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A certain property of a unified family of P_{J}-hierarchies (J=I, II, IV, 34) with a large parameter

By

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Abstract

We study a unified family of P_{J}-hierarchies (J=I, II, IV, 34) with a large parameter. The explicit forms of the deformation equation and the Schrödinger equation associated with the unified family of P_{J}-hierarchies are derived from the underlying Lax pair.

§ 1. Introduction

In the series of papers ([1], [6]–[14], [16]–[19]), the exact WKB analysis for higher order Painlevé equations has been progressed and important results have been established. T. Kawai, T. Koike, Y. Nishikawa and Y. Takei ([10], [17]) proved that there is a closed connection between the Stokes geometries of P_{J}-hierarchies (J=I, II, IV, 34), the Noumi-Yamada system and those of their underlying Lax pairs. For example, the following important properties are shared.

(i) If $t = \tau^{\mathrm{I}}$ is a turning point of the first kind of a system of non-linear ordinary differential equations, a double turning point merges with a simple turning point in the Stokes geometry of the underlying Lax pair at $t = \tau^{\mathrm{I}}$.

(ii) If $t = \tau^{\mathrm{II}}$ is a turning point of the second kind of a system of non-linear ordinary differential equations, two double turning points in the Stokes geometry of the underlying Lax pair merge at $t = \tau^{\mathrm{II}}$.

(iii) Under generic assumptions, if t lies on a Stokes curve of a system of non-linear ordinary differential equations, two turning points are connected by a Stokes curve in the Stokes geometry of the underlying Lax pair.

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These three properties play an important role in analyzing the Stokes phenomenon on a Stokes curve of non-linear differential equations (see [10]-[12], [16] and [19] for P_{J}-hierarchies). The author has a question: Do three properties always hold for any system of non-linear ordinary differential equations which describes the compatibility condition of a Lax pair? To investigate the question, the author proved that (i), (ii), (iii) also hold for a unified family of P_{J}-hierarchies (J=I, II, IV, 34) in [20] . The unified family of P_{J}-hierarchies is introduced in [21]. The system has arbitrary coefficients. If we specify the coefficients, then the unified family is equivalent to the m-th member of P_{J}-hierarchies (J=I, II, IV, 34). However it is not certain whether the system contains other known Painlevé hierarchies or essentially new equations or not. Motivated by the problem, this paper makes clear the difference between P_{J}-hierarchies (J=I, II, IV, 34) and the unified family. The plan of the paper is as follows: In §2 and §3, we recall the explicit form (2.2) of a unified family of P_{J}-hierarchies (J=I, II, IV, 34) with a large parameter and the underlying Lax pair. In §4, we apply the method given by T. Koike in [13] and [14] to (2.2). We derive the deformation equation and the Schrödinger equation from the Lax pair in §3 and the difference between (2.2) and the P_{J}-hierarchies is clarified. In §5, we give a supplementary explanation.

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§ 2. A unified family of P_{J}-hierarchies (J=I, II, IV, 34) with a large parameter

In [21], a unified family of P_{J}-hierarchies (J=I, II, IV, 34) with a large parameter η is derived from some common structures between the m-th members $(P_{\mathrm{J}})_m$ $(m = 1, 2, \cdots)$ of P_{J}-hierarchies. Let us recall the explicit form of the system. Let θ be an independent variable and the notation $A \equiv B$ means that $A - B$ is zero modulo θ^{m+2}. We denote by $\mathcal{O}(t)[[\theta]]$ the set of formal power series in θ with coefficients in holomorphic functions of variable t. Let U, V and C denote generating functions of unknown functions u_k, v_k $(k = 1, 2, \ldots, m)$ of t and constants c_k as follows.

$$U(\theta) := \sum_{k=1}^{m+1} u_k\theta^k,\ V(\theta) := \sum_{k=1}^{m+1} v_k\theta^k,\ C(\theta) := \sum_{k=1}^{m} c_k\theta^k$$

with arbitrary polynomials u_{m+1}, $v_{m+1} \in \mathcal{O}(t)[u_1, \ldots, u_m, v_1, \ldots, v_m]$ on condition that u_{m+1} and v_{m+1} do not include η. Let us define $H(U, V)$ by the polynomial in U and V with arbitrary complex constants p_i of the form

$$H(U, V) := (p_1U^2 + p_2V^2)\theta + p_3UV + p_4CU + p_5CV + p_6U + p_7V + p_8C + p_9. \tag{2.1}$$

We consider the following system on $\dfrac{\mathcal{O}(t)[[\theta]]}{\mathcal{O}(t)[[\theta]]\theta^{m+2}}$.

$$\eta^{-1}\frac{d}{dt}\begin{pmatrix} U\theta \\ V\theta \end{pmatrix} \equiv \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \times (1-U) + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial H}{\partial U} \\ \frac{\partial H}{\partial V} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{H(U,V)}{1-U} \end{pmatrix}, \tag{2.2}$$

where f_1, f_2 are defined by

$$\begin{aligned} f_1 &:= p_7 + (\alpha u_1 + p_5 c_1)\,\theta + y_1\theta^m + (y_1 u_1 + y_2)\theta^{m+1}, \\ f_2 &:= -\beta - (2\beta u_1 + \alpha v_1 + \varepsilon c_1)\,\theta + z_1\theta^m + (2z_1 u_1 - y_1 v_1 + z_2)\theta^{m+1} \end{aligned} \tag{2.3}$$

with arbitrary holomorphic functions y_i, z_i of t. Here c_1 is the coefficient of the leading term of $C(\theta)$ and α, β, ε are defined by

$$\alpha := p_3 + p_7, \quad \beta := p_6 + p_9 \quad \text{and} \quad \varepsilon := p_4 + p_8, \tag{2.4}$$

respectively.

The system (2.2) is equivalent to the following form of the first order system with $2m$ unknown functions u_j, v_j:

$$\begin{cases} \eta^{-1}\dfrac{du_j}{dt} = -\alpha u_{j+1} - (\alpha u_1 + p_5 c_1)u_j - 2p_2 v_j - p_5 c_{j+1} + y_1\delta_{j,\,m-1} + y_2\delta_{j,\,m}, \\ \hfill j = 1, 2, \ldots, m, \\ \eta^{-1}\dfrac{dv_j}{dt} = \beta u_{j+1} + p_3 v_{j+1} + p_4 c_{j+1} + (2\beta u_1 + \alpha v_1 + 2p_1 + \varepsilon c_1)u_j \\ \qquad + w_{j+1} + z_1\delta_{j,\,m-1} + (z_1 u_1 - y_1 v_1 + z_2)\delta_{j,\,m}, \qquad j = 1, 2, \ldots, m. \end{cases} \tag{2.5}$$

Here $\delta_{j,\,m-1}$, $\delta_{j,\,m}$ stand for Kronecker's delta, $c_{m+1} = 0$ and w_j is recursively defined by

$$\begin{aligned} w_j &= \sum_{k=1}^{j-1} w_k u_{j-k} + p_1\sum_{k=1}^{j-2} u_k u_{j-k-1} + p_2\sum_{k=1}^{j-2} v_k v_{j-k-1} + p_3\sum_{k=1}^{j-1} u_k v_{j-k} \\ &\quad + p_4\sum_{k=1}^{j-1} u_k c_{j-k} + p_5\sum_{k=1}^{j-1} v_k c_{j-k} + \beta u_j + p_7 v_j + p_8 c_j \quad (1 \le j \le m+1). \end{aligned} \tag{2.6}$$

If p_j, y_i, z_i in (2.1) and (2.3) are specified by the following list, then (2.2) (also (2.5)) is equivalent to the m-th member $(P_{\mathrm{J}})_m$ of P_{J}-hierarchy (J $=$ I, II, IV, 34) which are studied by Kudryashov ([15]), Gordoa, Joshi and Pickering ([5]), Clarkson, Joshi and Pickering ([3]) and so on.

$(P_{\rm I})_m$	$\alpha = 0,\ \beta \neq 0,\ k = m+3,$ $p_2 = -1, p_8 = 2, p_9 = 1, z_2 = 2t.$
$(P_{34})_m$	$\alpha = 0,\ \beta \neq 0,\ k = m+2$ $p_2 = -1, p_8 = 2, p_9 = 1, z_1 = 2\gamma t, z_2 = 4\gamma t c_0\ (\gamma \neq 0).$
$(P_{\rm II})_m$	$\alpha \neq 0,\ \beta = 0,\ k = m+3,$ $p_2 = 1, p_3 = p_5 = 2.$
$(P_{\rm IV})_m$	$\alpha \neq 0,\ \beta = 0,\ k = m+2,$ $p_2 = 1, p_3 = p_5 = 2, y_1 = -2\gamma t\ (\gamma \neq 0).$

Remark that other p_i, y_i, z_i which are not listed are zero and the explicit forms of u_{m+1} and v_{m+1} are described in [21].

§ 3. Lax pair of the system

Firstly, the system (2.2) is expressed in the following form

$$\begin{cases} \eta^{-1}\dfrac{d}{dt}(U\theta) \equiv f_1(1-U) - \dfrac{\partial H}{\partial V} - \alpha u_{m+1}\theta^{m+1}, & (3.1)\\ \eta^{-1}\dfrac{d}{dt}(V\theta) \equiv f_2(1-U) + \dfrac{H(U,V)}{1-U} + \dfrac{\partial H}{\partial U} + (2\beta u_{m+1} + \alpha v_{m+1})\theta^{m+1} & (3.2)\end{cases}$$

with

$$U(\theta) := \sum_{k=1}^{m} u_k\theta^k,\ V(\theta) := \sum_{k=1}^{m} v_k\theta^k,\ C(\theta) := \sum_{k=1}^{m} c_k\theta^k. \tag{3.3}$$

To admit terms of negative power in θ, we formally calculate the following underlying Lax pair on $\mathcal{O}(t)[[\theta,\theta^{-1}]] := \left\{ \sum_{k=-\infty}^{\infty} f_k\theta^k \mid f_k \in \mathcal{O}(t) \right\}$.

$$\left(\gamma\theta^k\frac{\partial}{\partial\theta} - \eta A\right)\psi(\theta, t) = 0, \qquad \left(\frac{\partial}{\partial t} - \eta B\right)\psi(\theta, t) = 0 \tag{3.4}$$

with

$$A := \begin{pmatrix} \triangle_1 & (1-U)\theta \\ \triangle_2 & -\triangle_1 \end{pmatrix}, \qquad B := \begin{pmatrix} \Box_1 & 1 \\ \Box_2 & -\Box_1 \end{pmatrix}, \tag{3.5}$$

where $\triangle_j$ and $\Box_j$ $(j = 1, 2)$ are defined by

$$\begin{aligned} \triangle_1 &:= -\frac{1}{2}\frac{\partial H}{\partial V} - \frac{p_3}{2}(1-U) + \frac{1}{2}(y_1\theta^m + y_2\theta^{m+1}) - \frac{\alpha}{2}u_{m+1}\theta^{m+1}, \\ \triangle_2 &:= p_2 \times \Bigg(-\frac{\partial H}{\partial U} - \frac{H(U,V)}{1-U} - (z_1\theta^m + (z_1u_1 - y_1v_1 + z_2)\theta^{m+1}) \\ &\qquad - (2\beta u_{m+1} + \alpha v_{m+1})\theta^{m+1}\Bigg), \\ \Box_1 &:= -\frac{1}{2\theta}(\alpha + (\alpha u_1 + p_5c_1)\theta), \quad \Box_2 := -\frac{p_2}{\theta}(\beta + (2\beta u_1 + \alpha v_1 + \varepsilon c_1)\theta). \end{aligned} \tag{3.6}$$

Then the compatibility condition of (3.4) is given in the form

$$\frac{\partial A}{\partial t} - \gamma\theta^k \frac{\partial B}{\partial \theta} + \eta(AB - BA) = \begin{pmatrix} M_1 & M_2 \\ M_3 & -M_1 \end{pmatrix} = \mathbf{0}. \tag{3.7}$$

Apparently the matrix B contains $\frac{1}{\theta}$, but the compatibility condition of $M_i = 0$ ($i =$ 1, 2, 3) does not contain terms of negative power in θ. Therefore we can consider the equations $M_i = 0$ on $\dfrac{\mathcal{O}(t)[[\theta]]}{\mathcal{O}(t)[[\theta]]\theta^{m+2}}$, that is $M_i \equiv 0$. The system (2.2) has arbitrary coefficients p_i, arbitrary holomorphic functions y_i, z_i, and arbitrary polynomials u_{m+1}, v_{m+1}. However, (2.2) dose not necessarily have the underlying Lax pair. As is shown in [20], if we choose u_{m+1}, v_{m+1}, p_i, y_i and z_i satisfying the conditions which will be given in Theorem 3.1, then the system (3.1), (3.2) has the underlying Lax pair.

Theorem 3.1. ([20]) *Assume that $p_1 = 0$ and k is $m+3$ or $m+2$. Let γ and p_2 be arbitrary nonzero constants. If we choose u_{m+1} and v_{m+1} so that they satisfy the conditions blow, then the system* (2.2) (*also* (2.5)) *is equivalent to the compatibility condition* ($M_i \equiv 0$ ($i = 1, 2, 3$)) *of* (3.4).

(3.8)
$$\begin{cases} \gamma\alpha\theta^{k-2} = y_1'\theta^m + y_2'\theta^{m+1} - \alpha\dfrac{\partial u_{m+1}}{\partial t}\theta^{m+1}, \\ \gamma\beta\theta^{k-2} = -\left(z_1'\theta^m + (z_1'u_1 - y_1'v_1 + z_2')\theta^{m+1} + \left(2\beta\dfrac{\partial u_{m+1}}{\partial t} + \alpha\dfrac{\partial v_{m+1}}{\partial t}\right)\theta^{m+1}\right). \end{cases}$$

Here $'$ denotes the derivative with respect to t.

§ 4. The deformation equation and the Schrödinger equation associated with (2.2)

Let us investigate the difference between (2.2) and $(P_{\mathrm{J}})_m$ (J=I, II, IV, 34). In what follows, equations are formally calculated on $\mathcal{O}(t)[[\theta, \theta^{-1}]]$ and we do not use relations which hold on $\frac{\mathcal{O}(t)[[\theta]]}{\mathcal{O}(t)[[\theta]]\theta^{m+2}}$ unless otherwise mentioned. Following T. Koike's idea (A.34) in [13], we take the change of unknown functions ψ and $\bar{\varphi}$ in (3.4):

$$\psi = \exp\left(-\frac{\eta}{2\gamma}\int^{\theta} h(t, \theta)d\theta\right)\bar{\varphi}, \tag{4.1}$$

where

$$h(t, \theta) := \frac{\alpha + p_5 C - y_1\theta^m - y_2\theta^{m+1} + \alpha u_{m+1}\theta^{m+1}}{\theta^k}. \tag{4.2}$$

Then (3.4) is transformed into

$$\left(\gamma\theta^k\frac{\partial}{\partial\theta} - \eta\tilde{A}\right)\bar{\varphi}(t, \theta) = 0, \qquad \left(\frac{\partial}{\partial t} - \eta\tilde{B}\right)\bar{\varphi}(t, \theta) = 0. \tag{4.3}$$

Here $\tilde{A}$ and $\tilde{B}$ are defined by

$$\tilde{A} := \begin{pmatrix} \triangle_1 + \frac{1}{2}\theta^k h(t,\,\theta) & (1-U)\theta \\ \triangle_2 & -\triangle_1 + \frac{1}{2}\theta^k h(t,\,\theta) \end{pmatrix}, \quad \tilde{B} := \begin{pmatrix} \square_1 + \frac{\alpha}{2\theta} & 1 \\ \square_2 & -\square_1 + \frac{\alpha}{2\theta} \end{pmatrix}, \tag{4.4}$$

where $\triangle_j$, $\square_j$ (j $= 1, 2$) are defined by (3.6). We can verify that the compatibility condition of (4.3) is completely the same as (3.7) under the conditions (3.8) in Theorem 3.1. From now on, we consider the Lax pair of (4.3) for (2.2).

Let us calculate the equations that the first component φ_1 of a solution $\bar{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$ for (4.3) satisfies. By the same argument in [13], Proposition A.2, the following two differential equations are derived from (4.3), if $(1-U)\theta \neq 0$ holds:

$$\begin{aligned} &\left(\gamma^2\theta^{2k}(1-U)\frac{\partial^2}{\partial\theta^2} + \eta\tilde{p}(t,\,\theta;\,\eta)\frac{\partial}{\partial\theta} + \eta^2\tilde{q}(t,\,\theta;\,\eta)\right)\varphi_1 = 0, \\ &(1-U)\frac{\partial\varphi_1}{\partial t} = \tilde{\mathcal{A}}\frac{\partial\varphi_1}{\partial\theta} + \eta\tilde{\mathcal{B}}\varphi_1 \end{aligned} \tag{4.5}$$

with

$$\tilde{\mathcal{A}} = \gamma\theta^{k-1}, \quad \tilde{\mathcal{B}} = -\frac{1}{2}(\alpha u_1 + p_5 c_1)(1-U) + p_2 V.$$

Here $\tilde{p}(t,\,\theta;\eta)$ and $\tilde{q}(t,\,\theta;\eta)$ are defined by

$$\begin{aligned} \tilde{p}(t,\,\theta;\eta) &= \eta^{-1}\gamma^2\theta^{2k-1}\left((k-1)(1-U) + \frac{\partial U}{\partial\theta}\theta\right) - \gamma\theta^{2k}h(t,\,\theta)(1-U), \\ \tilde{q}(t,\,\theta;\eta) &= \left(\frac{1}{4}h(t,\,\theta)^2\theta^{2k} - \triangle_1^2 - \triangle_2(1-U)\theta\right)(1-U) \\ &\quad + \eta^{-1}\gamma\theta^{k+1}p_2\left(\frac{\partial V}{\partial\theta}(1-U) + V\frac{\partial U}{\partial\theta}\right). \end{aligned}$$

Therefore, in $\mathcal{O}(t)[[\theta,\,\theta^{-1}]]$, (4.5) is rewritten in the following forms.

$$\begin{aligned} &\left(\frac{\partial^2}{\partial\theta^2} + p(t,\,\theta;\,\eta)\frac{\partial}{\partial\theta} + q(t,\,\theta;\,\eta)\right)\varphi_1 = 0, \\ &\frac{\partial\varphi_1}{\partial t} = \mathcal{A}\frac{\partial\varphi_1}{\partial\theta} + \mathcal{B}\varphi_1 \end{aligned} \tag{4.6}$$

with

$$\mathcal{A} = \frac{\gamma\theta^{k-1}}{1-U}, \quad \mathcal{B} = \eta\left(-\frac{1}{2}(\alpha u_1 + p_5 c_1) + \frac{p_2 V}{1-U}\right).$$

Here $p(t,\,\theta;\eta)$ and $q(t,\,\theta;\eta)$ are defined by

$$\begin{aligned} p(t,\,\theta;\eta) &= -\frac{\eta}{\gamma}h(t,\,\theta) + \frac{1}{\mathcal{A}}\left(\frac{\partial\mathcal{A}}{\partial\theta}\right), \\ q(t,\,\theta;\eta) &= \frac{\eta^2}{\gamma^2\theta^{2k}}\left(\frac{1}{4}h(t,\,\theta)^2\theta^{2k} - \triangle_1^2 - \triangle_2(1-U)\theta\right) + \frac{\eta}{\gamma\theta^{k-1}}p_2\left(\frac{\partial V}{\partial\theta} + V\frac{\frac{\partial U}{\partial\theta}}{1-U}\right). \end{aligned}$$

The compatibility condition of (4.6) is $\bar{\Theta}_1 = \bar{\Theta}_2 = 0$, where $\bar{\Theta}_1$, $\bar{\Theta}_2$ are defined by

$$
\begin{aligned}
\bar{\Theta}_1 &= \frac{1}{(1-U)^2\theta^2}\left(\left(\frac{\partial U}{\partial \theta}\theta - (1-U)\right)\Theta_1 + (1-U)\theta\frac{\partial \Theta_1}{\partial \theta}\right),\\
\bar{\Theta}_2 &= \frac{\eta}{\gamma\theta^{k+1}}\frac{p_2 V}{(1-U)^2}\left(\frac{\partial U}{\partial \theta} + (1-U)\right)\left(-\Theta_1 + \theta\frac{\partial \Theta_1}{\partial \theta}\right)\\
&\quad + \frac{\eta}{\gamma\theta^{k+1}}\frac{1}{1-U}\left(\left(\frac{\partial U}{\partial \theta}\theta - (1-U)\right)\Theta_2 + (1-U)\theta\frac{\partial \Theta_2}{\partial \theta}\right) + \frac{\eta^2}{\gamma^2\theta^{2k}}\Theta_3
\end{aligned}
\tag{4.7}
$$

with

$$
\begin{aligned}
\Theta_1 &:= \frac{\partial U}{\partial t}\theta + \eta\left(2p_2V\theta + h\theta^k - (\alpha + (\alpha u_1 + p_5 c_1)\theta)(1-U)\right),\\
\Theta_2 &:= p_2\frac{\partial V}{\partial t}\theta + \eta\left(\triangle_2 + p_2(\beta + (2\beta u_1 + \alpha v_1 + \varepsilon c_1)\theta)(1-U)\right),\\
\Theta_3 &:= p_2(y_1\theta^m + y_2\theta^{m+1} - \alpha u_{m+1}\theta^{m+1})\frac{\partial V}{\partial t}\theta\\
&\quad - p_2\left(z_1\theta^m + (z_1u_1 - y_1v_1 + z_2 + 2\beta u_{m+1} + \alpha v_{m+1})\theta^{m+1}\right)\frac{\partial U}{\partial t}\theta\\
&\quad + p_2\left(z_1\frac{du_1}{dt} - y_1\frac{dv_1}{dt}\right)(1-U)\theta^{m+2}
\end{aligned}
\tag{4.8}
$$

Note that, if $(U,\, V)$ is a solution of (2.2), then we see $\Theta_i \equiv 0$ $(i = 1,\, 2,\, 3)$ on $\frac{\mathcal{O}(t)[[\theta]]}{\mathcal{O}(t)[[\theta]]\theta^{m+2}}$.

Let us compute the Schrödinger equation associated with (2.2). We change the unknown function φ_1 by φ so that the second term in the first equation of (4.6) vanishes.

$$
\varphi_1 = e^{-\frac{1}{2}\int^{\theta} p(t,\,\theta;\,\eta)d\theta}\varphi
\tag{4.9}
$$

Then we have the Schrödinger equation associated with (2.2) by the first equation of (4.6):

$$
\begin{aligned}
\frac{\partial^2 \varphi}{\partial \theta^2} &= Q(t,\,\theta;\,\eta)\varphi,\\
Q(t,\,\theta;\,\eta) &= -q(t,\,\theta;\,\eta) + \frac{1}{4}p(t,\,\theta;\,\eta)^2 + \frac{1}{2}\frac{\partial p}{\partial \theta}(t,\,\theta;\,\eta).
\end{aligned}
\tag{4.10}
$$

The explicit form of the potential Q is given by

$$
\begin{aligned}
Q(t,\,\theta;\,\eta) = &-\frac{\eta^2}{\gamma^2\theta^{2k}}\det A\\
&- \frac{\eta}{\gamma}\left\{\frac{p_2}{\theta^{k-1}}\left(\frac{\partial V}{\partial \theta} + V\frac{\frac{\partial U}{\partial \theta}}{1-U}\right) + \frac{1}{2}h(t,\,\theta)\frac{1}{\mathcal{A}}\frac{\partial \mathcal{A}}{\partial \theta} + \frac{1}{2}\frac{\partial h}{\partial \theta}(t,\,\theta)\right\}\\
&+ \frac{1}{4}\left\{\left(\frac{1}{\mathcal{A}}\frac{\partial \mathcal{A}}{\partial \theta}\right)^2 + 2\frac{\partial}{\partial \theta}\left(\frac{1}{\mathcal{A}}\frac{\partial \mathcal{A}}{\partial \theta}\right)\right\}.
\end{aligned}
\tag{4.11}
$$

Here A is defined by (3.5). By the condition (3.8), we have

$$\frac{\partial h}{\partial t}(t,\,\theta) = -\frac{\gamma\alpha}{\theta^2} \quad \text{and} \quad \frac{\partial}{\partial t}\left(\int^{\theta} p(t,\,\theta;\,\eta)d\theta\right) = -\frac{\eta\alpha}{\theta} - \frac{\frac{\partial}{\partial t}(1-U)}{1-U}.$$

Therefore, by (4.9), the second equation of (4.6) is transformed into

$$\frac{\partial\varphi}{\partial t} = \mathcal{A}\frac{\partial\varphi}{\partial\theta} - \frac{1}{2}\left(\frac{\eta\alpha}{\theta} + \frac{\frac{\partial}{\partial t}(1-U)}{1-U} + \mathcal{A}\,p(t,\,\theta;\eta) - 2\mathcal{B}\right)\varphi. \tag{4.12}$$

The equation (3.1) is written in the form

$$\eta^{-1}\frac{d}{dt}(U\theta) = -2p_2V\theta + (\alpha + (\alpha u_1 + p_5c_1)\theta)\,(1-U) - h(t,\,\theta)\theta^k. \tag{4.13}$$

Using (4.13) in (4.12), we have

$$\frac{\partial\varphi}{\partial t} = \mathcal{A}\frac{\partial\varphi}{\partial\theta} - \frac{1}{2}\frac{\partial\mathcal{A}}{\partial\theta}\varphi. \tag{4.14}$$

Summing up, by (4.9), the deformation equation and the Schrödinger equation associated with (2.2) are obtained from (4.6):

$$\begin{cases} \dfrac{\partial^2\varphi}{\partial\theta^2} = Q(t,\,\theta;\,\eta)\varphi, \\ \dfrac{\partial\varphi}{\partial t} = \mathcal{A}\dfrac{\partial\varphi}{\partial\theta} - \dfrac{1}{2}\dfrac{\partial\mathcal{A}}{\partial\theta}\varphi, \qquad \mathcal{A} := \dfrac{\gamma\theta^{k-1}}{1-U}. \end{cases} \tag{4.15}$$

Here Q is defined by (4.11). In (4.15), we emphasize that $\mathcal{A}$ is independent of p_j,y_i,z_i. This means that the deformation equation associated with (2.2) is completely the same as $(P_{\mathrm{J}})_m$ (J=I, II, IV, 34). The difference between (2.2) and $(P_{\mathrm{J}})_m$ is the form of potential Q. This calculation implies the following. If the potential Q is deformed such as (4.11), the same geometric structures (specifically, (i),(ii),(iii) in §1) as $(P_{\mathrm{J}})_m$ hold.

§ 5. Remark

The expressions of $(P_{\mathrm{J}})_m$ in the series of papers [10]–[14] are derived by the transformation of $\theta = \frac{1}{x}$. Set

$$\mathcal{U}(x,\,t) := \sum_{j=1}^{m} u_j(t)x^{m-j},\ \mathcal{V}(x,\,t) := \sum_{j=1}^{m} v_j(t)x^{m-j},\ \mathcal{C}(x) := \sum_{j=1}^{m} c_jx^{m-j}. \tag{5.1}$$

By $\theta = \frac{1}{x}$, we have

$$U(\theta) = \left(\frac{1}{x}\right)^m \mathcal{U}(x,\,t), \quad V(\theta) = \left(\frac{1}{x}\right)^m \mathcal{V}(x,\,t), \quad C(\theta) = \left(\frac{1}{x}\right)^m \mathcal{C}(x).$$

By the relation, (3.1) and (3.2) are rewritten in

$$\begin{aligned}
&\eta^{-1}\frac{d\mathcal{U}}{dt} \equiv (p_7 x + (\alpha u_1 + p_5 c_1))\,(x^m - \mathcal{U}) - \frac{\partial \mathcal{H}}{\partial \mathcal{V}} + (y_1 x + y_2 - \alpha u_{m+1}),\\
(5.2)\quad &\eta^{-1}\frac{d\mathcal{V}}{dt} \equiv -(\beta x + (2\beta u_1 + \alpha v_1 + \varepsilon c_1))\,(x^m - \mathcal{U})\\
&\qquad + \frac{\partial \mathcal{H}}{\partial \mathcal{U}} + \frac{\mathcal{H}(\mathcal{U},\,\mathcal{V})}{x^m - \mathcal{U}} + (z_1 x + (z_1 u_1 - y_1 v_1 + z_2 + 2\beta u_{m+1} + \alpha v_{m+1}))
\end{aligned}$$

with

(5.3)

$$\mathcal{H}(\mathcal{U},\,\mathcal{V}) := p_2\mathcal{V}^2 + (p_3\mathcal{U}\mathcal{V} + p_4\mathcal{C}\mathcal{U} + p_5\mathcal{C}\mathcal{V})\,x + (p_6\mathcal{U} + p_7\mathcal{V} + p_8\mathcal{C})\,x^{m+1} + p_9 x^{2m+1}.$$

In the original equation (2.2), $A \equiv B$ means that $A - B$ is zero modulo θ^{m+2}. In the procedure of derivation of (5.2), we multiply both sides of the original equation by x^{m+1}. Hence we consider our problem with mode $\theta^{(m+2)-(m+1)}$. For that reason, the notation $\equiv$ in (5.2) means that we exclude the terms of x^j $(j \le -1)$, that is, we consider (5.2) on $\mathcal{O}(t)[[x]]$. If we compare the coefficients of θ^j $(j = 0,\,1,\,\ldots,\,m-1)$ in both sides of (5.2), we obtain (2.5).

By $\theta = \frac{1}{x}$, Theorem 3.1 is rewritten in the following.

Theorem 5.1. *Assume that $p_1 = 0$ and k is $m+3$ or $m+2$. Let γ and p_2 be arbitrary nonzero constants. Let us choose u_{m+1} and v_{m+1} of (5.2) by the following conditions.*

$$(5.4)\qquad \begin{cases} \gamma\alpha x^{m+3-k} = y_1' x + y_2' - \alpha\dfrac{\partial u_{m+1}}{\partial t},\\[2mm] \gamma\beta x^{m+3-k} = -\left(z_1' x + z_1' u_1 - y_1' v_1 + z_2' + 2\beta\dfrac{\partial u_{m+1}}{\partial t} + \alpha\dfrac{\partial v_{m+1}}{\partial t}\right). \end{cases}$$

Here $'$ denotes the derivative with respect to t. Then we have a Lax pair for (5.2) of the following form.

$$(5.5)\qquad \begin{aligned}
\eta^{-1}\frac{\partial\varphi}{\partial x} &= \frac{1}{\gamma}x^{k-(m+3)}\begin{pmatrix} p_2\mathcal{V} & -(x^m - \mathcal{U})\\ \hat{\triangle}_2 & -(p_2\mathcal{V} + 2h(t,\,x))\end{pmatrix}\varphi\\
\eta^{-1}\frac{\partial\varphi}{\partial t} &= \begin{pmatrix} -\frac{1}{2}(\alpha u_1 + p_5 c_1) & 1\\ -p_2(\beta x + (2\beta u_1 + \alpha v_1 + \varepsilon c_1)) & \frac{1}{2}(2\alpha x + (\alpha u_1 + p_5 c_1))\end{pmatrix}\varphi.
\end{aligned}$$

Here $\mathcal{H}(\mathcal{U},\,\mathcal{V})$ is defined by (5.3), $\hat{\triangle}_2$ and $h(t,\,x)$ are defined by

(5.6)

$$\begin{aligned}
\hat{\triangle}_2 &:= p_2\left(\frac{\partial\mathcal{H}}{\partial\mathcal{U}} + \frac{\mathcal{H}(\mathcal{U},\,\mathcal{V})}{x^m - \mathcal{U}} + z_1 x + (z_1 u_1 - y_1 v_1 + z_2 + 2\beta u_{m+1} + \alpha v_{m+1})\right),\\
h(t,\,x) &:= \frac{1}{2}\left(p_5\mathcal{C}x + \alpha x^{m+1} - y_1 x - y_2 + \alpha u_{m+1}\right),
\end{aligned}$$

respectively.

By the same arguments in §4, we have

$$\begin{aligned}\left(\frac{\partial^2}{\partial x^2}+p(t,\,x;\,\eta)\frac{\partial}{\partial x}+q(t,\,x;\,\eta)\right)\varphi_1=0\\ \frac{\partial\varphi_1}{\partial t}=\hat{\mathcal{A}}\frac{\partial\varphi_1}{\partial x}-\hat{\mathcal{B}}\varphi_1.\end{aligned}\tag{5.7}$$

Here $p(t,\,x;\eta)$, $q(t,\,x;\eta)$, $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ are defined by

$$\begin{aligned}p(t,\,x;\eta)&=\frac{2\eta}{\gamma}x^{k-(m+3)}h(t,\,x)-\frac{k-(m+3)}{x}-\frac{\frac{\partial}{\partial x}(x^m-\mathcal{U})}{x^m-\mathcal{U}},\\ q(t,\,x;\eta)&=\frac{\eta^2}{\gamma^2}x^{2(k-(m+3))}p_2\Big(\beta x^{2m+1}+\varepsilon\mathcal{C}x^{m+1}+\mathcal{V}(y_1x+y_2-\alpha u_{m+1})\\ &\qquad+(x^m-\mathcal{U})(z_1x+(z_1u_1-y_1v_1+z_2+2\beta u_{m+1}+\alpha v_{m+1}))\Big)\\ &\quad+\frac{\eta}{\gamma}x^{k-(m+3)}p_2\left(\frac{\frac{\partial}{\partial x}(x^m-\mathcal{U})}{x^m-\mathcal{U}}\mathcal{V}-\frac{\partial\mathcal{V}}{\partial x}\right),\\ \hat{\mathcal{A}}&=-\frac{\gamma x^{m+3-k}}{x^m-\mathcal{U}}\quad\text{and}\quad\hat{\mathcal{B}}=\frac{1}{2}\eta\left((\alpha u_1+p_5c_1)-\frac{2p_2\mathcal{V}}{x^m-\mathcal{U}}\right),\end{aligned}\tag{5.8}$$

respectively. By the following transformation

$$\varphi_1=e^{-\frac{1}{2}\int^x p(t,\,x;\,\eta)dx}\psi,\tag{5.9}$$

the system (5.7) is transformed into

$$\begin{cases}\dfrac{\partial^2\psi}{\partial x^2}=Q(t,\,x;\,\eta)\psi,\\ Q(t,\,x;\,\eta)=-q(t,\,x;\,\eta)+\dfrac{1}{4}p(t,\,x;\,\eta)^2+\dfrac{1}{2}\dfrac{\partial p}{\partial x}(t,\,x;\,\eta),\\ \dfrac{\partial\psi}{\partial t}=\mathcal{D}\dfrac{\partial\psi}{\partial x}-\dfrac{1}{2}\dfrac{\partial\mathcal{D}}{\partial x}\psi,\qquad\mathcal{D}:=-\dfrac{\gamma x^{m+3-k}}{x^m-\mathcal{U}}.\end{cases}\tag{5.10}$$

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