RIMS Kôkyûroku Bessatsu
B75 (2019), 139–150

On a non-hereditary turning point of a tangential system of the Pearcey system

By

Sampei Hirose*

Abstract

We report the presence of a hitherto unknown ordinary turning point in a tangential system of the Pearcey system. It is a double turning point which does not originate from the turning points of the Pearcey system, and we name it a non-hereditary turning point. Thanks to a result of Takei [4], a non-hereditary turning point is irrelevant to the Stokes phenomena of WKB solutions of the tangential system near the point.

§ 1. Introduction

In this paper, we discuss the exact WKB analysis for a tangential system of the Pearcey system and its ordinary turning point. Here the Pearcey system is the completely integrable system of two variables $x = (x_1, x_2)$ as follows:

$$\begin{cases} \eta^{-1}\dfrac{\partial}{\partial x_1}\Psi = P(x,\eta)\Psi, \quad P(x,\eta) = \displaystyle\sum_{n=0}^{\infty}\eta^{-n}P_n(x), \\ \eta^{-1}\dfrac{\partial}{\partial x_2}\Psi = Q(x,\eta)\Psi, \quad Q(x,\eta) = \displaystyle\sum_{n=0}^{\infty}\eta^{-n}Q_n(x), \end{cases}$$

where

$$P_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -x_1/4 & -x_2/2 & 0 \end{pmatrix}, \quad P_n = 0\ (n = 1, 2, \ldots),$$

$$Q_0 = P_0^2, \quad Q_1 = \frac{\partial P_0}{\partial x_1}, \quad Q_n = 0\ (n = 2, 3, \ldots).$$

Received December 6, 2017. Revised November 22, 2018.
2010 Mathematics Subject Classification(s): 34E20, 34M60
Key Words: non-hereditary turning point, Pearcey system, WKB-theoretic transformation.
Supported by JSPS KAKENHI Grant No.15K17556.
*Department of Engineering and Design, Shibaura Institute of Technology, 307 Fukasaku, Minuma-ku, Saitama-shi, Saitama 337-8570, Japan.
e-mail: hirose3@shibaura-it.ac.jp

The tangential system under consideration is the restriction of the Pearcey system to the hyperplane

$$L_{c,\theta} = \left\{ x = (x_1, x_2) \in \mathbb{C}^2 \; ; \; (-\sin\theta)x_1 + (\cos\theta)x_2 = c \right\},$$

and it is denoted by $\mathcal{M}_{c,\theta}$. The Stokes geometry of the Pearcey system and its relationship with the tangential system $\mathcal{M}_{c,0}$ was discussed in [2]. Note that the tangential system $\mathcal{M}_{c,0}$ is equivalent to the equation which Berk-Nevins-Roberts [1] discussed to study new Stokes' line. In [2], we showed that the set of ordinary turning points for the tangential system $\mathcal{M}_{c,0}$ coincides with the set of turning points for the Pearcey system restricted to $L_{c,0}$. This means that the turning points for the Pearcey system are inherited to the tangential system $\mathcal{M}_{c,0}$. In this paper we show that the set of turning points for the Pearcey system restricted to $L_{c,\theta}$ is included in the set of ordinary turning points for the tangential system $\mathcal{M}_{c,\theta}$, and further, we show that the tangential system $\mathcal{M}_{c,\theta}$ has an ordinary turning point which is not in the set of turning points for the Pearcey system restricted to $L_{c,\theta}$. That is, this ordinary turning point does not originate from a turning point of the Pearcey system. We call this ordinary turning point a non-hereditary turning point. The purpose of this paper is to show why such a turning point appears and how it is related to the singularity structure of the Borel transform of WKB solutions of $\mathcal{M}_{c,\theta}$.

This paper is constructed as follows: In §2 we recall the definition of an ordinary turning point, and show the existence of a non-hereditary turning point for $\mathcal{M}_{c,\theta}$. In §3, we present characteristic properties of a non-hereditary turning point. Finally in §4 we show, thanks to the results of Takei [4], that a non-hereditary turning point is not relevant to the Stokes phenomena of WKB solutions of $\mathcal{M}_{c,\theta}$.

The author would like to express his thanks to Professor Takahiro Kawai and Professor Yoshitsugu Takei for their many valuable advices and encouragements. This work was supported by the Research Institute for Mathematical Sciences, a Joint Usage/Research Center located in Kyoto University.

Remark. The paper [3] discusses a non-hereditary turning point from the viewpoint of the integral representation of solutions of the Pearcey system. This paper discusses it from the viewpoint of the tangential system of the Pearcey system.

§ 2. Ordinary turning points of the tangential system $\mathcal{M}_{c,\theta}$

We consider the tangential system $\mathcal{M}_{c,\theta}$ of the Pearcey system to the hyperplane $L_{c,\theta}$ and its ordinary turning points. We first give the explicit form of the tangential system $\mathcal{M}_{c,\theta}$. By the coordinate transformation

$$z = xR_\theta, \quad z = (z_1, z_2), \quad x = (x_1, x_2), \quad R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

the Pearcey system is transformed into the following form:

$$\mathcal{M} : \begin{cases} \eta^{-1} \dfrac{\partial}{\partial z_1} \Psi = R(z, \theta, \eta) \Psi, \quad R(z, \theta, \eta) = \displaystyle\sum_{n=0}^{\infty} \eta^{-n} R_n(z, \theta), \\ \eta^{-1} \dfrac{\partial}{\partial z_2} \Psi = S(z, \theta, \eta) \Psi, \quad S(z, \theta, \eta) = \displaystyle\sum_{n=0}^{\infty} \eta^{-n} S_n(z, \theta), \end{cases} \tag{2.1}$$

where

$$\begin{aligned} R(z, \theta, \eta) &= \{(\cos\theta) P(x, \eta) + (\sin\theta) Q(x, \eta)\} \big|_{x = zR_{-\theta}}, \\ S(z, \theta, \eta) &= \{-(\sin\theta) P(x, \eta) + (\cos\theta) Q(x, \eta)\} \big|_{x = zR_{-\theta}}. \end{aligned}$$

We call this system also the Pearcey system. The tangential system of the Pearcey system to the hyperplane

$$\begin{aligned} L_{c,\theta} &= \big\{ x = (x_1, x_2) \in \mathbb{C}^2 \ ; \ -(\sin\theta) x_1 + (\cos\theta) x_2 = c \big\} \\ &= \big\{ z = (z_1, z_2) \in \mathbb{C}^2 \ ; \ z_2 = c \big\} \end{aligned}$$

is given by

$$\mathcal{M}_{c,\theta} : \eta^{-1} \frac{d}{dz_1} \Psi = R(z_1, c, \theta, \eta) \Psi, \quad R(z_1, c, \theta, \eta) = \sum_{n=0}^{\infty} \eta^{-n} R_n(z_1, c, \theta),$$

where

$$R(z_1, c, \theta, \eta) = \{(\cos\theta) P(x, \eta) + (\sin\theta) Q(x, \eta)\} \big|_{x = (z_1, c) R_{-\theta}}.$$

Here and in what follows we always assume $c \neq 0$.

The definition of an ordinary turning point for the tangential system $\mathcal{M}_{c,\theta}$ is as follows:

Definition 2.1. A point $a_1 \in \mathbb{C}$ is called an ordinary turning point for the tangential system $\mathcal{M}_{c,\theta}$ if there exist $i, i' \in \{1, 2, 3\}$ $(i \neq i')$ for which

$$\zeta_{1,i}(a_1, c) = \zeta_{1,i'}(a_1, c)$$

holds, where $\zeta_{1,i}(z_1, c)$ is a root of the characteristic equation

$$\det(\zeta_1 - R_0(z_1, c, \theta)) = 0.$$

That is, an ordinary turning point for the tangential system $\mathcal{M}_{c,\theta}$ is a zero of the discriminant of the characteristic equation for the variable ζ_1. The characteristic equation of the tangential system $\mathcal{M}_{c,\theta}$ is

$$\det(\zeta_1 - R_0(z_1, c, \theta)) = \zeta_1^3 + a_1(z_1, c, \theta)\zeta_1^2 + a_2(z_1, c, \theta)\zeta_1 + a_3(z_1, c, \theta),$$

where

$$\begin{aligned} a_1(z_1, c, \theta) &= \left(\sin^2 \theta\right) z_1 + c\cos\theta \sin\theta, \\ a_2(z_1, c, \theta) &= \left(\frac{1}{4}\sin^4 \theta\right) z_1^2 + \left(\frac{1}{2}c\cos\theta\sin^3\theta + \frac{5}{4}\cos^2\theta\sin\theta\right) z_1 \\ &\quad + \frac{1}{4}c^2\cos^2\theta\sin^2\theta + \frac{1}{2}c\cos^3\theta - \frac{3}{4}c\cos\theta\sin^2\theta, \\ a_3(z_1, c, \theta) &= \left(\frac{1}{16}\cos^2\theta\sin^3\theta\right) z_1^2 + \left(\frac{1}{8}c\cos^3\theta\sin^2\theta + \frac{1}{4}\cos^4\theta\right) z_1 \\ &\quad - \frac{1}{8}c^2\cos^2\theta\sin^3\theta - \frac{1}{16}c^2\sin^5\theta - \frac{1}{4}c\cos^3\theta\sin\theta, \end{aligned}$$

and its discriminant is

$$D_1(z_1, c, \theta)D_2(z_1, c, \theta)^2,$$

where

$$\begin{aligned} D_1(z_1, c, \theta) =& D_{13}(c, \theta)z_1^3 + D_{12}(c, \theta)z_1^2 + D_{11}(c, \theta)z_1 + D_{10}(c, \theta), \\ D_2(z_1, c, \theta) =& D_{21}(c, \theta)z_1 + D_{20}(c, \theta), \end{aligned}$$

and

$$\begin{aligned} D_{13}(c, \theta) =& 8\sin^3\theta, \\ D_{12}(c, \theta) =& 24c\cos\theta\sin^2\theta + 27\cos^2\theta, \\ D_{11}(c, \theta) =& 24c^2\cos^2\theta\sin\theta - 54c\cos\theta\sin\theta, \\ D_{10}(c, \theta) =& 8c^3\cos^3\theta + 27c^2\sin^2\theta, \\ D_{21}(c, \theta) =& 3\cos\theta\sin^3\theta, \\ D_{20}(c, \theta) =& 2c\cos^2\theta\sin^2\theta - c\sin^4\theta + 4\cos^3\theta. \end{aligned}$$

Hence the set of ordinary turning points for the tangential system $\mathcal{M}_{c,\theta}$ is explicitly given by

$$\begin{aligned} T_{c,\theta} &= \left\{z_1 \in \mathbb{C} \; ; \; D_1(z_1, c, \theta)D_2(z_1, c, \theta)^2 = 0\right\} \\ &= \{z_1 \in \mathbb{C} \; ; \; D_1(z_1, c, \theta) = 0\} \cup \{z_1 \in \mathbb{C} \; ; \; D_2(z_1, c, \theta) = 0\}. \end{aligned}$$

Note that $D_2(z_1, c, \theta)$ is a non-zero constant function for $\theta = m\pi/2$ ($m \in \mathbb{Z}$). In what follows, we assume $\theta \neq m\pi/2$ ($m \in \mathbb{Z}$).

On the other hand, the definition of a turning point for the Pearcey system $\mathcal{M}$ is as follows:

Definition 2.2. A point $a = (a_1, a_2) \in \mathbb{C}^2$ is called a turning point for the Pearcey system $\mathcal{M}$ if there exist $i, i' \in \{1, 2, 3\}$ $(i \neq i')$ for which

$$\zeta_{1,i}(a) = \zeta_{1,i'}(a), \quad \zeta_{2,i}(a) = \zeta_{2,i'}(a)$$

hold, where $\zeta_{1,i}(z)$ (resp., $\zeta_{2,i}(z)$) is a root of the algebraic equation

$$\det(\zeta_1 - R_0(z, \theta)) = 0 \quad (\text{resp.}, \det(\zeta_2 - S_0(z, \theta)) = 0)$$

satisfying

$$\frac{\partial \zeta_{1,i}}{\partial z_2} = \frac{\partial \zeta_{2,i}}{\partial z_1}.$$

As was shown in [2], the set of turning points for the Pearcey system $\mathcal{M}$ is explicitly given by

$$T = \left\{ x = (x_1, x_2) \in \mathbb{C}^2 \mid 27x_1^2 + 8x_2^3 = 0 \right\}.$$

Note that we have

$$D_1(z, \theta) = (27x_1^2 + 8x_2^3)\big|_{x = zR_{-\theta}},$$

and, in the coordinate system z, T is expressed as follows:

$$T = \left\{ z = (z_1, z_2) \in \mathbb{C}^2 \mid D_1(z, \theta) = 0 \right\}.$$

This means that the set T of turning points for the Pearcey system restricted to $L_{c,\theta}$ is $\{z_1 \in \mathbb{C} \,;\, D_1(z_1, c, \theta) = 0\}$ and does not coincide with $T_{c,\theta}$. That is, the tangential system $\mathcal{M}_{c,\theta}$ has an ordinary turning point $z_1 = -D_{20}(c, \theta)/D_{21}(c, \theta)$ which is not in the set T restricted to $L_{c,\theta}$. Here and in what follows, we identify $(z_1, c) \in L_{c,\theta}$ with $z_1 \in \mathbb{C}$. We call this ordinary turning point a non-hereditary turning point. Note that this ordinary turning point is a double turning point, because of the form of the relevant factor $D_2(z_1, c, \theta)^2$ in the discriminant. For the reference of the reader we present in Figure 1 the configuration of ordinary and virtual turning points for the tangential system $\mathcal{M}_{c,\theta}$ and Stokes curves emanating from them for $\theta = \pi/4, c = 1/2 - \sqrt{-1}$, ignoring the non-hereditary turning point. The Stokes geometry of $\mathcal{M}_{c,\theta}$ with the non-hereditary turning point added will be later given in Figure 3.

§ 3. Characteristic properties of a non-hereditary turning point

We discuss the non-hereditary turning point for the tangential system $\mathcal{M}_{c,\theta}$ from the viewpoint of the relation between the characteristic variety of $\mathcal{M}_{c,\theta}$ and that of the

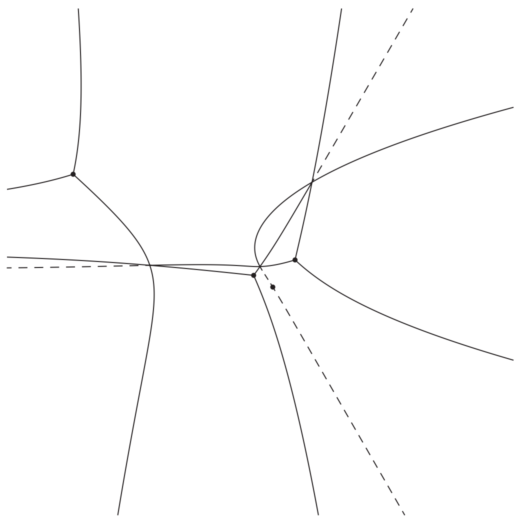

Figure 1. The ordinary and virtual turning points for the tangential system $\mathcal{M}_{c,\theta}$ and the Stokes curves emanating from them for $\theta = \pi/4, c = 1/2 - \sqrt{-1}$. Here we ignore a non-hereditary turning point. The dotted part of a Stokes curve indicates that it is inert.

Pearcey system $\mathcal{M}$ in (2.1). Let $a_1 \in \mathbb{C}$ be a non-hereditary turning point. Since a_1 is not a turning point for the Pearcey system $\mathcal{M}$ but an ordinary turning point for the tangential system $\mathcal{M}_{c,\theta}$, there exist $i, i' \in \{1, 2, 3\}$ $(i \neq i')$ such that

$$\zeta_{1,i}(a_1, c) = \zeta_{1,i'}(a_1, c), \quad \zeta_{2,i}(a_1, c) \neq \zeta_{2,i'}(a_1, c).$$

This condition characterizes a non-hereditary turning point.

Theorem 3.1. *Let $a_1 \in \mathbb{C}$ be a point which satisfies*

$$\zeta_{1,i}(a_1, c) = \zeta_{1,i'}(a_1, c), \quad \zeta_{2,i}(a_1, c) \neq \zeta_{2,i'}(a_1, c).$$

Then a_1 is a non-hereditary turning point for the tangential system $\mathcal{M}_{c,\theta}$, that is, $a_1 \in \{z_1 \in \mathbb{C}\ ;\ D_2(z_1, c, \theta) = 0\}$.

Proof. By the relations

$$\begin{aligned} R_0(z,\theta) &= \{(\cos\theta)P_0(x) + (\sin\theta)Q_0(x)\}\,\big|_{x=zR_{-\theta}}, \\ S_0(z,\theta) &= \{-(\sin\theta)P_0(x) + (\cos\theta)Q_0(x)\}\,\big|_{x=zR_{-\theta}}, \\ Q_0(x) &= P_0(x)^2, \end{aligned}$$

we have

$$\zeta_{1,i}(z) = \left\{(\cos\theta)\xi_i(x) + (\sin\theta)\xi_i(x)^2\right\}\big|_{x=zR_{-\theta}},$$
$$\zeta_{2,i}(z) = \left\{-(\sin\theta)\xi_i(x) + (\cos\theta)\xi_i(x)^2\right\}\big|_{x=zR_{-\theta}}$$

for $i = 1, 2, 3$. Here $\xi_i(x)$ is a root of the algebraic equation

$$\det(\xi - P_0(x)) = \xi^3 + \frac{x_2}{2}\xi + \frac{x_1}{4} = 0.$$

For simplicity, $\xi_i(x)\big|_{x=zR_{-\theta}}$ is denoted by $\xi_i(z)$. Note that, since $(a_1, c) \in \mathbb{C}^2$ is not a turning point for the Pearcey system, (a_1, c) satisfies

$$\xi_i(a_1, c) \neq \xi_{i'}(a_1, c).$$

By the relation $\zeta_{1,i}(a_1, c) = \zeta_{1,i'}(a_1, c)$, we have

$$(\cos\theta)\xi_i(a_1, c) + (\sin\theta)\xi_i(a_1, c)^2 = (\cos\theta)\xi_{i'}(a_1, c) + (\sin\theta)\xi_{i'}(a_1, c)^2.$$

Hence

$$(\xi_i(a_1, c) - \xi_{i'}(a_1, c))\left\{\cos\theta + \sin\theta(\xi_i(a_1, c) + \xi_{i'}(a_1, c))\right\} = 0$$

holds. Using $\xi_i(a_1, c) \neq \xi_{i'}(a_1, c)$, we obtain

$$\cos\theta + \sin\theta(\xi_i(a_1, c) + \xi_{i'}(a_1, c)) = 0.$$

Since $\xi_i + \xi_{i'} + \xi_{i''} = 0$ for three roots $\xi_i, \xi_{i'}, \xi_{i''}$ of the algebraic equation

$$\det(\xi - P_0(x)) = \xi^3 + \frac{x_2}{2}\xi + \frac{x_1}{4} = 0,$$

we have

$$\xi_{i''}(a_1, c) = \frac{\cos\theta}{\sin\theta}.$$

Hence, we obtain

$$D_2(a_1, c, \theta) = 4\sin^3\theta\left(\xi_{i''}^3 + \frac{x_2}{2}\xi_{i''} + \frac{x_1}{4}\right)\bigg|_{x=(a_1,c)R_{-\theta}} = 0,$$

that is, $a_1 \in \{z_1 \in \mathbb{C}\,;\, D_2(z_1, c, \theta) = 0\}$. □

Schematic correspondence between characteristic points of $\mathcal{M}$ and that of $\mathcal{M}_{c,\theta}$ is illustrated in Figure 2. The correspondence ρ is a well-known one in microlocal analysis.

§ 4. Redundancy of a non-hereditary turning point

The purpose of this section is to show that a non-hereditary turning point of $\mathcal{M}_{c,\theta}$ is irrelevant to the Stokes phenomena of WKB solutions of $\mathcal{M}_{c,\theta}$ near the point. The

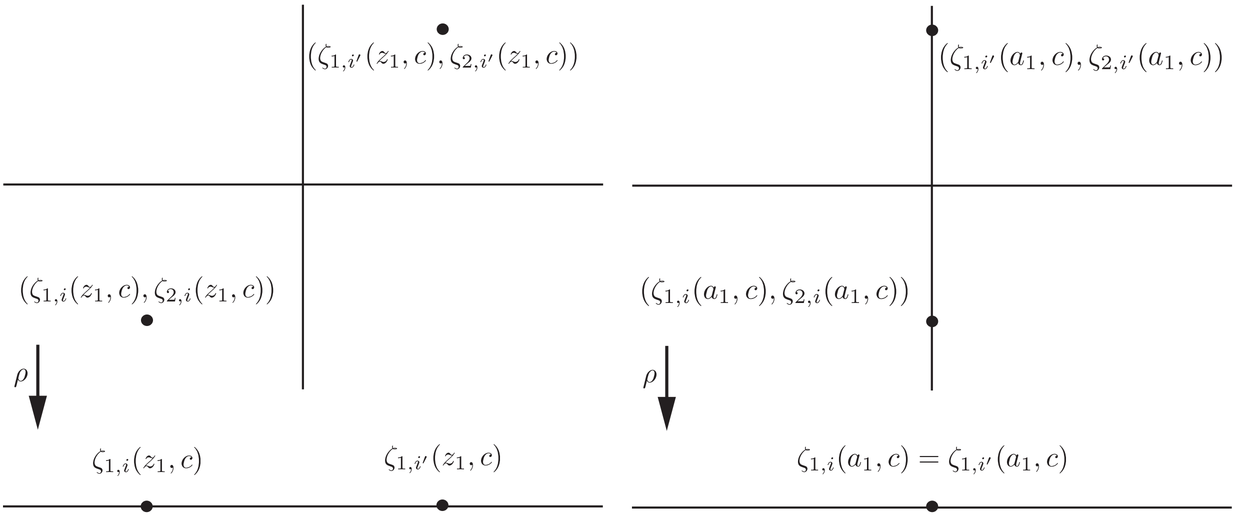

Figure 2. Schematic correspondence between characteristic points of $\mathcal{M}$ and that of $\mathcal{M}_{c,\theta}$ near a non-hereditary turning point $z_1 = a_1$. Left (resp., Right) figure is a configuration at $z_1 \neq a_1$ (resp., $z_1 = a_1$).

first step of our reasoning is to use a result of Takei ([4, , Proposition 1]) on the block-diagonalization of completely integrable systems so that the analysis of the Pearcey system $\mathcal{M}$ can be reduced to the study of a completely integrable 2×2 system near a point $z = (a_1(c), c)$ where $a_1(c) = -D_{20}(c,\theta)/D_{21}(c,\theta)$ is a non-hereditary turning point of $\mathcal{M}_{c,\theta}$.

We first block-diagonalize the first equation of (2.1) near $z = (a_1(c), c)$: by an appropriate choice of transformation

$$\widetilde{\Psi} = T(z,\eta)\Psi, \quad T(z,\eta) = \sum_{n=0}^{\infty} \eta^{-n} T_n(z) \tag{4.1}$$

where $T_n(z)$ $(n = 0, 1, \ldots)$ are 3×3 matrices with holomorphic entries near $z = (a_1(c), c)$ satisfying $\det T_0(z) \neq 0$, we can reduce the first equation of (2.1) to an equation of the following form:

$$\eta^{-1}\frac{\partial}{\partial z_1}\widetilde{\Psi} = \begin{pmatrix} \widetilde{R}(z,\eta) & \\ & \widetilde{r}(z,\eta) \end{pmatrix} \widetilde{\Psi},$$

where $\widetilde{R}(z,\eta)$ is a 2×2 matrix whose entries are power series in η^{-1}

$$\widetilde{R}(z,\eta) = \sum_{n=0}^{\infty} \eta^{-n} \widetilde{R}_n(z),$$

and $\widetilde{r}(z,\eta)$ is a scalar power series in η^{-1}

$$\widetilde{r}(z,\eta) = \sum_{n=0}^{\infty} \eta^{-n} \widetilde{r}_n(z),$$

and satisfy the following properties:

- the eigenvalues $(\zeta_{1,i}(z), \zeta_{1,i'}(z))$ of $\widetilde{R}_0(z)$ satisfy

$$\zeta_{1,i}(a_1(z_2), z_2) = \zeta_{1,i'}(a_1(z_2), z_2)$$

near $z_2 = c$,

- $\widetilde{r}_0(z)$ is distinct from the eigenvalues $(\zeta_{1,i}(z), \zeta_{1,i'}(z))$ of $\widetilde{R}_0(z)$.

Further $T(z,\eta), \widetilde{R}(z,\eta)$ and $\widetilde{r}(z,\eta)$ are all Borel-transformable.

Then Takei's result guarantees that transformation (4.1) automatically block-diagonalizes the second equation of (2.1) to the following form:

$$\eta^{-1}\frac{\partial}{\partial z_2}\widetilde{\Psi} = \begin{pmatrix} \widetilde{S}(z,\eta) & \\ & \widetilde{s}(z,\eta) \end{pmatrix} \widetilde{\Psi},$$

where $\widetilde{S}(z,\eta)$ is a 2×2 matrix whose entries are power series in η^{-1}

$$\widetilde{S}(z,\eta) = \sum_{n=0}^{\infty} \eta^{-n}\widetilde{S}_n(z),$$

and $\widetilde{s}(z,\eta)$ is a scalar power series in η^{-1}. Then it follows from the definition of a non-hereditary turning point that eigenvalues of $\widetilde{S}_0(z)$ are distinct, that is, eigenvalues $(\zeta_{2,i}(z), \zeta_{2,i'}(z))$ of $\widetilde{S}_0(z)$ satisfy

$$\zeta_{2,i}(a_1(z_2), z_2) \neq \zeta_{2,i'}(a_1(z_2), z_2) \tag{4.2}$$

near $z_2 = c$.

Thus, in order to study the structure of the Pearcey system near a non-hereditary turning point, it suffices to study the following 2×2 system:

$$\begin{cases} \eta^{-1}\dfrac{\partial}{\partial z_1}\Phi = \widetilde{R}(z,\eta)\Phi, \\ \eta^{-1}\dfrac{\partial}{\partial z_2}\Phi = \widetilde{S}(z,\eta)\Phi. \end{cases} \tag{4.3}$$

Then, again thanks to a result of Takei ([4, Theorem 3]), we find the following.

Theorem 4.1. *The completely integrable system* (4.3) *can be transformed to*

$$\begin{cases} \eta^{-1}\dfrac{\partial}{\partial \widetilde{z}_1}\widetilde{\Phi} = \begin{pmatrix} -\widetilde{z}_1 & 0 \\ 0 & \widetilde{z}_1 \end{pmatrix}\widetilde{\Phi}, \\ \eta^{-1}\dfrac{\partial}{\partial \widetilde{z}_2}\widetilde{\Phi} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\widetilde{\Phi} \end{cases}$$

by a change of coordinate $(\widetilde{z}_1, \widetilde{z}_2) = (\widetilde{z}_1(z), \widetilde{z}_2(z))$ *and a Borel-transformable transformation of the form*

$$\widetilde{\Phi} = \exp(\eta f(z)) \left(\sum_{n=0} \eta^{-n} \widetilde{T}_n(z) \right) \Phi,$$

where $f(z)$ *is a holomorphic function near* $z = (a_1(c), c)$ *and* $\widetilde{T}_n(z)$ $(n = 0, 1, \ldots)$ *are* 2×2 *matrices with holomorphic entries near* $z = (a_1(c), c)$ *satisfying* $\det \widetilde{T}_0(z) \neq 0$.

Proof. It follows from the definition of $D_2(z, \theta)$ that $z_1 = a_1(z_2)$ is a double turning point of the first equation of (4.3). Hence what remains to show is to confirm

$$\operatorname{rank}(\widetilde{R}_0(a_1(z_2), z_2) - \zeta_{1,i}(a_1(z_2), z_2)) = 0. \tag{4.4}$$

(Cf. [4, Theorem 3].)

To confirm this, we first note that (4.2) entails the existence of 2×2 matrix $\widetilde{T}(z_2)$ with holomorphic entries near $z_2 = c$ such that

$$\widetilde{T}(z_2)^{-1} \widetilde{S}_0(a_1(a_2), z_2) \widetilde{T}(z_2) = \begin{pmatrix} \zeta_{2,i}(a_1(z_2), z_2) & 0 \\ 0 & \zeta_{2,i'}(a_1(z_2), z_2) \end{pmatrix}.$$

Since the first and second equations of (4.3) are compatible, we have

$$\frac{\partial \widetilde{R}}{\partial z_2} - \frac{\partial \widetilde{S}}{\partial z_1} + \eta[\widetilde{R}, \widetilde{S}] = 0.$$

Hence $[\widetilde{R}_0, \widetilde{S}_0] = 0$. Then we obtain

$$\widetilde{T}(z_2)^{-1} \widetilde{R}_0(a_1(z_2), z_2) \widetilde{T}(z_2) = \begin{pmatrix} \zeta_{1,i}(a_1(z_2), z_2) & 0 \\ 0 & \zeta_{1,i'}(a_1(z_2), z_2) \end{pmatrix}.$$

Note that $\zeta_{1,i}(a_1(z_2), z_2) = \zeta_{1,i'}(a_1(z_2), z_2)$ means $\zeta_{1,i}(a_1(z_2), z_2)$ is the value at $z = (a_1(z_2), z_2)$ of the two merging eigenvalues of $\widetilde{R}_0(z)$. We thus have

$$\begin{aligned} &\operatorname{rank}(\widetilde{R}_0(a_1(z_2), z_2) - \zeta_{1,i}(a_1(z_2), z_2)) \\ =&\operatorname{rank}\left(\begin{pmatrix} \zeta_{1,i}(a_1(z_2), z_2) & 0 \\ 0 & \zeta_{1,i'}(a_1(z_2), z_2) \end{pmatrix} - \begin{pmatrix} \zeta_{1,i}(a_1(z_2), z_2) & 0 \\ 0 & \zeta_{1,i}(a_1(z_2), z_2) \end{pmatrix} \right) \\ =&\operatorname{rank}\begin{pmatrix} 0 & 0 \\ 0 & \zeta_{1,i'}(a_1(z_2), z_2) - \zeta_{1,i}(a_1(z_2), z_2) \end{pmatrix} \\ =&0. \end{aligned}$$

Thus Theorem 3 of [4] proves Theorem 4.1. □

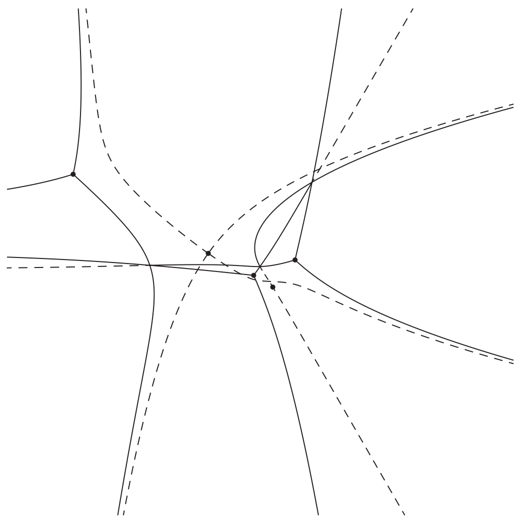

Figure 3. The non-hereditary turning point and the Stokes curve emanating from it being added to Figure 1. The dotted part of a Stokes curve indicates that it is inert.

Theorem 4.1 means that the completely integrable system (4.3) is locally isomorphic to a direct sum of two trivial systems, that is,

$$\begin{cases} \eta^{-1}\dfrac{\partial}{\partial \tilde{z}_1}\tilde{\phi}_1 = -\tilde{z}_1\tilde{\phi}_1, \\ \eta^{-1}\dfrac{\partial}{\partial \tilde{z}_2}\tilde{\phi}_1 = -\tilde{\phi}_1, \end{cases}$$

and

$$\begin{cases} \eta^{-1}\dfrac{\partial}{\partial \tilde{z}_1}\tilde{\phi}_2 = \tilde{z}_1\tilde{\phi}_2, \\ \eta^{-1}\dfrac{\partial}{\partial \tilde{z}_2}\tilde{\phi}_2 = \tilde{\phi}_2. \end{cases}$$

Hence we do not anticipate any Stokes phenomena of WKB solutions of $\mathcal{M}_{c,\theta}$ near its non-hereditary turning point. Actually, if only the first equation of (4.3) were studied (i.e., without the second equation of (4.3)), even if assuming (4.4) in addition, then, as Theorem 1 of [4] shows, $\widetilde{R}_n(z)$ $(n = 1, 2, \ldots)$ should contain off-diagonal components in general, which are tied up with the Stokes phenomena of WKB solutions of the first equation of (4.3). Furthermore, as Figure 3 shows, the Stokes curves emanating from the non-hereditary turning point remain inert. Thus we see that the non-hereditary turning point is a redundant one.

References

[1] H. L. Berk, W. M. Nevins and K. V. Roberts, New Stokes' line in WKB theory, *J. Math. Phys.*, **23** (1982), 988-1002.

[2] S. Hirose, On the Stokes geometry for the Pearcey system and the (1,4) hypergeometric system, *RIMS Kôkyûroku Bessatsu*, **B40** (2013), 243-292.
[3] S. Hirose, T. Kawai and Y. Takei, The article in this proceedings.
[4] Y. Takei, On a double turning point problem for systems of linear ordinary differential equations, preprint.