\beta}}{4}q^4.$$

By replacing q and p with constant times of respective variable, if necessary, and by introducing new unknown quantities $q_2 := q+p$ and $p_2 := q-p$ we can write the system in the Hamiltonian system with Hamitonian

$$H = \lambda q_2 p_2 + ce^{z\beta}(q_2 + p_2)^4, \tag{2.2}$$

where $\lambda \neq 0$ and c are certain constants. One may assume $\lambda = 1$ without loss of generality.

We set $q_1 = z$ and $\tilde{c}(z) = 4ce^{\beta z}$ in (2.2). Consider the autonomous system $p_1 + H$ with $H := q_2 p_2 + ce^{z\beta}(q_2 + p_2)^4$. The resonance term appears from the nonlinear part. Define

$$a \equiv a(q_2 p_2) := 6cq_2^2 p_2^2. \tag{2.3}$$

We shall find a formal Birkhoff transformation which brings the nonlinear part of H to the one $a(q_2p_2) + c(q_2^4 + p_2^4)$. In fact, we have

Theorem 2.1. *There exists a formal symplectic transformation which transforms $p_1 + H$ to $p_1 + q_2p_2 + a(q_2p_2) + c(q_2^4 + p_2^4)$.*

Proof. The proof is essentially Birkhoff's reduction. For the sake of completeness we give the proof. By the general theory of the transformation of a vector field, we know that the vector field $X_0 + R$ is transformed to $X_0 + S$ by certain coordinate change generated by Y, if we determine Y by the homology equation (adjoint equation) $[X_0, Y] = R - S$, where $[X_0, Y]$ denotes the Lie bracket and S is a certain term containing the resonance. We shall solve the homology equation $[X_0, Y] = R - S$ formally in terms of the infinite composition of symplectic transformations. Because we cannot show the convergence of the infinite compositions, the transformation is formal. As to the solvability and properties of the adjoint equation in the Hamiltonian setting we refer to Lemma 6 in [3]. Let χ_g denotes the Hamiltonian vector field with the Hamiltonian function g with respect to a standard symplectic structure. We denote by $\{\cdot, \cdot\}$ the Poisson bracket.

Let $x = (\tilde{q}_1, \tilde{p}_1, \tilde{q}_2, \tilde{p}_2)$ and $y = (q_1, p_1, q_2, p_2)$ be the original and the transformed coordinates, respectively. For simplicity we sometimes write $y = (y_1, \ldots, y_4)$. We consider the transformation $x = u(y)$ for some $u = (u_1, \ldots, u_4)$. For $\phi \equiv \phi(q_1, q_2, p_2)$, define

$$X_0 := \chi_{p_1 + q_2 p_2} \quad Y := \chi_\phi = \sum_j u_j \frac{\partial}{\partial y_j}. \tag{2.4}$$

Then the component of X_0 on the basis $\partial/\partial q_1$, $\partial/\partial p_1$, $\partial/\partial q_2$, $\partial/\partial p_2$ with this order is given by $\Lambda(y) = (1, 0, q_2, -p_2)$. Set

$$f(x) := \tilde{c}(\tilde{q}_1)(\tilde{q}_2 + \tilde{p}_2)^4/4, \tag{2.5}$$

and define

$$R := \chi_f, \qquad S := \chi_{a(q_2 p_2) + c(q_2^4 + p_2^4)}. \tag{2.6}$$

Write $R = r(x)\frac{\partial}{\partial x}$ and $S = s(y)\frac{\partial}{\partial y}$. Note that the first and the second components of $s(y)$ vanish and $r(x)$ is independent of x_2. We have

$$\chi_{\{p_1 + q_2 p_2, \phi\}} = [X_0, Y] = X_0 Y - Y X_0. \tag{2.7}$$

Note that the component of the right-hand side is given by $\Lambda(y)\nabla u - u\nabla\Lambda(y)$. Then, by simple computation the equation $[X_0, Y] = R - S$ is written as

$$\Lambda(y)\nabla u - u\nabla\Lambda(y) = r(u) - s(y)\nabla u. \tag{2.8}$$

If we set $u_1 = v_1$, $u_2 = v_2$, $u_j(y) = y_j + v_j(y)$, then we see that (2.8) is written in the form

$$\Lambda(y)\nabla v - v\nabla\Lambda(y) = r(v_1, y_j + v_j) - s(y) - s(y)\nabla v. \tag{2.9}$$

Clearly, if we can solve (2.8) or (2.9) formally, then we have our theorem. Approximate the right-hand side of (2.8) with $r(y) - s(y)$. Consider the linear equation of ϕ

$$\{p_1 + q_2p_2, \phi\} = f(y) - a(q_2p_2) - c(q_2^4 + p_2^4). \tag{2.10}$$

The right-hand side is homogeneous degree of four. Consider the monomial of the form $e^{\beta q_1} K q_2^i p_2^j$ for some constants β and K in the right-hand side. Set $\phi = e^{\beta q_1}\psi(q_2, p_2)$. Clearly, if $\beta + i - j \neq 0$, then one can determine ϕ. If otherwise, then we set $\phi = q_1 e^{\beta q_1}\psi(q_2, p_2)$. In general, we consider $\alpha(q_1)\beta(q_2, p_2)$, where $\beta(q_2, p_2)$ is a monomial. We have, for $\phi = \omega(q_1)\psi(q_2, p_2)$

$$\{p_1 + q_2p_2, \phi\} = \{p_1, \omega\}\psi + \{q_2p_2, \psi\}\omega = \alpha(q_1)\beta(q_2, p_2).$$

Clearly, $\{q_2p_2, \psi\}$ can be chosen as the constant times of $\beta(q_2, p_2)$. Hence ω satisfies the equation like $\omega' + c_1\omega = c_2\alpha$ for some constants c_1 and c_2. Therefore one can solve (2.10).

One can verifies that ϕ gives the terms of homogeneous degree 4 of v in (2.9). Set $u = y + v$. Then, the change of variables $x = u(y)$ eliminate the terms of degree 4 in f except for those which remain in S. After the change of variables we make the same argument in such a way that we eliminate the terms of homogeneous degree 5 in the right-hand side of (2.9). We repeat the same argument. One can delete every homogeneous term in the right-hand side of (2.9). Finally, by the infinite compositions of transformations like $u = y + v$ one can solve (2.9) in category of formal power series. □

Next we give a meaning to a certain normal form obtained in Theorem 2.1. First we look for the alternative expression of the homology equation. In fact, we have

Lemma 2.2. *Suppose that u satisfy*

$$\Lambda(y)\nabla u + s(y)\nabla u = r(u) + \Lambda(u). \tag{2.11}$$

Then, the transformation $x = u(y)$ *maps the vector field* $(\Lambda(x) + r(x))\frac{\partial}{\partial x}$ *to* $(\Lambda(y) + s(y))\frac{\partial}{\partial y}$.

Proof.

$$(\Lambda(x) + r(x))\frac{\partial}{\partial x} = (\Lambda(u) + r(u))(\nabla u)^{-1}\frac{\partial}{\partial y} = (\Lambda(y) + s(y))\frac{\partial}{\partial y}.$$

□

We remark that $\Lambda(u)$ in (2.11) and $u\nabla\Lambda(y)$ in (2.8) is identical except for the term 1 in the first component, which causes a minor change of variables in the original equation. We shall simplify (2.11). We first consider the third equation of (2.11). Set $w = u_3$, $u = (u_1, \cdots, u_4)$. Then we have

$$\frac{\partial w}{\partial q_1} + \delta w + q_2 p_2 \delta + p_2^3 \frac{\partial w}{\partial q_2} - q_2^3 \frac{\partial w}{\partial p_2} - w - \mathcal{R}(u) = 0 \tag{2.12}$$

where $\delta = q_2 \frac{\partial}{\partial q_2} - p_2 \frac{\partial}{\partial p_2}$ and $\mathcal{R}(u) = \tilde{c}(u_1)(u_3 + u_4)^3$. Because we consider a singular solution we shall solve (2.12) near $q_2 = \infty$, $p_2 = \infty$. Hence we replace $q_2 \mapsto q_2^{-1}$, $\frac{\partial}{\partial q_2} \mapsto -q_2^2 \frac{\partial}{\partial q_2}$ and so on in (2.12) and we multiply the equation with $q_2^3 p_2^3$. Then we obtain

$$q_2^3 p_2^3 \frac{\partial w}{\partial q_1} - q_2^3 p_2^3 \delta w - q_2^2 p_2^2 \delta w - q_2^5 \frac{\partial w}{\partial q_2} + p_2^5 \frac{\partial w}{\partial p_2} - q_2^3 p_2^3 w - q_2^3 p_2^3 \mathcal{R}(u) = 0. \tag{2.13}$$

We shall look for $w = w(q_1, q_2 + p_2)$. Set $r = q_2 + p_2$ and define

$$q_2 = \alpha\zeta, \quad p_2 = \eta\zeta, \tag{2.14}$$

where $\alpha + \eta \neq 0$ and ζ is a complex parameter. Then we have $\delta w = (\alpha - \eta)\zeta \frac{\partial w}{\partial r}$ and

$$-q_2^5 \frac{\partial w}{\partial q_2} + p_2^5 \frac{\partial w}{\partial p_2} = (-\alpha^5 + \eta^5)\zeta^5 \frac{\partial w}{\partial r}.$$

Hence, by (2.13) we have

$$\zeta \frac{\partial w}{\partial q_1} + (\zeta^2(\eta - \alpha) + \alpha^{-1} - \eta^{-1}) \frac{\partial w}{\partial r} + (-\alpha^2\eta^{-3} + \alpha^{-3}\eta^2) \frac{\partial w}{\partial r} - \zeta w - \zeta \mathcal{R}(u) = 0. \tag{2.15}$$

Suppose that $\alpha^{-1} - \eta^{-1} - \alpha^2\eta^{-3} + \alpha^{-3}\eta^2 \neq 0$. This is equivalent to $(\eta^3 - \alpha^3)(\eta^2 + \alpha^2) \neq 0$. We assume

$$\eta^3 - \alpha^3 \neq 0, \quad \eta^2 + \alpha^2 \neq 0, \quad \eta + \alpha \neq 0. \tag{2.16}$$

We make the change of variables $q_1 \mapsto \lambda q_1$ and $r \mapsto rK$ for some $\lambda \neq 0$ and $K \neq 0$ such that (2.15) is transformed to the following

$$r \frac{\partial w}{\partial q_1} + (r^2 + 1) \frac{\partial w}{\partial r} - \epsilon_0 r w - \epsilon_0 r \mathcal{R}(u) = 0, \tag{2.17}$$

where ϵ_0 is some constant. Similarly, one can verify that u_4 satisfies (2.17) with $-\epsilon_0 r w$ replaced by $\epsilon_0 r w$.

Next we consider the equation for u_1 in (2.11), which is given by the first column of (2.11). Because the first component of $r(u)$ vanishes, the equation for $w := u_1$ is exactly the equation (2.17) with $-\epsilon_0 r w - \epsilon_0 r \mathcal{R}(u)$ replaced by $-\epsilon_0 r$. One can easily verify that it has a solution

$$\tilde{q}_1 \equiv u_1 = q_1 - \frac{1 - \epsilon_0}{2} \log(1 + r^2). \tag{2.18}$$

On the other hand, the equation for u_2 has the same form as (2.17), where the nonlinear term $\mathcal{R}(u)$ does not contain u_2. Hence the equation is linear. One can integrate it by the method which we show in the following if we detemine u_3 and u_4.

Insert u_1 into $\mathcal{R}(u) = \tilde{c}(u_1)(u_3 + u_4)^3$ and consider the system of equations for u_3 and u_4 in (2.17). We note that one can delete $\epsilon_0 rw$ by introducing the unknown quantity $w_3 = we^{q_1\epsilon_0}$. Then $\tilde{c}$ in $\mathcal{R}$ is multiplied by some power of $e^{q_1\epsilon_0}$, which we denote by $\mathcal{R}_3$. Defining $w_4 = we^{-q_1\epsilon_0}$ and $\mathcal{R}_4$ in the same way as $\mathcal{R}_3$ we obtain

$$r\frac{\partial w_3}{\partial q_1} + (1+r^2)\frac{\partial w_3}{\partial r} = \epsilon_0 r\mathcal{R}_3(w_3, w_4). \tag{2.19}$$

$$r\frac{\partial w_4}{\partial q_1} + (1+r^2)\frac{\partial w_4}{\partial r} = \epsilon_0 r\mathcal{R}_4(w_3, w_4). \tag{2.20}$$

We shall solve the reduced homology equation (2.19) and (2.20) by iteration. Consider

$$r\frac{\partial W}{\partial q_1} + (1+r^2)\frac{\partial W}{\partial r} = f(r, q_1) \tag{2.21}$$

where $W = (w_3, w_4)$ and $f(r, q_1)$ is a given vector function.

The equation (2.21) can be solved by the method of characteristics. Consider

$$\frac{dq_1}{r} = \frac{dr}{1+r^2} = d\sigma. \tag{2.22}$$

By integration we have $\tan^{-1} r = \sigma + \tan^{-1} r^0$ and $q_1 = q_1^0 - \log\cos\sigma$. For given r and q_1 we set $r^0 = 0$ and define σ_0 and q_1^0 by

$$\tan^{-1} r = \sigma_0, \quad q_1 = q_1^0 - \log\cos\sigma_0. \tag{2.23}$$

Then the solution of (2.21) is given by

$$W(r, q_1) = Pf := \int_0^{\sigma_0} f\left(\tan\sigma, q_1^0 - \log\cos\sigma\right) d\sigma. \tag{2.24}$$

We shall show that in the integrand of (2.24) the function $\tilde{c}(\tilde{q}_1)$ is bounded, where $\tilde{q}_1 = q_1 - \frac{1-\epsilon_0}{2}\log(1+r^2)$. Because $\tilde{c}(\tilde{q}_1)$ is a constant times of $e^{\beta\tilde{q}_1}$ we consider the latter function. By using $q_1^0 - \log\cos\sigma = q_1 + \log\cos\sigma_0 - \log\cos\sigma$ and $r = \tan\sigma$ we have

$$e^{\beta\tilde{q}_1} = e^{\beta q_1}(1+r^2)^{-\beta/2+\epsilon_0/2} = e^{\beta q_1}(\cos\sigma_0)^{\beta}(\cos\sigma)^{-\epsilon_0}. \tag{2.25}$$

Hence the integrand is bounded in σ.

Define

$$\Omega_0 := \{(q_1, r) \,|\, |q_1| < \eta_0, \ |r| < \eta_0\}. \tag{2.26}$$

We shall solve (2.19) and (2.20) in the set of functions analytic in Ω_0 and continuous up to the boundary. Set $K := \overline{\Omega_0}$ and define

$$\|W\| := \|w_3\| + \|w_4\|, \quad \|w_j\| := \sup_K |w_j(r, q_1)|. \tag{2.27}$$

Define $Q \equiv Q(W) := \epsilon_0 r(\mathcal{R}_3, \mathcal{R}_4)$. We define the approximate sequence $\{W_n\}_n$ by $W_{-1} = 0$, $W_0 = \left(q_1 - \frac{1}{2}\log(1+r^2), q_1 - \frac{1}{2}\log(1+r^2)\right)$ and

$$W_n = W_{n-1} + PQ(W_{n-1}) - PQ(W_{n-2}), \quad n = 1, 2, \ldots \tag{2.28}$$

One can easily see that the sequence is well defined on Ω_0. If the limit, $W := \lim W_n = W_0 + \sum_{n=1}^{\infty}(W_n - W_{n-1})$ exists , then we have

$$W - W_0 = P\left(\sum_{n=1}^{\infty}\left(Q(W_{n-1}) - Q(W_{n-2})\right)\right) = P\left(\lim Q(W_n)\right) = PQ(W). \tag{2.29}$$

By the definition of W_0, W satisfies (2.21).

We shall show an apriori estimate. Given an $\epsilon > 0$. Take $C_1 > 0$ such that $|e^{\beta\tilde{q}_1}| \le C_1$ on Ω_0. If η_0 is sufficiently small, then we have $\|W_0\| \le \epsilon/2$. Consider $W_1 - W_0 = PQ(W_0)$. We have $|r| < \epsilon$ if η_0 is sufficiently small. Because $\tilde{c}(\tilde{q}_1)$ in the integrand is bounded, we get, from the formula of P that $\|W_1 - W_0\| < \tilde{K}\epsilon^2$ for some $\tilde{K} > 0$ depending only on the equation. Take ϵ such that $\tilde{K}\epsilon \le 1/4$. Then we have $\|W_1\| \le (1/2 + 1/4)\epsilon < \epsilon$.

Next, we shall estimate $\|W_2 - W_0\|$. Writing $W_2 - W_0 = W_2 - W_1 + W_1 - W_0$ we consider $\|W_2 - W_1\|$, where

$$W_2 - W_1 = P\left(Q(W_1) - Q(W_0)\right). \tag{2.30}$$

By the definition of Q we see that $Q(W_1) - Q(W_0)$ is estimated by the constant times of $\|W_1 - W_0\|(\|W_1\|^2 + \|W_1\|\|W_0\| + \|W_0\|^2)$. By the estimate of W_1 the last term is bounded by $3\epsilon^2\|W_1 - W_0\| \le 3\epsilon^3/4$. Hence , by (2.30) we have $\|W_2 - W_1\| \le \epsilon^3 K$ for some K depending only on the equation. Therefore we have

$$\|W_2 - W_0\| \le \epsilon/4 + \epsilon^3 K \le \epsilon(1/4 + \epsilon^2 K). \tag{2.31}$$

If $\epsilon^2 K \le 1/8$, then we have $\|W_2 - W_0\| < \epsilon/2$. Especially, we have $\|W_2\| \le \epsilon$.

We proceed in the same way for $W_3 - W_0 = W_3 - W_2 + W_2 - W_1 + W_1 - W_0$. We can show that $\|W_3 - W_0\| \le 2^{-1}\epsilon(1/2 + 1/4 + 1/8) < \epsilon/2$. In general we have $\|W_n - W_0\| \le \epsilon/2$ for all n. Hence we obtain the apriori estimate

$$\|W_n\| \le \epsilon, \qquad n = 0, 1, \ldots \tag{2.32}$$

By the apriori estimate and (2.28) we have

$$\|W_n - W_{n-1}\| \le 3\epsilon^2 \|W_{n-1} - W_{n-2}\| \le \frac{1}{2}\|W_{n-1} - W_{n-2}\| \tag{2.33}$$

if $3\epsilon^2 \le 1/2$. This proves the convergence. Therefore we have proved

Theorem 2.3. *Suppose that α and η satisfy (2.16). Then there exists an $\eta_0 > 0$ such that if p_1 is in some neighborhood of the origin and (q_1, q_2, p_2) is given by (2.14) with q_2 and p_2 replaced by q_2^{-1} and p_2^{-1}, respectively, and $\zeta = (\alpha+\eta)^{-1} r$, $(q_1, r) \in \Omega_0$, then the vector field $(\Lambda(x) + r(x))\frac{\partial}{\partial x}$ is transformed to $(\Lambda(y) + s(y))\frac{\partial}{\partial y}$ by an analytic change of coordinates.*

§ 3. Movable Singularity

In this section we shall study the movable singularity of solutions of (1.1) with $\ell = 3$. In view of Theorem 2.3 we consider the reduced Hamiltonian $p_1 + q_2 p_2 + c q_2^2 p_2^2 + \tilde{c}(q_2^4 + p_2^4)$, where $c \neq 0$ and $\tilde{c} \neq 0$ are constants. By setting $q_2 = q$ and $p_2 = p$ we consider the Hamiltonian

$$\tilde{H} := qp + \frac{\varepsilon}{2} q^2 p^2 - \frac{\eta}{8}(q^2 - p^2)^2, \tag{3.1}$$

where ϵ and $\eta \neq 0$ are constants. Because $\tilde{c}$ can be chosen arbitrarily by changing the transformation in the proof of Theorem 2.3, we may assume $\epsilon \neq 0$, $\epsilon + \eta \neq 0$ without loss of generality. Suppose that (q, p) is the solution of the Hamiltonian system for $\tilde{H}$. Then there exists a constant C_2 such that $\tilde{H} = C_2$. Define

$$\zeta = \frac{q+p}{2}, \ \xi = \frac{q-p}{2i}. \tag{3.2}$$

Then we have

$$\begin{aligned} C_2 = \tilde{H} &= (\zeta^2 + \xi^2) + \frac{\epsilon}{2}(\zeta^2 + \xi^2)^2 + 2\eta \zeta^2 \xi^2 \\ &= \frac{\epsilon+\eta}{2}(\zeta^2 + \xi^2 + \frac{1}{\epsilon+\eta})^2 - \frac{1}{2(\epsilon+\eta)} - \frac{\eta}{2}(\zeta^2 - \xi^2)^2. \end{aligned} \tag{3.3}$$

Hence we have

$$1 = \frac{(\epsilon+\eta)^2}{A}(\zeta^2 + \xi^2 + \frac{1}{\epsilon+\eta})^2 - \frac{\eta(\epsilon+\eta)}{A}(\zeta^2 - \xi^2)^2 \tag{3.4}$$

where $A = 1 + 2C_2(\epsilon+\eta)$. We determine $\theta = \theta(z)$ such that

$$\sin^2\theta = \frac{(\epsilon+\eta)^2}{A}(\zeta^2 + \xi^2 + \frac{1}{\epsilon+\eta})^2, \quad \cos^2\theta = -\frac{\eta(\epsilon+\eta)}{A}(\zeta^2 - \xi^2)^2. \tag{3.5}$$

Then, by (3.5) and simple computations we have

$$\zeta \equiv \zeta(z) = \sqrt{\frac{\sqrt{\frac{A(\epsilon+\eta)}{-\eta}}\cos\theta + \sqrt{A}\sin\theta - 1}{2(\epsilon+\eta)}}, \tag{3.6}$$

$$\xi \equiv \xi(z) = \sqrt{\frac{-\sqrt{\frac{A(\epsilon+\eta)}{-\eta}}\cos\theta + \sqrt{A}\sin\theta - 1}{2(\epsilon+\eta)}}. \tag{3.7}$$

Set $X(z) = \sin\theta(z) + \eta\epsilon^{-1}/\sqrt{A}$ and define

$$\mathcal{A} = \sqrt{\mathcal{E} + \frac{i}{2}\sqrt{\mathcal{F}}}, \quad \mathcal{B} = \sqrt{\mathcal{E} - \frac{i}{2}\sqrt{\mathcal{F}}}, \tag{3.8}$$

where

$$\mathcal{E} = \frac{1}{2(\epsilon+\eta)}\left(\sqrt{A}X(z) - \eta\epsilon^{-1} - 1\right), \tag{3.9}$$

$$\mathcal{F} = \frac{A}{\eta(\epsilon+\eta)}\left(1 - (X(z) - \eta\epsilon^{-1}/\sqrt{A})^2\right). \tag{3.10}$$

Then we see that $\zeta = \mathcal{A}$ and $\xi = \mathcal{B}$. Therefore, by (3.2) we obtain

$$q(z) = \mathcal{A} + i\mathcal{B}, \quad p(z) = \mathcal{A} - i\mathcal{B}. \tag{3.11}$$

We will show the following lemma.

Lemma 3.1. *$X(z)$ is an elliptic function.*

Proof. By (3.1) the Hamitonian equation with Hamiltonian $\tilde{H}$ is given by

$$\dot{q} = q + \epsilon q^2 p + \frac{\eta}{2}p(q^2 - p^2), \quad \dot{p} = -p - \epsilon q p^2 + \frac{\eta}{2}q(q^2 - p^2).$$

In terms of (3.2), these equations are written as

$$\dot{\zeta} = i\xi + i\epsilon\xi(\zeta^2 + \xi^2) + 2i\eta\zeta^2\xi, \quad \dot{\xi} = -i\zeta - i\epsilon\zeta(\zeta^2 + \xi^2) - 2i\eta\zeta\xi^2. \tag{3.12}$$

In view of (3.2) we have

$$\frac{d}{dz}(\zeta^2 + \xi^2) = 4\eta i\zeta\xi(\zeta^2 - \xi^2). \tag{3.13}$$

By (3.6) and (3.7) we have

$$\zeta^2 - \xi^2 = \frac{\sqrt{\frac{A(\epsilon+\eta)}{-\eta}}\cos\theta}{\epsilon+\eta}. \tag{3.14}$$

$$
\begin{aligned}
(3.15) \qquad \zeta^2\xi^2 &= \frac{(\sqrt{A}\sin\theta - 1)^2 + \eta^{-1}A(\epsilon+\eta)\cos^2\theta}{4(\epsilon+\eta)^2} \\
&= \frac{1}{4(\epsilon+\eta)^2}\Big(-\eta^{-1}A\epsilon\sin^2\theta - 2\sqrt{A}\sin\theta + 1 + \eta^{-1}A(\epsilon+\eta)\Big) \\
&= \frac{1}{4(\epsilon+\eta)^2}\Big(-\eta^{-1}A\epsilon(\sin\theta + \eta\epsilon^{-1}A^{-1/2})^2 + \eta\epsilon^{-1} + 1 + \eta^{-1}A(\epsilon+\eta)\Big) \\
&= \frac{1}{4(\epsilon+\eta)^2}\left(-\eta^{-1}A\epsilon X^2 + \eta\epsilon^{-1} + 1 + \eta^{-1}A(\epsilon+\eta)\right).
\end{aligned}
$$

$$
(3.16) \qquad \frac{d}{dz}(\zeta^2+\xi^2) = \frac{d\theta}{dz}\frac{d}{d\theta}\left(\frac{\sqrt{A}\sin\theta - 1}{\epsilon+\eta}\right) = \frac{\sqrt{A}\cos\theta}{\epsilon+\eta}\frac{d\theta}{dz} = \frac{\sqrt{A}}{\epsilon+\eta}\frac{dX}{dz}.
$$

By (3.13), (3.14), (3.15) and (3.16) we have

$$
\begin{aligned}
(3.17) \qquad \frac{dX}{dz} &= \sqrt{\frac{\eta}{\epsilon+\eta}}\sqrt{1-(X-\eta\epsilon^{-1}A^{-1/2})^2} \\
&\quad \times \sqrt{-\eta^{-1}A\epsilon X^2 + \eta\epsilon^{-1} + 1 + \eta^{-1}(\epsilon+\eta)A}.
\end{aligned}
$$

Hence, $(\frac{dX}{dz})^2$ is at most the fourth order polynomial of X. By the general theory of the elliptic function we see that $X(z)$ is Jacobi's elliptic function. (See [2] and [5]). □

Proof of Theorem 1.1. First we recall that $qp = \zeta^2 + \xi^2 = C_0X(z) + C_1$ for some constants $C_0 \neq 0$ and C_1 by what we have proved in the above, where $X(z)$ is the elliptic function. In view of representation formula of q and p, q and p are singular at the poles of the elliptic function $X(z)$. It may occur that another branch point appears for p or q. One can easily verify that the singular point is the movable singularity because it does not appear in the coefficients of the equations of q and p.

In order to show that the generalized Emden equation has movable singularities, we transform the equation to a normal form as in Theorem 2.3 in some domain near the infinity. Indeed, the assumption (2.16) is the condition on q_2 and p_2, which can be satisfied by changing parameters in view of the expression of singular solution in §3.

Recall that the elliptic function has a pole on each paralleogram. By suitable choice of parameters of the transformed equation, the image of the fundamental paralleogram lies in the domain on which the normalizing transformation is defined. Hence we have a singular solution of the generalized Emden equation which is parametrized by the elliptic function. Finally, by (2.18) one can change the independent variable of the transformed equation to that of the generalized Emden equation, and one obtains a singular solution of the generalized Emden equation. There exist infinitely many movable singularities in view of the periodicity of the elliptic function.

□

References

[1] Filipuk, G. and Halburd, R. G., Movable algebraic singularities of second-order ordinary differential equations, Journal of Mathematical Physics **50**, 023509 (2009); doi: http://dx.doi.org/10.1063/1.3068414

[2] Hurwitz, A., Vorlesungen ueber Allgemeine Funktionen-theorie und Elliptische Funktionen, *Springer-Verlag, Berlin Heidelberg.* 1964.

[3] Ito, H., Integrable symplectic maps and their Birkhoff normal forms, Tohoku Math. J. 49 (1997), 73-114.

[4] Kycia, R. A. and Filipuk, G., On the generalized Emden-Fowler and isothermal spheres equations, Applied Mathematics and Computation, **265**, (2015) 1003-1010.

[5] Whittaker, E.T. and Watson, G.N., A course of modern analysis, Cambridge Univ. Press, London and New York, 1952.