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A note on G_q-summability of formal solutions of some linear q-difference-differential equations

By

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Abstract

Let $q > 1$ and $\delta > 0$. For a function $f(t,z)$, the q-shift operator σ_q in t is defined by $\sigma_q(f)(t,z) = f(qt,z)$. This article discusses a linear q-difference-differential equation $\sum_{j+\delta|\alpha|\le m} a_{j,\alpha}(t,z)(\sigma_q)^j \partial_z^\alpha X = F(t,z)$ in the complex domain, and shows a result on the G_q-summability of formal solutions (which may be divergent) in the framework of q-Laplace and q-Borel transforms by Ramis-Zhang.

§ 1. Introduction

Let (t,z) be the variable in $\mathbb{C}_t \times \mathbb{C}_z^d$. Let $q > 1$. For a function $f(t,z)$ we define a q-shift operator σ_q in t by $\sigma_q(f)(t,z) = f(qt,z)$.

In this note, we consider a linear q-difference-differential equation

$$\sum_{j+\delta|\alpha|\le m} a_{j,\alpha}(t,z)(\sigma_q)^j \partial_z^\alpha X = F(t,z) \tag{1.1}$$

under the following assumptions:

(1) $q > 1$, $\delta > 0$ and $m \in \mathbb{N}^* (= \{1,2,\ldots\})$;

(2) $a_{j,\alpha}(t,z)$ $(j+\delta|\alpha| \le m)$ and $F(t,z)$ are holomorphic functions in a neighborhood of $(0,0) \in \mathbb{C}_t \times \mathbb{C}_z^d$;

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(3) (1.1) has a formal power series solution

$$X(t,z) = \sum_{n\geq 0} X_n(z)t^n \in \mathcal{O}_R[[t]] \tag{1.2}$$

where $\mathcal{O}_R$ denotes the set of all holomorphic functions on $D_R = \{z \in \mathbb{C}^d \,;\, |z_i| < R \ (i = 1,\dots,d)\}$.

Our basic problem is:

Problem 1.1. Under what condition can we get a true solution $W(t,z)$ of (1.1) which admits $\hat{X}(t,z)$ as a q-Gevrey asymptotic expansion of order 1 (in the sense of Definition 1.2 given below) ?

For $\lambda \in \mathbb{C}\setminus\{0\}$ and $\epsilon > 0$ we set

$$\mathscr{Z}_\lambda = \{-\lambda q^m \in \mathbb{C}\,;\, m \in \mathbb{Z}\},$$
$$\mathscr{Z}_{\lambda,\epsilon} = \bigcup_{m\in\mathbb{Z}} \{t \in \mathbb{C}\setminus\{0\}\,;\, |1+\lambda q^m/t| \leq \epsilon\}.$$

It is easy to see that if $\epsilon > 0$ is sufficiently small the set $\mathscr{Z}_{\lambda,\epsilon}$ is a disjoint union of closed disks. For $r > 0$ we write $D_r^* = \{t \in \mathbb{C}\,;\, 0 < |t| < r\}$. The following definition is due to Ramis-Zhang [8].

Definition 1.2. (1) Let $\hat{X}(t,z) = \sum_{n\geq 0} X_n(z)t^n \in \mathcal{O}_R[[t]]$ and let $W(t,z)$ be a holomorphic function on $(D_r^* \setminus \mathscr{Z}_\lambda) \times D_R$ for some $r > 0$. We say that $W(t,z)$ *admits* $\hat{X}(t,z)$ *as a q-Gevrey asymptitoc expansion of order 1*, if there are $M > 0$ and $H > 0$ such that

$$\left| W(t,z) - \sum_{n=0}^{N-1} X_n(z)t^n \right| \leq \frac{MH^N}{\epsilon} q^{N(N-1)/2}|t|^N$$

holds on $(D_r^* \setminus \mathscr{Z}_{\lambda,\epsilon}) \times D_R$ for any $N = 0,1,2,\dots$ and any sufficiently small $\epsilon > 0$.

(2) If there is a $W(t,z)$ as above, we say that the formal solution $\hat{X}(t,z)$ is G_q-summable in the direction λ.

A partial answer to Problem 1.1 was given in Tahara-Yamazawa [11]: in this paper, we will give an improvement of the result in [11]. As in [11], we will use the framework of q-Laplace and q-Borel transforms via Jacobi theta function, developped by Ramis-Zhang [8] and Zhang [10].

Similar problems are discussed by Zhang [9], Marotte-Zhang [5] and Ramis-Sauloy-Zhang [7] in the q-difference equations, and by Malek [3, 4], Lastra-Malek [1] and Lastra-Malek-Sanz [2] in the case of q-difference-differential equations. But, their equations are different from ours.

§ 2. Main results

For a holomorphic function $f(t,z)$ in a neighborhood of $(0,0) \in \mathbb{C}_t \times \mathbb{C}_z^d$, we define the order of the zeros of the function $f(t,z)$ at $t=0$ (we denote this by $\mathrm{ord}_t(f)$) by

$$\mathrm{ord}_t(f) = \min\{k \in \mathbb{N}\,;\, (\partial_t^k f)(0,z) \not\equiv 0 \ \text{ near } z=0\}$$

where $\mathbb{N} = \{0,1,2,\ldots\}$.

For $(a,b) \in \mathbb{R}^2$ we set $C(a,b) = \{(x,y) \in \mathbb{R}^2\,;\, x \leq a, y \geq b\}$. We define the t-Newton polygon $N_t(1.1)$ of equation (1.1) by

$$N_t(1.1) = \text{the convex hull of} \bigcup_{j+\delta|\alpha| \leq m} C(j, \mathrm{ord}_t(a_{j,\alpha})).$$

In this note, we will consider the equation (1.1) under the following conditions (A_1) and (A_2):

(A_1) There is an integer m_0 such that $0 \leq m_0 < m$ and

$$N_t(1.1) = \{(x,y) \in \mathbb{R}^2\,;\, x \leq m,\ y \geq \max\{0, x-m_0\}\}.$$

(A_2) Moreover, we have

$$|\alpha| > 0 \Longrightarrow (j, \mathrm{ord}_t(a_{j,\alpha})) \in int(N_t(1.1)),$$

where $int(N_t(1.1))$ denotes the interior of the set $N_t(1.1)$ in $\mathbb{R}^2$.

The figure of $N_t(1.1)$ is as in Figure 1. In Figure 1, the boundary of $N_t(1.1)$ consists of a horizontal half-line Γ_0, a segment Γ_1 and a vertical half-line Γ_2, and k_i is the slope of Γ_i for $i = 0,1,2$.

Lemma 2.1. *If* (A_1) *and* (A_2) *are satisfied, we have*

$$\mathrm{ord}_t(a_{j,\alpha}) \geq \begin{cases} \max\{0, j-m_0\}, & \text{if } |\alpha| = 0, \\ \max\{1, j-m_0+1\}, & \text{if } |\alpha| > 0. \end{cases} \tag{2.1}$$

By the condition (2.1), we have the expression

$$a_{j,0}(t,z) = t^{j-m_0} b_{j,0}(t,z) \quad \text{for } m_0 < j \leq m \tag{2.2}$$

for some holomorphic functions $b_{j,0}(t,z)$ $(m_0 < j \leq m)$ in a neighborhood of $(0,0) \in \mathbb{C} \times \mathbb{C}_z^d$. We suppose:

$$a_{m_0,0}(0,0) \neq 0 \quad \text{and} \quad b_{m,0}(0,0) \neq 0. \tag{2.3}$$

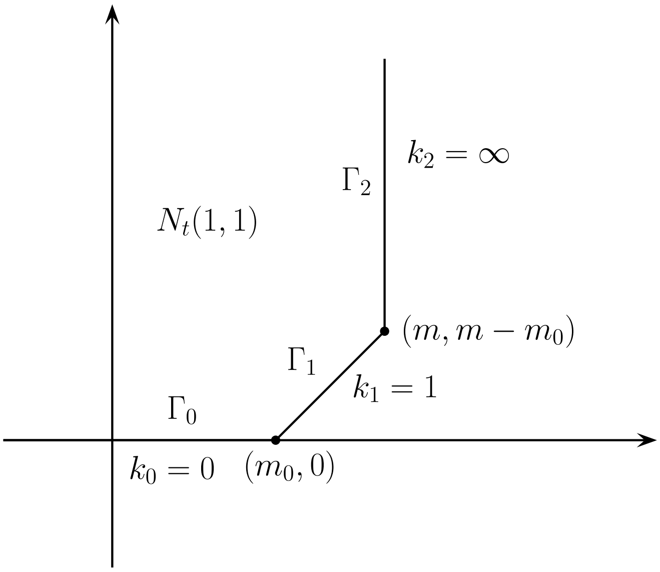

Figure 1. t-Newton polygon of $N_t(1.1)$

We set

$$P(\tau, z) = \sum_{m_0 < j \leq m} \frac{b_{j,0}(0,z)}{q^{j(j-1)/2}} \tau^{j-m_0} + \frac{a_{m_0,0}(0,z)}{q^{m_0(m_0-1)/2}} \tag{2.4}$$

and denote by $\tau_1, \ldots, \tau_{m-m_0}$ the roots of $P(\tau, 0) = 0$. By (2.3) we have $\tau_i \neq 0$ for all $i = 1, 2, \ldots, m - m_0$. The set S of singular directions at $z = 0$ is defined by

$$S = \bigcup_{i=1}^{m-m_0} \{t = \tau_i \eta \,;\, \eta > 0\}.$$

In [11], we have shown the following result.

Theorem 2.2 (Theorem 2.3 in [11]). (1) *Suppose the conditions* (A_1), (A_2) *and* (2.3). *Then, if equation* (1.1) *has a formal solution* $\hat{X}(t,z) = \sum_{n \geq 0} X_n(z) t^n \in \mathcal{O}_R[[t]]$, *we can find* $A > 0$, $h > 0$ *and* $0 < R_1 < R$ *such that* $|X_n(z)| \leq A h^n q^{n(n-1)/2}$ *on* D_{R_1} *for any* $n = 0, 1, 2, \ldots$.

(2) *In addition, if the condition*

$$\mathrm{ord}_t(a_{j,\alpha}) \geq j - m_0 + 2, \quad \text{if } |\alpha| > 0 \text{ and } m_0 \leq j < m \tag{2.5}$$

is satisfied, for any $\lambda \in \mathbb{C} \setminus (\{0\} \cup S)$ *the formal solution* $\hat{X}(t,z)$ *is* G_q*-summable in the direction* λ. *In other words, there are* $r > 0$, $R_1 > 0$ *and a holomorphic solution* $W(t,z)$ *of (1.1) on* $(D_r^* \setminus \mathscr{Z}_\lambda) \times D_{R_1}$ *such that* $W(t,z)$ *admits* $\hat{X}(t,z)$ *as a q-Gevrey asymptitoc expansion of order 1.*

In this paper, we remove the additional condition (2.5) from the part (2) of Theorem 2.2. We have

Theorem 2.3. *Suppose the conditions* (A_1), (A_2) *and* (2.3). *Then, for any* $\lambda \in \mathbb{C} \setminus (\{0\} \cup S)$ *the formal solution* $\hat{X}(t,z)$ *(in* (1.2)*) is* G_q*-summable in the direction* λ.

To prove this, we use the framework of q-Laplace and q-Borel transforms developped by Rramis-Zhang [8]. By (1) of Theorem 2.2 we know that the formal q-Borel transform of $\hat{X}(t,z)$ in t

$$u(\xi,z) = \sum_{k \geq 0} \frac{X_k(z)}{q^{k(k-1)/2}} \xi^k \tag{2.6}$$

is convergent in a neighborhood of $(0,0) \in \mathbb{C}_\xi \times \mathbb{C}_z^n$. For $\lambda \in \mathbb{C} \setminus \{0\}$ and $\theta > 0$ we write $S_\theta(\lambda) = \{\xi \in \mathbb{C} \setminus \{0\}\,;\, |\arg \xi - \arg \lambda| < \theta\}$. Then, to show Theorem 2.3 it is enough to prove the following result.

Proposition 2.4. *For any* $\lambda \in \mathbb{C} \setminus (\{0\} \cup S)$ *there are* $\theta > 0$, $R_1 > 0$, $C > 0$ *and* $H > 0$ *such that* $u(\xi,z)$ *has an analytic extension* $u^*(\xi,z)$ *to the domain* $S_\theta(\lambda) \times D_{R_1}$ *satisfying the following condition:*

$$|u^*(\lambda q^m, z)| \leq C H^m q^{m^2/2} \quad \text{on } D_{R_1}, \quad m = 0,1,2,\ldots. \tag{2.7}$$

§ 3. Some lemmas

Before the proof of Proposition 2.4, let us give some lemmas which are needed in the proof of Proposition 2.4.

The following is the key lemma of the proof of Proposition 2.4.

Lemma 3.1. *Let* $q > 1$. *Let* $f(t,z)$ *be a function in* (t,z).

(1) *We have* $\sigma_q(f)(t,z) = (\sigma_{\sqrt{q}})^2(f)(t,z)$.

(2) *We set* $F(t,z) = f(t^2,z)$*: then we have* $\sigma_q(f)(t^2,z) = \sigma_{\sqrt{q}}(F)(t,z)$. *Similarly, we have* $(\sigma_q)^m(f)(t^2,z) = (\sigma_{\sqrt{q}})^m(F)(t,z)$ *for any* $m = 1,2,\ldots$.

Proof. (1) is clear. (2) is verified as follows: $\sigma_q(f)(t^2,z) = f(qt^2,z) = f((\sqrt{q}t)^2,z) = F(\sqrt{q}t,z) = \sigma_{\sqrt{q}}(F)(t,z)$. The equality $(\sigma_q)^m(f)(t^2,z) = (\sigma_{\sqrt{q}})^m(F)(t,z)$ can be proved in the same way. □

The following result is proved in [Proposition 2.1 in [6]]:

Proposition 3.2. *Let $\hat{f}(t)=\sum_{n\geq 0} a_n t^n \in \mathbb{C}[[t]]$. The following two conditions are equivalent:*

(1) *There are $A>0$ and $H>0$ such that*

$$|a_n| \leq \frac{AH^n}{q^{n(n-1)/2}}, \quad n=0,1,2,\ldots.$$

(2) *$\hat{f}(t)$ is the Taylor expansion at $t=0$ of an entire function $f(t)$ satisfying the estimate*

$$|f(t)| \leq M \exp\Big(\frac{(\log|t|)^2}{2\log q}+\alpha\log|t|\Big) \quad on\ \mathbb{C}\setminus\{0\}$$

for some $M>0$ and $\alpha \in \mathbb{R}$.

§ 4. Proof of Proposition 2.4

We set $q_1=q^{1/4}$, replace t by t^2 in (1.1), and apply Lemma 3.1 to the equation (1.1): then (1.1) is rewritten into the form

$$\sum_{j+\delta|\alpha|\leq m} A_{j,\alpha}(t,z)(\sigma_{q_1})^{2j}\partial_z^{\alpha}Y=G(t,z) \tag{4.1}$$

where

$$\begin{aligned}
&A_{j,\alpha}(t,z)=a_{j,\alpha}(t^2,z) \quad (j+\delta|\alpha|\leq m),\\
&Y(t,z)=X(t^2,z)=\sum_{k\geq 0} X_k(z)t^{2k},\\
&G(t,z)=F(t^2,z).
\end{aligned}$$

We can regards (4.1) as a q_1-difference-differential equation, and in this case, the order of the equation is $2m$ in t. Therefore, the t-Newton polygon $N_t(4.1)$ of (4.1) (as a q_1-difference equation) is

$$N_t(4.1)=\{(x,y)\in\mathbb{R}^2\,;\,x\leq 2m,\ y\geq \max\{0,x-2m_0\}\}$$

which is as in Figure 2.

Moreover, we have

$$\operatorname{ord}_t(A_{j,\alpha}) \geq \begin{cases} \max\{0,2j-2m_0\}, & \text{if } |\alpha|=0,\\ \max\{2,2j-2m_0+2\}, & \text{if } |\alpha|>0. \end{cases} \tag{4.2}$$

By (2.2) we have

$$A_{j,0}(t,z)=t^{2j-2m_0}B_{j,0}(t,z) \quad \text{for } m_0<j\leq m$$

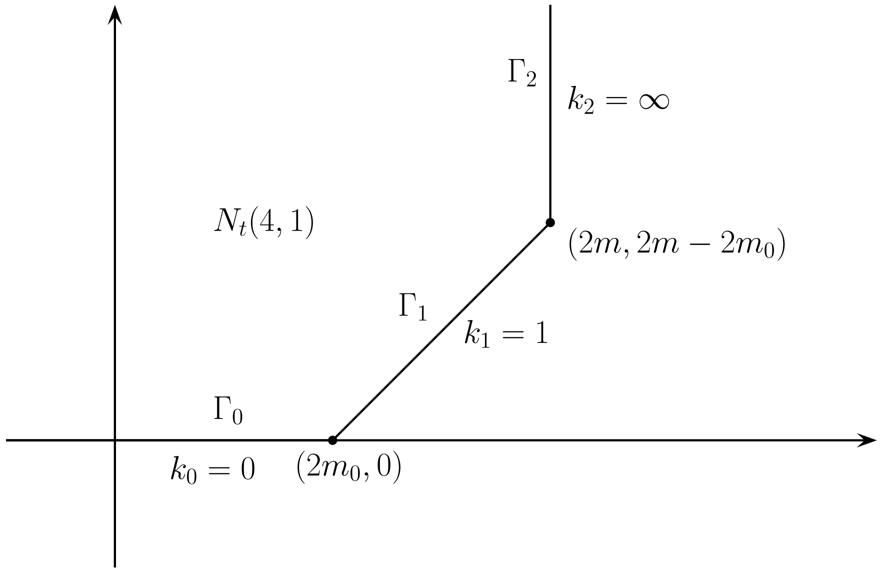

Figure 2. t-Newton polygon of (4.1)

for $B_{j,0}(t,z) = b_{j,0}(t^2,z)$ $(m_0 < j \le m)$. The set S_1 of singular directions of (4.1) is defined by using

$$P_1(\rho,z) = \sum_{m_0<j\le m} \frac{B_{j,0}(0,z)}{{q_1}^{2j(2j-1)/2}}\rho^{2j-2m_0} + \frac{A_{m_0,0}(0,z)}{{q_1}^{2m_0(2m_0-1)/2}}$$
$$= \sum_{m_0<j\le m} \frac{b_{j,0}(0,z)}{{q_1}^{2j(2j-1)/2}}\rho^{2j-2m_0}. + \frac{a_{m_0,0}(0,z)}{{q_1}^{2m_0(2m_0-1)/2}}.$$

Let $\rho_1, \dots, \rho_{2m-2m_0}$ be the roots of $P_1(\rho, 0) = 0$: then S_1 is defined by

$$S_1 = \bigcup_{i=1}^{2m-2m_0} \{t = \rho_i \eta\,;\, \eta > 0\}.$$

Let $u_1(\xi, x)$ be the q_1-formal Borel transform of $Y(t,x)$, that is,

$$u_1(\xi,z) = \sum_{k\ge 0} \frac{X_k(z)}{{q_1}^{2k(2k-1)/2}}\xi^{2k}.$$

Since $q_1 = q^{1/4}$ we can easily see:

$$u_1(\xi,z) = u(q^{-1/4}\xi^2, z), \tag{4.3}$$
$$P_1(\lambda,z) = q^{-m_0/4}P(q^{-1/4}\lambda^2, z), \tag{4.4}$$

where $u(\xi,z)$ and $P(\tau,z)$ are the ones in (2.6) and (2.4), respectively.

By (4.3) we see that $u_1(\xi, z)$ is convergent in a neighborhood of $(\xi, z) = (0, 0)$. The equality (4.4) implies that $\lambda \in \mathbb{C} \setminus (\{0\} \cup S_1)$ is equivalent to the condition $\lambda^2 \in \mathbb{C} \setminus (\{0\} \cup S)$.

Since $\mathrm{ord}_t(A_{j,\alpha}) \geq 2j - 2m_0 + 2$ holds for any (j, α) with $m_0 \leq j < m$ and $|\alpha| > 0$, the q_1-difference equation (4.1) satisfies the condition (2.5) (with j, m_0, m replaced by $2j$, $2m_0$, $2m$, respectively). Therefore, we can apply (2) of Theorem 2.2 and its proof to the equation (4.1).

In particular, by the proof of [Proposition 5.6 in [11]] we have

Proposition 4.1. *For any $\rho \in \mathbb{C} \setminus (\{0\} \cup S_1)$ we can find $\theta_1 > 0$ and $R_1 > 0$ which satisfy the following conditions* (1) *and* (2)*:*

(1) *$u_1(\xi, z)$ has an analytic extension $u_1^*(\xi, z)$ to the domain $S_{\theta_1}(\rho) \times D_{R_1}$.*

(2) *There are $\mu > 0$ and holomorphic functions $w_n(\xi, z)$ $(n \geq \mu)$ on $S_{\theta_1}(\rho) \times D_{R_1}$ which satisfy*

$$u_1^*(\xi, z) = \sum_{n \geq 2\mu} w_n(\xi, z) + \sum_{0 \leq k < \mu} \frac{X_k(z)}{{q_1}^{2k(2k-1)}} \xi^{2k} \quad \text{on } S_{\theta_1}(\rho) \times D_{R_1} \tag{4.5}$$

and

$$|w_n(\xi, z)| \leq \frac{AH^n |\xi|^n}{{q_1}^{n(n-1)/2}} \quad \text{on } S_{\theta_1}(\rho) \times D_{R_1}, \quad n \geq 2\mu$$

for some $A > 0$ and $H > 0$.

Therefore, by applying Proposition 3.2 to (4.5) we have the estimate

$$|u_1^*(\xi, x)| \leq M \exp\Big(\frac{(\log |\xi|)^2}{2 \log q_1} + \alpha \log |\xi| \Big) \quad \text{on } S_{\theta_1}(\rho) \times D_{R_1} \tag{4.6}$$

for some $M > 0$ and $\alpha \in \mathbb{R}$.

Completion of the proof of Proposition 2.4. Take any $\lambda = re^{\sqrt{-1}\theta} \in \mathbb{C} \setminus (\{0\} \cup S)$. We set $\rho = \sqrt{r} e^{\sqrt{-1}\theta/2}$: then we have $\rho \in \mathbb{C} \setminus (\{0\} \cup S_1)$. Therefore, by Proposition 4.1 we can get $\theta_1 > 0$, $R_1 > 0$, $M > 0$ and $\alpha \in \mathbb{R}$ such that $u_1(\xi, z)$ has an analytic extension $u_1^*(\xi, z)$ to the domain $S_{\theta_1}(\rho) \times D_{R_1}$ satisfying the estimate (4.6) on $S_{\theta_1}(\rho) \times D_{R_1}$.

Since $u_1(\xi, z) = u(q^{-1/4}\xi^2, z)$ holds, this shows that $u(\xi, z)$ has also an analytic continuation $u^*(\xi, x)$ to the domain $S_\theta(\lambda) \times D_{R_1}$ (with $\theta = 2\theta_1$), and we have $u^*(\xi, z) = u_1^*(q^{1/8}\xi^{1/2}, z)$ on $S_\theta(\lambda) \times D_{R_1}$. Therefore, by (4.6) we have the estimate

$$\begin{aligned} |u^*(\xi, x)| &\leq M \exp\Big(\frac{(\log(q^{1/8}|\xi|^{1/2}))^2}{2 \log q^{1/4}} + \alpha \log(q^{1/8}|\xi|^{1/2}) \Big) \\ &= M_1 |\xi|^\beta \exp\Big(\frac{(\log |\xi|)^2}{2 \log q} \Big) \quad \text{on } S_\theta(\lambda) \times D_{R_1} \end{aligned}$$

(with $M = M_1 q^{1/32 + \alpha/8}$ and $\beta = 1/4 + \alpha/2$).

Thus, by setting $\xi = \lambda q^m$ we obtain

$$|u^*(\lambda q^m, x)| \leq M_1 |\lambda q^m|^\beta \exp\Big(\frac{(\log|\lambda q^m|)^2}{2\log q}\Big)$$
$$= M_1|\lambda|^\beta \exp\Big(\frac{(\log|\lambda|)^2}{2\log q}\Big)(|\lambda|q^\beta)^m q^{m^2/2}, \quad m = 0, 1, 2, \ldots.$$

This proves (2.7). $\square$

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