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Toward exact WKB analysis of nonlinear eigenvalue problems

By

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Abstract

In this paper, we study the nonlinear first order ordinary differential equation which Bender-Fring-Komijani studied as a model equation of nonlinear eigenvalue problems. We announce our recent results on the Borel summability of 0-parameter solutions of the equation in question, and, by using this result, we show that the Borel sum of a 0-parameter solution gives a separatrix, a solution which satisfies the boundary condition imposed by Bender-Fring-Komijani.

§ 1. Introduction

Bender, Fring and Komijani introduced *nonlinear eigenvalue problems* in [BFK]. As a typical example, they studied

$$y'(x) = \cos[\pi x y(x)]. \tag{1.1}$$

They pointed out that, for each n, the boundary condition

$$y(x) \sim \frac{m+1/2}{x} \quad (\text{with } m = 2n-1) \text{ as } x \to \infty \tag{1.2}$$

determines a unique solution of (1.1), and argued that the initial value $a_n := y(0)$ can be considered as the corresponding eigenvalue (see also §2.2). Based on the complex WKB method with some physically reasonable intuition, they derived the asymptotic behavior of eigenvalues:

$$a_n \sim 2^{5/6}\sqrt{n} \quad \text{as} \quad n \to \infty. \tag{1.3}$$

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One of our goals is to give a mathematically rigorous proof of the formula (1.3) by employing exact WKB analysis (see, e.g., [KT]). Although we have not succeeded in proving it, we obtain

(1) the Borel summability of the so-called 0-parameter solution of the equation (3.3) associated to (1.1), and

(2) the solution of (1.1) which satisfies the boundary condition (1.2) by an exact WKB theoretic argument

at this stage. In this paper we will give an announcement on (1) ([KoS]. Some results are already given in [S]), and give a proof of (2) (where the result (1) will be used).

This paper is organized as follows: In §2 we will briefly recall the argument given in [BFK]. We will also study more details about the asymptotic expansion of solutions near the infinity. In §3, after scaling the variable and the unknown function, we will construct 0-parameter solutions and study their properties. Their Borel summability will be explained in §4. This section is also an announcement of a result in our forthcoming paper [KoS]. In §5 we will show that one of the 0-parameter solution satisfies the boundary condition (1.2). In §6 we will give some remarks on the solution which satisfies (1.2) with an even integer m, and also the Borel summability of 0-parameter solution for $\arg\eta \neq 0$.

§ 2. Nonlinear eigenvalue problems

In [BFK], Bender, Fring and Komijani studied

$$y'(x) = \cos[\pi x y(x)] \tag{2.1}$$

as one example of nonlinear eigenvalue problems. To this equation we expect that the solution behaves like

$$y(x) \sim \frac{m+1/2}{x} \tag{2.2}$$

with some integer m (so that $y'(x)$ tends to zero), as x tends to the infinity. Figure 1 shows a result of numerical computations[1] of the initial value problem of (2.1). We can see from this figure that each solution approaches to a curve $xy =$ (const). An interesting observation made by [BFK] (see also [BO]) is that these asymptotic curves are $\{xy = m + 1/2\}$ with an *even* integer m (cf. Figure 2). In fact there exists one and only one solution which satisfies (2.2) with an *odd* integer m. In this section, basically following [BFK] (see also [K]), we recall some results on solutions of (2.1) with (2.2).

[1]Numerical computations in this paper were done by Mathematica.

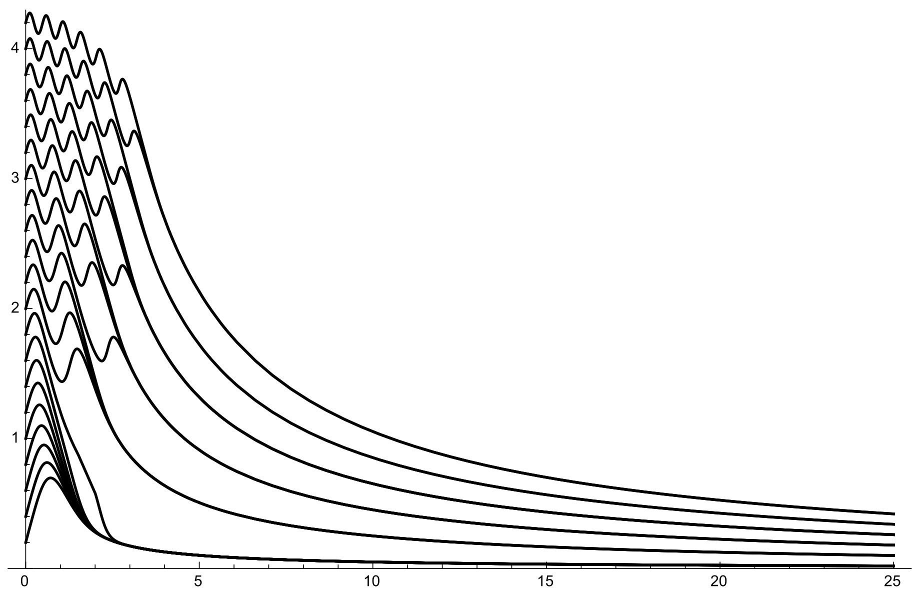

Figure 1. Graph of the solutions of (2.1) with $y(0) = 0.2k$ $(k = 1, 2, \ldots 21)$.

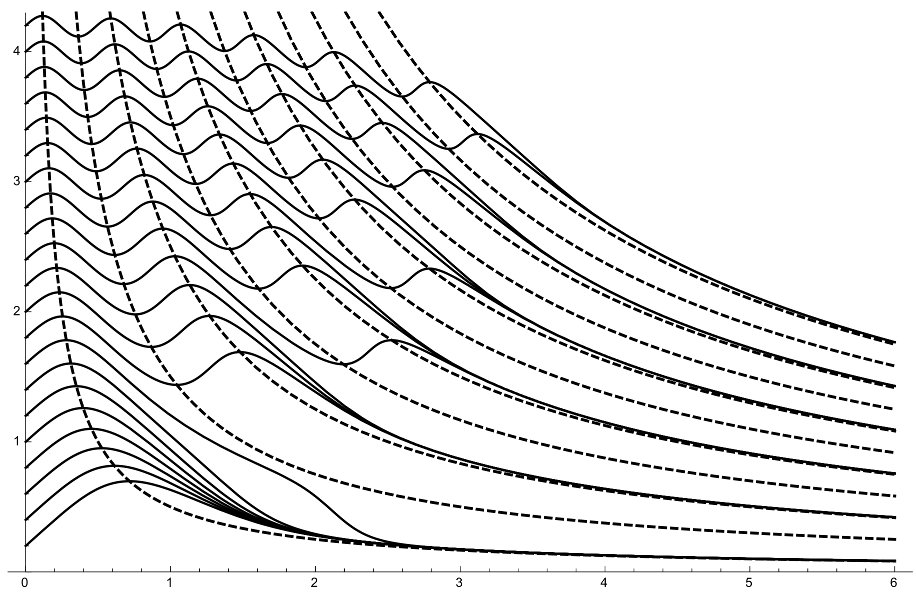

Figure 2. The solid curves are graph of the solutions of (2.1) with $y(0) = 0.2k$ $(k = 1, 2, \ldots 21)$. The dashed curves are $y = (m + 1/2)/x$ $(m = 0, 1, \ldots, 10)$.

§ 2.1. Asymptotic expansions of the solution

In [BFK, §I] it is claimed that

Proposition 2.1. *For any solution $y(x)$ of (2.1) with a positive initial value at the origin, there exists an positive integer m such that $y(x)$ has an asymptotic expansion*

$$y(x) \sim \frac{m+1/2}{x} + \sum_{k=1}^{\infty} \frac{c_k}{x^{2k+1}} \tag{2.3}$$

as x tends to $+\infty$ along the positive real axis.

We can determine coefficients of the asymptotic expansion (2.3) uniquely and recursively by substituting (2.3) into (2.1), and comparing both sides degree by degree. First four coefficients are

$$c_1 = \frac{(-1)^m}{\pi}(m+1/2), \tag{2.4}$$

$$c_2 = \frac{3}{\pi^2}(m+1/2), \tag{2.5}$$

$$c_3 = (-1)^m \left[\frac{(m+1/2)^3}{6\pi} + \frac{15(m+1/2)}{\pi^3} \right], \tag{2.6}$$

$$c_4 = \frac{8(m+1/2)^3}{3\pi^2} + \frac{105(m+1/2)}{\pi^4}, \tag{2.7}$$

and, in general, we obtain

Proposition 2.2. *The coefficients c_k in (2.3) for $k \geq 1$ are determined by the recursive relations $c_1 = ((-1)^m/\pi)c_0$ and*

$$c_k = -\frac{1}{\pi} \sum_{1 \leq l \leq (k-1)/2} \frac{(-1)^l \pi^{2l+1}}{(2l+1)!} \sum_{\substack{k_1+\cdots+k_{2l+1}=k \\ k_1,\ldots,k_{2l+1} \geq 1}} c_{k_1} \cdots c_{k_{2l+1}} + \frac{(-1)^m(2k-1)}{\pi} c_{k-1} \tag{2.8}$$

for $k \geq 2$ with $c_0 = m + 1/2$.

Proof. We set $y(x) = c_0/x + w(x)$ with $w(x) = \sum_{k=1}^{\infty} c_k/x^{2k+1}$ and substitute it into (2.1). Because

$$y'(x) = -c_0/x^2 + w'(x) = -c_0/x^2 + O(1/x^4) \ (x \to \infty) \tag{2.9}$$

and

$$\begin{aligned} \cos[\pi x y(x)] &= \cos(c_0\pi)\cos(\pi x w(x)) - \sin(c_0\pi)\sin(\pi x w(x)) \\ &= \cos(c_0\pi) - (\sin(c_0\pi))\pi c_1/x^2 + O(1/x^4) \ (x \to \infty), \end{aligned} \tag{2.10}$$

we obtain $\cos(c_0\pi) = 0$ as a leading term of (2.1). Therefore $c_0 = m+1/2$ for an integer m. Then (2.1) becomes

$$w'(x) - (m + 1/2)/x^2 = (-1)^{m+1} \sin(\pi x w(x)). \tag{2.11}$$

After expanding the right-hand side of (2.11), we obtain (2.8). □

Let us determine the asymptotic behavior of c_k as m tends to the infinity.

Proposition 2.3. *Let $\{c_k\}_{k\geq 1}$ be a sequence satisfying (2.8). We regard each c_k as a function of c_0. Then*

(i) *c_k $(k \geq 1)$ is a polynomial of c_0 without a constant term. Its degree is k when k is odd, and $k-1$ when k is even.*

(ii) *The coefficient of the linear term of c_k $(k \geq 1)$ with respect to c_0 is*

$$(-1)^{mk} \left(\frac{2}{\pi}\right)^k \frac{\Gamma(k+1/2)}{\Gamma(1/2)}. \tag{2.12}$$

(iii) *Let d_k be the coefficient of the highest degree term of c_k. Then, for $l \geq 1$, we obtain*

$$d_{2l-1} = \frac{1}{2l-1}\binom{-1/2}{l-1}, \tag{2.13}$$

$$d_{2l} = \frac{(-1)^{l-1}}{\pi^2} \sum_{j=0}^{l-1} \frac{4l-4j-1}{2l-2j-1} \binom{1/2}{l-j-1}\binom{-1/2}{j}. \tag{2.14}$$

In particular, each d_k is non-zero. Here $\binom{\alpha}{k}$ is a binomial coefficient, that is, $\binom{\alpha}{k} = \alpha(\alpha-1)\cdots(\alpha-k+1)/k!$.

Because $c_0 = m + 1/2$ in our case, the asymptotic behavior of c_k as m tends to the infinity is

$$c_k = \begin{cases} d_k m^k \left\{1 + O\left(\dfrac{1}{m}\right)\right\} & (k\text{: odd}), \\ d_k m^{k-1} \left\{1 + O\left(\dfrac{1}{m}\right)\right\} & (k\text{: even}) \end{cases} \tag{2.15}$$

with a non-zero constant d_k.

Proof. The induction shows c_k is a polynomial of c_0 with the degree described in (i). If $c_0 = 0$, then we get $c_k = 0$ for any k from the recursion relation (2.8). Therefore c_k $(k \geq 1)$ has no constant term. Thus we obtain (i).

Because c_k has no constant term, products of c_k's do not contain a linear term. Hence the recursion relation (2.8) gives

$$\bigl(\text{the linear term of } c_k\bigr) = (-1)^m \frac{2k-1}{\pi}\bigl(\text{the linear term of } c_{k-1}\bigr) \tag{2.16}$$

for $k \geq 1$. Thus we obtain (ii).

Finally, let us prove (iii). Because $c_1 = (-1)^m c_0/\pi$, we have $d_1 = (-1)^m/\pi$. Since

$$\sum_{\substack{k_1+\cdots+k_{2l+1}=k \\ k_1,\ldots,l_{2l+1}\geq 1}} c_{k_1}\cdots c_{k_{2l+1}} = \left(\sum_{\substack{k_1+\cdots k_{2l+1}=k \\ k_1,\ldots,k_{2l+1}\geq 1:\text{odd}}} d_{k_1}\cdots d_{k_{2l+1}}\right) {c_0}^k + \bigl(\text{lower order terms}\bigr). \tag{2.17}$$

holds when $k \geq 3$ is odd, we obtain

$$d_k = -\frac{1}{\pi}\sum_{1\leq l\leq (k-1)/2} \frac{(-1)^l \pi^{2l+1}}{(2l+1)!} \sum_{\substack{k_1+\cdots k_{2l+1}=k \\ k_1,\ldots,k_{2l+1}\geq 1:\text{odd}}} d_{k_1}\cdots d_{k_{2l+1}}, \tag{2.18}$$

or

$$\sum_{0\leq l\leq (k-1)/2} \frac{(-1)^l \pi^{2l+1}}{(2l+1)!} \sum_{\substack{k_1+\cdots k_{2l+1}=k \\ k_1,\ldots,k_{2l+1}\geq 1:\text{odd}}} d_{k_1}\cdots d_{k_{2l+1}} = 0. \tag{2.19}$$

Let $F(X) = \sum_{k\geq 0} d_{2k+1} X^{2k+1}$. Then (2.19) gives

$$\sin\bigl((-1)^m \pi F(X)\bigr) = X. \tag{2.20}$$

Thus

$$F(X) = \frac{(-1)^m}{\pi}\sin^{-1}(X) = \frac{(-1)^m}{\pi}\sum_{j=0}^{\infty}\frac{1}{2j+1}\binom{-1/2}{j} X^{2j+1}. \tag{2.21}$$

In a similar manner, since

$$\sum_{0\leq l\leq (k-2)/2} \frac{(-1)^l \pi^{2l+1}}{(2l)!} \sum_{\substack{k_1+\cdots k_{2l+1}=k \\ k_1,\ldots,k_{2l}\geq 1:\text{odd} \\ k_{2l+1}\geq 2:\text{even}}} d_{k_1}\cdots d_{k_{2l+1}} = (-1)^m (2k-1) d_{k-1} \tag{2.22}$$

holds for even k, $G(X) = \sum_{k\geq 1} d_{2k} X^{2k}$ is given by

$$\begin{aligned} G(X) &= \frac{(-1)^m}{\pi}\frac{X}{\sqrt{1-X^2}}\bigl\{F(X) + 2XF'(X)\bigr\} \\ &= \frac{1}{\pi^2}\left(\frac{X\sin^{-1}(X)}{\sqrt{1-X^2}} + \frac{2X^2}{1-X^2}\right). \end{aligned} \tag{2.23}$$

This completes the proof of (iii). □

§ 2.2. The separatrix and the eigenvalue

As Figure 1 and 2 illustrate, a solution which behaves like

$$y(x) \sim \frac{m+1/2}{x} \quad (x \to +\infty) \tag{2.24}$$

with $m = 2n-1$ ($n = 1, 2, \cdots$) plays a special role:

Proposition 2.4 ([BFK], § I-A.). *For $n = 1, 2, \cdots$, there exists a unique solution $y_n(x)$ of* (2.1) *which satisfies* (2.24) *with $m = 2n-1$.*

This $y_n(x)$ is called the *n-th separatrix* in [BFK], and $a_n := y_n(0)$ the *n-th eigenvalue.* To study the asymptotic behavior of the eigenvalue a_n, they introduce the scaling of variables (cf. [BFK, §III]):

$$x = \sqrt{2n - 1/2} \cdot t, \ \ y(x) = \sqrt{2n-1/2} \cdot z(t). \tag{2.25}$$

Then (2.1) is transformed to

$$z'(t) = \cos[\lambda t z(t)] \tag{2.26}$$

with

$$\lambda = (2n - 1/2)\pi. \tag{2.27}$$

Because $z(t)$ also depends on λ, we denote it by $z(t, \lambda)$ from now on. Bender-Fring-Komijani then claimed that the limit $Z(t) = \lim_{\lambda\to\infty} z(t, \lambda)$ exists, and $Z(0) = 2^{1/3}$ holds. Hence they concluded that

$$a_n = y_n(0) = \sqrt{2n-1/2} \cdot z(0, (2n-1/2)\pi) \sim 2^{1/3}\sqrt{2n} = 2^{5/6}\sqrt{n}. \tag{2.28}$$

holds as n tends to the infinity.

Bender-Fring-Komijani compared their results with the usual eigenvalue problem of the Schrödinger equation. See Table 1. Because of this analogy, they call a_n the eigenvalue of the problem.

§ 3. Exact WKB analysis and 0-parameter solutions

As is mentioned in Introduction, our goal is to prove the asymptotic behavior (2.28) of the eigenvalue a_n. Following the conventional notation of exact WKB analysis, we use η instead of λ as a large parameter (i.e., we replace λ with η in (2.26)). An immediate consequence of Proposition 2.1, Proposition 2.3, (2.25) and (2.27) is

	NEP	Schrödinger
Equation	$y'(x) = \cos[\pi x y(x)]$	$-\psi''(x) + x^4\psi(x) = E\psi(x)$
Boundary condition	$y(x) \sim \frac{2n-1/2}{x}$ $(x \to \infty)$	$\lim_{x\to\pm\infty} \psi(x) = 0$
Eigenvalue	$a_n = y(0)$	$E_n = E$
Asymptotic behavior $(n \to \infty)$	$a_n \sim 2^{5/6}\sqrt{n}$	$E_n \sim 3\Gamma(3/4)\sqrt{\pi}n^{4/3}/\Gamma(1/4)$
Number of nodes of n-th eigenfunction	n nodes	n nodes

Table 1. Nonlinear eigenvalue problems (NEP) and well-known eigenvalue problems of Schrödinger equation.

Proposition 3.1. *The equation* (2.26) *has a formal solution of the form*

$$z(t,\eta) = \frac{1}{t}(1 + \eta^{-1}\tilde{u}(t,\eta)) \quad \textit{with} \quad \tilde{u}(t,\eta) = \sum_{l=1}^{\infty} \tilde{c}_l(\eta)\frac{1}{t^{2l}}. \tag{3.1}$$

Here $\tilde{c}_l(\eta)$ is a polynomial in η^{-1} of degree $l-1$.

Because of this property, we transform the unknown function by

$$z(t,\eta) = \frac{1}{t}(1 + \eta^{-1}u(t,\eta)) \tag{3.2}$$

to remove the term $1/t$ from $z(t,\eta)$. Then the resulting equation

$$\eta^{-1}\frac{\partial}{\partial t}u(t,\eta) = t\sin u(t,\eta) + \frac{1}{t} + \eta^{-1}\frac{u(t,\eta)}{t} \tag{3.3}$$

has a suitable form for WKB analysis. We now construct the so-called 0-parameter solution (cf. [KT]) of (3.3).

Proposition 3.2. *The equation* (3.3) *has a formal power series solution*

$$\hat{u}(t,\eta) = \sum_{j=0}^{\infty} \eta^{-j}u_j(t) \tag{3.4}$$

with respect to η. Here $u_0(t)$ satisfies

$$\sin u_0(t) = -\frac{1}{t^2}, \tag{3.5}$$

and $u_j(t)$ $(j \geq 1)$ are (multi-valued) holomorphic functions in $U^* = \mathbb{C} \setminus \{0, \pm 1, \pm i\}$, which are determined uniquely and recursively once we fix a solution of (3.5). *Furthermore, for any compact set K in U^*, there exist positive constants A_K, C_K such that*

$$\sup_{t\in K}|u_{j+1}(t)| \leq A_K {C_K}^j j! \ (j = 0, 1, 2, \ldots). \tag{3.6}$$

Proof. By substituting (3.4) into (3.3), we obtain

$$\sum_{j=0}^{\infty}\eta^{-j-1}\frac{\partial}{\partial t}u_j(t) = t\left(\sin u_0(t)\cos\sum_{j=1}^{\infty}\eta^{-j}u_j(t)+\cos u_0(t)\sin\sum_{j=1}^{\infty}\eta^{-j}u_j(t)\right) + \frac{1}{t}+\frac{1}{t}\sum_{j=1}^{\infty}\eta^{-j-1}u_j(t). \tag{3.7}$$

The leading term of (3.7) with respect to η gives (3.5). Therefore

$$u_0(t) = -\sin^{-1}\left(\frac{1}{t^2}\right) = i\log\left(\frac{i}{t^2}+\sqrt{1-\frac{1}{t^4}}\right). \tag{3.8}$$

We fix a branch of $\sin^{-1}$ in (3.8) so that

$$\sin^{-1}\frac{1}{t^2} = N\pi + (-1)^N\sum_{n=0}^{\infty}\frac{(2n)!}{(n!)^2 4^n(2n+1)}\left(\frac{1}{t^2}\right)^{2n+1} \tag{3.9}$$

holds for $t>1$, where N is an integer. As t tends to $+\infty$, $u_0(t)\to -N\pi$, and hence $\cos u_0(t)\to(-1)^N$. Therefore

$$\cos u_0(t) = (-1)^N\sqrt{1-\sin^2 u_0(t)} = (-1)^N\frac{\sqrt{t^4-1}}{t^2}, \tag{3.10}$$

where we have chosen a branch of square root such that $\sqrt{t^4-1}>0$ for $t>1$.

The terms of η^{-1} in (3.7) give

$$u_0'(t) = t(\cos u_0(t))u_1(t)+\frac{u_0(t)}{t}. \tag{3.11}$$

By (3.10) and

$$u_0'(t) = (-1)^N\frac{2}{t\sqrt{t^4-1}}, \tag{3.12}$$

we obtain

$$u_1(t) = \frac{2}{t^4-1}+(-1)^N\frac{1}{\sqrt{t^4-1}}\sin^{-1}\frac{1}{t^2}. \tag{3.13}$$

Finally the terms of η^{-j-1} $(j\geq 1)$ in (3.7) give

$$u_j'(t) = t(\sin u_0(t))\Phi_{1,j}(t)+t(\cos u_0(t))(u_{j+1}(t)+\Phi_{2,j}(t))+\frac{u_j(t)}{t}. \tag{3.14}$$

Here $\Phi_{1,j}(t)$ is a coefficient of η^{-j-1} in

$$\cos\left(\sum_{j=1}^{\infty}\eta^{-j}u_j(t)\right), \tag{3.15}$$

and $\Phi_{2,j}(t)$ is a coefficient of η^{-j-1} in

$$\sin\left(\sum_{j=1}^{\infty}\eta^{-j}u_j(t)\right)-\sum_{j=1}^{\infty}\eta^{-j}u_j(t). \tag{3.16}$$

Their explicit forms are

$$\Phi_{1,j}(t)=\sum_{1\le k\le (j+1)/2}\left\{\frac{(-1)^k}{(2k)!}\sum_{\substack{l_1+\cdots+l_{2k}=j+1\\ l_i\ne 0}}u_{l_1}(t)\cdots u_{l_{2k}}(t)\right\} \tag{3.17}$$

and

$$\Phi_{2,j}(t)=\sum_{1\le k\le j/2}\left\{\frac{(-1)^k}{(2k+1)!}\sum_{\substack{l_1+\cdots+l_{2k+1}=j+1\\ l_i\ne 0}}u_{l_1}(t)\cdots u_{l_{2k+1}}(t)\right\}. \tag{3.18}$$

Therefore each $u_{j+1}(t)$ is determined from (3.14) recursively and uniquely once we fix a branch of $u_0(t)$.

It also follows from (3.14) and the Cauchy integral formula that, for any point $t_0\in U^*$ and for any positive number r such that a closed disk $\{t\in\mathbb{C}\mid |t-t_0|\le r\}$ is included in U^*, there exist positive constants C_1, C_2 for which

$$\sup_{|t-t_0|<r-\varepsilon}\left|u_{j+1}(t)\right|\le C_1\left(\frac{C_2}{\varepsilon}\right)^j j! \tag{3.19}$$

holds for any $j=0,1,2,\cdots$ and any positive ε small enough. Since this is somewhat a routine task, and we omit its detail here. Estimates (3.6) follow from (3.19). □

Here we summarize the results obtained in this proof:

$$u_0(t)=-\sin^{-1}\frac{1}{t^2},\quad u_1(t)=\frac{2}{t^4-1}+(-1)^N\frac{1}{\sqrt{t^4-1}}\sin^{-1}\frac{1}{t^2}, \tag{3.20}$$

$$u_{j+1}(t)=(-1)^N\frac{1}{\sqrt{t^4-1}}\{tu_j'(t)-u_j(t)+\Phi_{1,j}(t)\}-\Phi_{2,j}(t). \tag{3.21}$$

It follows from (3.20) and (3.21) that each $u_j(t)$ is holomorphic except at $t\ne 0,\pm1,\pm i$. It is also holomorphic near the infinity. By substituting the Taylor expansion of $u_j(t)$ near $t=\infty$ into (3.4), $u(t,\eta)$ becomes a double power series of t^{-1} and η^{-1}.

Proposition 3.3. *The solution $\hat{u}(t,\eta)$ in Proposition 3.2 coincides with $\tilde{u}(t,\eta)$ in Proposition 3.1 as a formal power series in t^{-1} and η^{-1}. Here the branch of $u_0(t)$ in $\hat{u}(t,\eta)$ is chosen as the principal value, that is, $N=0$ in* (3.9).

Proof. Because both $\hat{u}(t,\eta)$ and $\tilde{u}(t,\eta)$ are formal solutions of (3.3), and because they have the form

$$u(t,\eta) = \sum_{j,k=0}^{\infty} \hat{u}_{j,k}\eta^{-j}t^{-k} \quad \text{with} \quad \hat{u}_{0,0} = 0, \tag{3.22}$$

it is enough to prove the uniqueness of the formal solution of the form (3.22).

By substituting (3.22) into (3.3) and comparing both sides degree by degree, we obtain

$$\hat{u}_{1,0} = \hat{u}_{0,1} = 0, \tag{3.23}$$

$$\hat{u}_{2,0} = \hat{u}_{1,1} = 0,\ \hat{u}_{0,2} = -1, \tag{3.24}$$

$$\hat{u}_{3,0} = \hat{u}_{2,1} = \hat{u}_{1,2} = \hat{u}_{0,3} = 0 \tag{3.25}$$

and for $l \geq 3$,

$$\begin{aligned}\hat{u}_{l+1}(t,\eta) = &-\sum_{j+k+2=l, j\geq 0, k\geq 1} k\hat{u}_{j,k}\eta^{-j-1}t^{-k-2} + \eta^{-1}t^{-2}\hat{u}_{l-2}(t,\eta) \\ &+ \sum_{1\leq m\leq l/2} \frac{(-1)^m}{(2m+1)!} \sum_{j_1+\cdots+j_{2m+1}=l+1, j_n\geq 1} \hat{u}_{j_1}(t,\eta)\cdots\hat{u}_{j_{2m+1}}(t,\eta).\end{aligned} \tag{3.26}$$

Here we set $\hat{u}_l(t,\eta) = \sum_{j+k=l,\ j,k\geq 0} \hat{u}_{j,k}\eta^{-j}t^{-k}$. By (3.26), $\hat{u}_{l+1}(t,\eta)$ is determined uniquely from $\{\hat{u}_{j,k}\}_{j+k\leq l-1}$. Therefore once $\{\hat{u}_{j,k}\}_{j+k\leq l}$ is given, then $\{\hat{u}_{j,k}\}_{j+k=l+1}$ is determined uniquely. By induction with respect to l, we conclude that there exists a unique formal solution (3.22) of (3.3). □

§ 4. Borel resummation method and Stokes geometry

§ 4.1. Borel resummation method

To give an analytic meaning to the 0-parameter solutions constructed in Proposition 3.2, we employ the Borel resummation method. In this subsection we recall its definition and basic properties. See, e.g., [KT] for more detailed explanation.

Definition 4.1 (Borel transform). For a power series

$$f(\eta) = \sum_{j=1}^{\infty} f_j\eta^{-j} \tag{4.1}$$

of a large parameter η, we define the *Borel transform* of f by

$$f_B(y) := \sum_{j=1}^{\infty} \frac{f_j}{(j-1)!}y^{j-1}. \tag{4.2}$$

If f_B converges in a neighborhood of $y = 0$, f is said to be *Borel transformable.*

Definition 4.2 (Borel sum). When a power series (4.1) satisfies the following three conditions, f is said to be *Borel summable.*

(i) A power series f is Borel transformable.

(ii) There exists $\delta > 0$ such that the Borel transform $f_B(y)$ can be analytically continued into

$$\Sigma(\delta) := \{z \in \mathbb{C} \mid \operatorname{dist}(z, \mathbb{R}_+) < \delta\}. \tag{4.3}$$

(iii) For each $0 < \delta' < \delta$, there exist positive constants B_1 and B_2 such that

$$\bigl|f_B(y)\bigr| \leq B_1 e^{B_2|y|} \tag{4.4}$$

holds for any $y \in \overline{\Sigma(\delta')}$.

When f is Borel summable, we define its *Borel sum* $F(\eta)$ as the Laplace transform

$$F(\eta) := \int_0^\infty e^{-\eta y} f_B(y) dy \tag{4.5}$$

of $f_B(y)$. Here the path of integration is chosen to be the real axis.

If a power series has a constant term with respect to η^{-1} like $u = \sum_{j=0}^\infty u_j \eta^{-j}$, we define

$$(\text{the Borel transform of } u) := v_B, \quad \text{where} \quad v := u - u_0, \tag{4.6}$$

$$(\text{the Borel sum of } u) := u_0 + (\text{the Borel sum of } v). \tag{4.7}$$

If f is a convergent power series, f coincides with its Borel sum. When f is a divergent power series, we obtain

Theorem 4.3. *If* (4.1) *is Borel summable, the power series f is the asymptotic expansion of Borel sum $F(\eta)$ in $\{\eta \in \mathbb{C} \mid \operatorname{Re}\eta > B_2\}$: for any $\varepsilon > 0$, there exists positive constants A, B such that*

$$\left| F(\eta) - \sum_{j=1}^N f_j \eta^{-j} \right| < AB^{N+1}(N+1)!|\eta|^{-N-1} \tag{4.8}$$

holds for $N = 0, 1, \ldots$ and $\operatorname{Re}\eta \geq B_2 + \varepsilon$.

We refer to [E, Theorem 4.4] for the proof.

Our formal solution (3.4) is a formal power series with respect to η whose coefficients depend on t. To discuss a power series of this kind, we define

Definition 4.4. Let $f(t,\eta) = \sum_{j\geq 1} f_j(t)\eta^{-j}$ be a formal power series whose coefficient $f_j(t)$ is a holomorphic function in some domain $\Omega \subset \mathbb{C}$. We say $f(t,\eta)$ is Borel summable with respect to η uniformly in Ω if

(i) $f(t,\eta)$ is Borel transformable for each $t \in \Omega$.

(ii) There exists $\delta > 0$ such that the Borel transform $f_B(t,y)$ can be analytically continued into $\Omega \times \Sigma(\delta)$.

(iii) For any $0 < \delta' < \delta$ and any compact set $K \subset \Omega$, there exist positive constants B_1 and B_2 such that $|f_B(t,y)| \leq B_1 e^{B_2|y|}$ holds for any $(t,y) \in K \times \overline{\Sigma(\delta')}$.

§ 4.2. Stokes geometry

The leading order part of the linearized equation of (3.3) at $u_0(t)$ (i.e., an equation obtained by substituting $u(t,\eta) = u_0(t) + (\Delta u)(t,\eta)$ into (3.3) and eliminating both non-linear terms with respect to Δu and the lower order terms with respect to η) is

$$\eta^{-1}(\Delta u)' = t\cos\big(u_0(t)\big)\Delta u = (-1)^N \frac{\sqrt{t^4-1}}{t}\Delta u. \tag{4.9}$$

Here we have used (3.10). In analogy with the Riccati equations, we define Stokes geometry of (3.3) by

$$Q(t) := \left\{(-1)^N \frac{\sqrt{t^4-1}}{t}\right\}^2 = \frac{t^4-1}{t^2}, \tag{4.10}$$

that is, we define a **turning point** as a zero of $Q(t)$ and a **Stokes curves** as a curve emanating from a turning point a satisfying

$$\operatorname{Im} \int_a^t \sqrt{Q(s)}ds = 0. \tag{4.11}$$

Note that, in our case, Stokes geometry does not depend on N, i.e., a choice of a branch of $u_0(t)$, as $Q(t)$ does not depend on N. We also define a *Stokes region* as a domain surrounded by Stokes curves. Figure 3 shows the Stokes geometry of (3.3). There exist five Stokes regions. This Stokes geometry is degenerate in the sense that there exists Stokes curve which connect two turning points. As in the case of second order linear differential equations, the sign of $\operatorname{Re} \int_a^t \sqrt{Q(s)}ds$ does not change on a Stokes curve emanating from a turning point a (cf. [KT]).

Let $X = \mathbb{C} \setminus \{0, \pm 1, \pm i\}$, a set of points which are not turning points nor a singular point. Following [KoSc] we introduce two notions. First one is

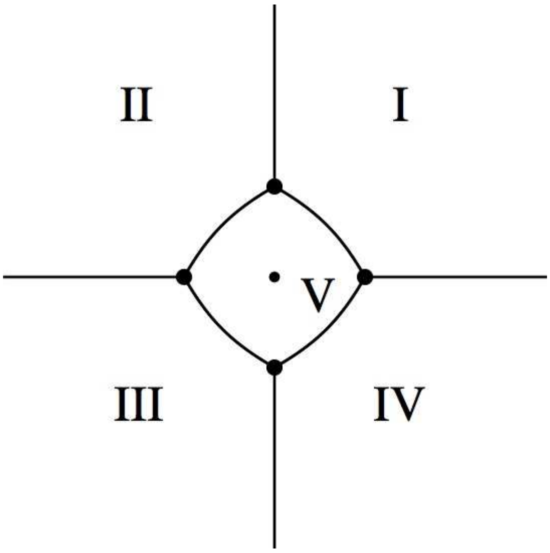

Figure 3. Stokes curves of (3.3).

Definition 4.5. For any $t_0 \in X$, the *level curve* Γ_{t_0} is defined as a curve passing through t_0 and satisfying

$$\operatorname{Im} \int_{t_0}^{t} \sqrt{Q(s)}ds = 0. \tag{4.12}$$

We also define its positive (resp., negative) component of the level curve Γ_{t_0} by

$$\Gamma_{t_0}^{(+)} := \left\{ t \in \Gamma_{t_0} \,\middle|\, \operatorname{Im} \int_{t_0}^{t} \sqrt{\frac{s^4-1}{s^2}}ds = 0, \operatorname{Re} \int_{t_0}^{t} \sqrt{\frac{s^4-1}{s^2}}ds \geq 0 \right\} \tag{4.13}$$

$$\left(\text{resp.,} \quad \Gamma_{t_0}^{(-)} := \left\{ t \in \Gamma_{t_0} \,\middle|\, \operatorname{Im} \int_{t_0}^{t} \sqrt{\frac{s^4-1}{s^2}}ds = 0, \operatorname{Re} \int_{t_0}^{t} \sqrt{\frac{s^4-1}{s^2}}ds \leq 0 \right\} \right). \tag{4.14}$$

See Figure 4 for examples of level curves of $Q(t)$. (We choose one point from each Stokes regions, and draw a level curve passing through it.)

The second notion we introduce is

Definition 4.6. For a domain $\Omega \subset X$, we define a *Stokes closure* of Ω by

$$\hat{\Omega} := \bigcup_{t \in \Omega} \Gamma_t. \tag{4.15}$$

We also define the *positive* (resp., *negative*) *component* of $\hat{\Omega}$ by

$$\hat{\Omega}^{(+)} := \bigcup_{t \in \Omega} \Gamma_t^{(+)} \quad \left(\text{resp.,} \quad \hat{\Omega}^{(-)} := \bigcup_{t \in \Omega} \Gamma_t^{(-)} \right). \tag{4.16}$$

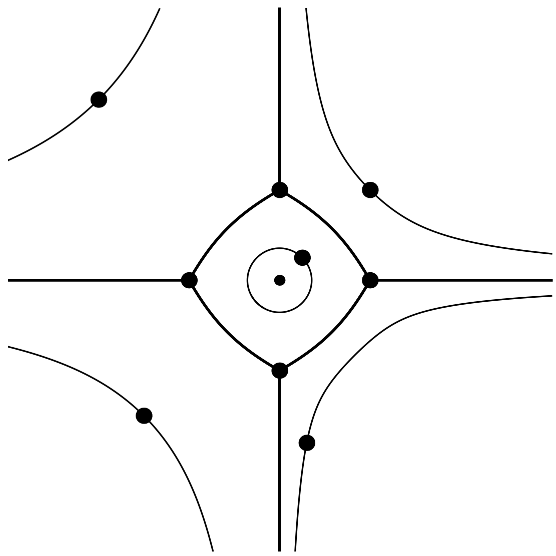

Figure 4. Stokes curves and level curves of (3.3).

§ 4.3. Borel summability of 0-parameter solutions

In this subsection we state some results on the Borel summability of the formal solution (3.4) of (3.3). The following Theorem 4.8 is an announcement of [KoS].

A first result on the Borel summability is

Theorem 4.7 ([S]). *In each Stokes region* I,II,III *or* IV *in Figure 3, the formal solution* (3.4) *of* (3.3) *is Borel summable uniformly.*

To state this theorem more precise, we first fix the branch of $u_0(t)$: we place cuts as in Figure 5 and choose an integer N such that

$$u_0(t) = -\sin^{-1}(1/t^2) \to -N\pi \tag{4.17}$$

as t tends to the infinity along the positive real axis. Second we define $w(t)$ by

$$\eta^{-1}w(t,\eta) = u(t,\eta) - u_0(t) - \eta^{-1}u_1(t). \tag{4.18}$$

Then Theorem 4.7 follows from

Theorem 4.8 ([KoS]). *Let* Ω *be an open or closed region in* $X = \mathbb{C}\backslash\{0, \pm 1, \pm i\}$.

(I) ***In the case when*** N ***is even***:

We further assume that (i) the level curve $\Gamma_{t_0}^{(+)}$ *flows into* ∞ *for each* $t_0 \in \Omega$, *and (ii) the (usual) closure of* $\hat{\Omega}^{(+)}$ *does not contain turning points* $\pm 1, \pm i$. *Then there exist* $\delta, B_1, B_2 > 0$ *such that the Borel transform* $w_B(t, y)$ *satisfies*

$$|w_B(t,y)| \le \frac{B_0}{|t|^2} e^{B_1|y|} \quad ((t,y) \in \hat{\Omega}^{(+)} \times \Sigma(\delta)). \tag{4.19}$$

Especially the formal solution (3.4) *is Borel summable uniformly in* $\hat{\Omega}^{(+)}$.

(II) ***In the case when*** N ***is odd***:

We further assume that (i) the level curve $\Gamma_{t_0}^{(-)}$ *flows into* ∞ *for each* $t_0 \in \Omega$, *and (ii) the (usual) closure of* $\hat{\Omega}^{(-)}$ *does not contain turning points* $\pm 1, \pm i$. *Then there exist* $\delta, B_1, B_2 > 0$ *such that the Borel transform* $w_B(t, y)$ *satisfies*

$$|w_B(t,y)| \le \frac{B_0}{|t|^2} e^{B_1|y|} \quad ((t,y) \in \hat{\Omega}^{(-)} \times \Sigma(\delta)). \tag{4.20}$$

Especially the formal solution (3.4) *is Borel summable uniformly in* $\hat{\Omega}^{(-)}$.

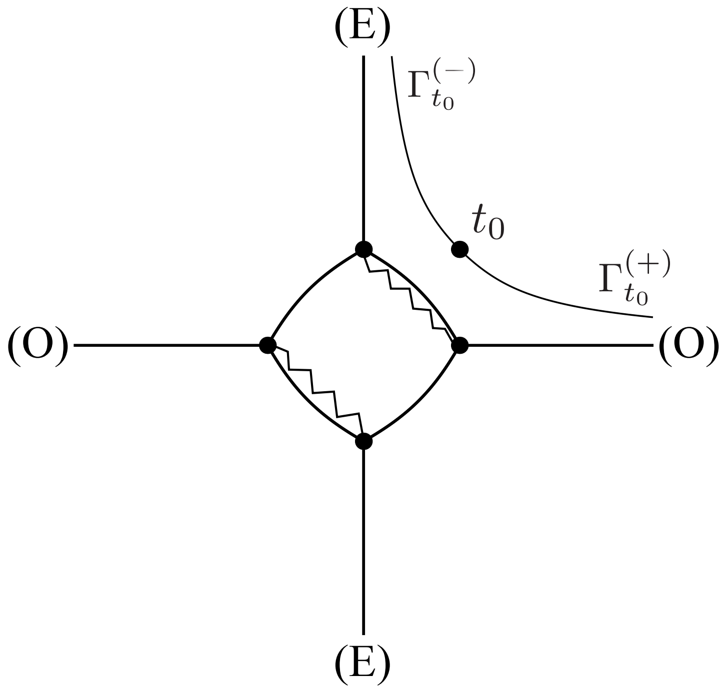

Figure 5. The level curve Γ_{t_0} and its positive and negative component.

We can show that any compact set K included in Region I, II, III or IV satisfies the assumption of Theorem 4.8: In fact, let us take a point t_0 from Region I. Then the positive and negative components of the level curve Γ_{t_0} run as shown in Figure 5, i.e., both of them flow into the infinity. Furthermore, because K is a compact set in Region I, its Stokes closure does not contain ± 1, $\pm i$. Thus Theorem 4.7 follows in this case.

Even in the case of t being on Stokes curves, the formal solution (3.4) can be Borel summable: For example when N is even, $\Gamma^{(+)}_{t_0}$ for t_0 on the Stokes curve (O) indicated in Figure 5 flows into the infinity. Hence the Borel summability of (3.4) follows from Theorem 4.8. On the other hand, when t_0 lies on the Stokes curve (E), the corresponding level curve $\Gamma^{(+)}_{t_0}$ flows into a turning point. Therefore we cannot say anything about the Borel summability in this case from Theorem 4.8.

Here we give a sketch of the proof of Theorem 4.8. Because the Borel transformability of $w(t,\eta)$ is proved in Proposition 3.2, what remains for the proof is the analytic continuation of the Borel transform of $w(t,\eta)$, and its estimates. Our argument below is based on the idea used in [S]. We refer to [KoS] for a full proof of the theorem, where a slightly different idea is used in which we follow [KoSc]. To make the argument below simple, we restrict ourselves to the case when $N = 0$.

We fix a point $t_0 \in \Omega$ and transform the variable from t to z by

$$z(t) = \int_{t_0}^{t} \sqrt{\frac{s^4-1}{s^2}}\,ds. \tag{4.21}$$

Note that the level curve Γ_{t_0} (resp., its positive component $\Gamma^{(+)}_{t_0}$) in t-plane is mapped to the real axis (resp., the positive real axis) in z-plane. We also transform the unknown functions by

$$\Phi(z(t),y) = t^2 w_B(t,y). \tag{4.22}$$

By this transformation of the variable and the unknown function, we can integrate the Borel transform of the equation (3.3) to obtain an integral-convolution equation:

$$\begin{aligned}\Phi(z,y) - \tilde{u_2}(z+y) = &-\int_0^y \tilde{F}_1(z+y-y')\Phi(z+y-y',y')dy' \\ &-\int_0^y \tilde{F}_2(z+y-y')(\Phi*\Phi)(z+y-y',y')dy' \\ &-\int_0^y \sum_{n=0}^{\infty}(\tilde{G}_{n,B}*\Phi^{*n})(z+y-y',y')dy',\end{aligned} \tag{4.23}$$

where we set $\tilde{u_2}(z(t)) = t^2 u_2(t)$, and coefficients of this equation are given as follows:

$$\tilde{F}_1(z(t)) = \frac{1}{\sqrt{t^4-1}}(3+u_1(t)), \tag{4.24}$$

$$\tilde{F}_2(z(t)) = \frac{1}{2t^2\sqrt{t^4-1}}, \tag{4.25}$$

$$\tilde{G}_{0,B}(z(t),y) = t^2\sum_{n=1}^{\infty}(-1)^n \frac{y^{2n-2}}{(2n-2)!}\left\{\frac{u_1(t)^{2n+1}}{(2n+1)!} + \frac{1}{\sqrt{t^4-1}}\frac{u_1(t)^{2n+2}}{(2n+2)!}\right\}, \tag{4.26}$$

$$\tilde{G}_{1,B}(z(t),y) = \sum_{n=1}^{\infty}(-1)^n \left\{\frac{y^{2n-2}}{(2n-2)!}\frac{u_1(t)^{2n}}{(2n)!} + \frac{1}{\sqrt{t^4-1}}\frac{y^{2n-1}}{(2n-1)!}\frac{u_1(t)^{2n+1}}{(2n+1)!}\right\}, \tag{4.27}$$

$$\tilde{G}_{2,B}(z(t),y) = \frac{1}{2t^2}\sum_{n=1}^{\infty}(-1)^n \left\{\frac{y^{2n-2}}{(2n-2)!}\frac{u_1(t)^{2n-1}}{(2n-1)!} + \frac{1}{\sqrt{t^4-1}}\frac{y^{2n-1}}{(2n-1)!}\frac{u_1(t)^{2n}}{(2n)!}\right\}, \tag{4.28}$$

and, for $m \geq 1$,

$$\tilde{G}_{2m+1,B}(z(t),y) = \frac{(-1)^m}{(2m+1)!t^{4m}}\sum_{n=0}^{\infty}(-1)^n \left\{\frac{y^{2n+2m-2}}{(2n+2m-2)!}\frac{u_1(t)^{2n}}{(2n)!} + \frac{1}{\sqrt{t^4-1}}\frac{y^{2n+2m-1}}{(2n+2m-1)!}\frac{u_1(t)^{2n+1}}{(2n+1)!}\right\}, \tag{4.29}$$

$$\tilde{G}_{2m+2,B}(z(t),y) = \frac{(-1)^{m+1}}{(2m+2)!t^{4m+2}}\sum_{n=0}^{\infty}(-1)^n \left\{\frac{y^{2n+2m}}{(2n+2m)!}\frac{u_1(t)^{2n+1}}{(2n+1)!} - \frac{1}{\sqrt{t^4-1}}\frac{y^{2n+2m-1}}{(2n+2m-1)!}\frac{u_1(t)^{2n}}{(2n)!}\right\}. \tag{4.30}$$

Here the convolution $f * g$ is defined as

$$(f*g)(t,y) := \int_0^y f(t,y-y')g(t,y')dy' \tag{4.31}$$

for $f(t,y)$ and $g(t,y)$, and

$$f^{*n}(t,y) := (\underbrace{f*f\cdots*f}_{n\text{ times}})(t,y). \tag{4.32}$$

To obtain a solution of (4.23), we use the successive approximation: Let us define

$\{\Phi_n(z,y)\}_{n\geq 0}$ by

$$\Phi_0(z,y) = \tilde{u}_2(z+y) - \int_0^y \tilde{G}_{0,B}(z+y-y',y')dy', \tag{4.33}$$

$$\Phi_1(z,y) = -\int_0^y \tilde{F}_1(z+y-y')\Phi_0(z+y-y',y')dy', \tag{4.34}$$

$$\begin{aligned}\Phi_n(z,y) = &-\int_0^y \tilde{F}_1(z+y-y')\Phi_{n-1}(z+y-y',y')dy' \\ &-\sum_{j=0}^{n-2}\int_0^y \tilde{F}_2(z+y-y')(\Phi_j * \Phi_{n-2-j})(z+y-y',y')dy' \\ &-\sum_{k=1}^{n-1}\sum_{\substack{m_1+\cdots+m_k \\ =n-k-1}}\int_0^y (\tilde{G}_{k,B} * \Phi_{m_1} * \cdots * \Phi_{m_k})(z+y-y',y')dy'\end{aligned} \tag{4.35}$$

for $n \geq 2$. Then, by induction, we can show

Proposition 4.9. *There exist positive constants C_1, C_2, C_3 and r for which*

$$\big|\Phi_n(z,y)\big| < C_1 \frac{C_2^n |y|^n}{n!} e^{C_3|y|} \tag{4.36}$$

holds for any $(z,y) \in z(\Omega) \times \{y \mid \mathrm{dist}(y,\mathbb{R}_+) < r\}$.

By this estimates we find that

$$\Phi(z,y) := \sum_{n\geq 0} \Phi_n(z,y) \tag{4.37}$$

converges and gives a solution of (4.23). We can also show that it coincides with $t^2 w_B(t,y)$ after the change of variable (4.21), and can prove the statements of the theorem.

Remark 4.10.

(i) Estimates (4.19) and (4.20) in Theorem 4.8 become stronger than those obtained in [S] in the sense that we obtain $|t|^{-2}$ in the right-hand side. The above argument is a stronger version of that used in [S]. Theorem 4.7, however, follows from the weaker result of [S]. This vanishing factor $|t|^{-2}$ will be used in the next section.

(ii) A difference between (3.3) and the Riccati equation appears when we study (3.3) near the origin: the coefficient $u_j(t)$ in the 0-parameter solution (3.4) contains some power of $\log t$, and its degree becomes greater and greater as j increases. Such a singular point does not observed for the Riccati equation associated with the Schrödinger equation with rational potentials, and hence it is not discussed in

[KoSc]. As we will see in §6, when we vary $\arg\eta$, some of Stokes curves and level curves flow into the origin. By the exactly same reason, it is not straightforward to include Region V in Theorem 4.7. These problems will be studied in [KoS].

§ 5. A consequence of Theorem 4.8

We can construct the solution of (2.1) satisfying the boundary condition (2.2) with an odd integer m from the Borel sum of the 0-parameter solution (3.4) of (3.3), which we will see below. Note that the solution $y(x)$ of (2.1) is transformed as

(5.1)
$$\begin{aligned} y(x) &= \sqrt{2n-1/2}\left\{\frac{\sqrt{2n-1/2}}{x} + \frac{1}{\pi x\sqrt{2n-1/2}}\cdot u\left(\frac{x}{\sqrt{2n-1/2}}, (2n-1/2)\pi\right)\right\} \\ &= \frac{2n-1/2}{x} + \frac{1}{\pi x}\cdot u\left(\frac{x}{\sqrt{2n-1/2}}, (2n-1/2)\pi\right) \end{aligned}$$

in Section 2.2 and (3.2).

Theorem 5.1. *Let $U(t,\eta)$ be the Borel sum of $u(t,\eta)$ defined by* (3.4)*. Then there exist $M>0$ and $R>0$ such that*

$$y_n(x) = \frac{2n-1/2}{x} + \frac{1}{\pi x}\cdot U\left(\frac{x}{\sqrt{2n-1/2}}, (2n-1/2)\pi\right) \tag{5.2}$$

satisfies

$$\left|y_n(x) - \frac{2n-1/2}{x}\right| \le \frac{Mn}{x^3} \quad (x > Rn) \tag{5.3}$$

for any sufficiently large positive integer n. Here we choose a branch of $u_0(t)$ as $N=0$ (*cf.* (4.17))*.*

Theorem 5.1 and the uniqueness of the solution satisfying $y(x) \sim (2n-1/2)/x$ as $x \to \infty$ guarantee that (5.2) should coincide the n-th separatrix of (2.1) for sufficiently large n. (Cf. Section 2.2).

Proof. Let Ω be a region in $\mathrm{I}\cup\mathrm{IV}\cup\{t>1\}$ so that $\Omega\cap\{t>1\}$ is not empty set ($\Omega = \{t\in\mathbb{C} \mid |t-2| < 1/2\}$ is enough). It follows from Theorem 4.8 that there exist $B_0, B_1 > 0$ and $R' > 1$ such that for any $t > R'$ and any $y > 0$,

$$|w_B(t,y)| \le \frac{B_0}{t^2}e^{B_1 y} \tag{5.4}$$

holds. We can also take a constant $C_0 > 0$ such that

$$(5.5) \qquad |u_j(t)| \le \frac{C_0}{t^2} \ (j = 0, 1)$$

holds for $t > R'$ (cf. (3.20)). Let $W(t, \eta)$ be the Borel sum of $w(t, \eta)$. For $\eta \ge 2B_1$ and $t > R'$, we obtain

$$(5.6) \qquad |W(t,\eta)| \le \frac{B_0}{t^2} \int_0^\infty e^{(B_1-\eta)y} dy \le \frac{B_0}{t^2} \int_0^\infty e^{-\eta y/2} dy \le \eta^{-1} \frac{2B_0}{t^2} \le \frac{B_0}{B_1} \frac{1}{t^2}.$$

Therefore, for $t > R'$ and $\eta > 2B_1$,

$$(5.7) \qquad \begin{aligned} |U(t,\eta)| &\le |U(t,\eta) - u_0(t)| + |u_0(t)| \\ &= \eta^{-1} |u_1(t) + W(t,\eta)| + |u_0(t)| \qquad (\because (4.18)) \\ &\le \frac{1}{2B_1} \big|W(t,\eta)\big| + \big|u_0(t)\big| + \frac{1}{2B_1} \big|u_1(t)\big| \qquad (\because \eta > 2B_1) \\ &\le \frac{1}{t^2} \left\{ \frac{1}{2B_1} \left(C_0 + \frac{B_0}{B_1} \right) + C_0 \right\} \qquad (\because (5.5) \text{ and } (5.6)) \\ &=: \frac{M'}{t^2}. \end{aligned}$$

Hence

$$(5.8) \qquad \begin{aligned} \left| y_n(x) - \frac{2n - 1/2}{x} \right| &= \left| \frac{1}{\pi x} \cdot U\left(\frac{x}{\sqrt{2n - 1/2}}, (2n - 1/2)\pi \right) \right| \\ &\le \frac{1}{\pi x} \cdot \frac{M'}{(x/\sqrt{2n-1/2})^2} \\ &= \frac{M'(2n - 1/2)}{\pi x^3} \end{aligned}$$

holds for $(2n - 1/2)\pi \ge 2B_1$ and $x > R'\sqrt{2n - 1/2}$. Let $M = 2M'/\pi$ and $R = \sqrt{2}R'$. Then we conclude that

$$(5.9) \qquad \left| y_n(x) - \frac{2n - 1/2}{x} \right| \le \frac{Mn}{x^3}$$

holds for $x > Rn$ and $n > B_1/\pi + 1/4$. □

§ 6. Concluding remarks

In ending this paper we give two remarks here.

The first remark concerns the Borel summability of the 0-parameter solution in the case when $m = 2n$ in (2.3): In the previous sections we have constructed the 0-parameter solution of (3.4), and have proven that this corresponds to the solution of

(2.1) which satisfies (2.3) with $m = 2n - 1$. By the exactly same way, we can construct 0-parameter solutions when $m = 2n$. In this case the transformation of variables

$$x = \sqrt{m + 1/2}t, \tag{6.1}$$

$$y(x) = z(t)\sqrt{m + 1/2}, \tag{6.2}$$

$$\eta = (m + 1/2)\pi, \tag{6.3}$$

together with (3.2), gives

$$\eta^{-1}\frac{\partial}{\partial t}u(t,\eta) = (-1)^{m-1}t\sin u(t,\eta) + \frac{1}{t} + \eta^{-1}\frac{u(t,\eta)}{t}. \tag{6.4}$$

When $m = 2n$, the coefficient of $t\sin u(t,\eta)$ in (6.4) is -1. Because of this minus sign, although Stokes curves coincide with those for $m = 2n - 1$, the dominance relation on each Stokes curve becomes reversed. Therefore, although we can prove the Borel summability of the 0-parameter solutions in Region I, II, III and IV similarly, we cannot prove the Borel summability on $\mathbb{R}_{>1}$. This is, however, what we expect: As we saw in Figure 2, the condition (2.3) with $m = 2n$ does not determine a solution of (2.1) uniquely. Therefore the 0-parameter solutions which corresponds to $m = 2n$ should not be Borel summable on $\mathbb{R}_{>1}$.

The second remark is on the Stokes geometry when $\arg\eta \neq 0$: One way to study the analytic structure of the Borel transform of 0-parameter solutions is to vary $\arg\eta$ from 0, and see what happens for such $\arg\eta$. This method is known as the "Voros' radar method", because rotating $\arg\eta$ corresponds to rotating the path of integration of the Laplace integral to define Borel sum. Stokes curves for $\arg\eta = \theta$ is defined by

$$\operatorname{Im}\left(e^{i\theta}\int_a^t \sqrt{Q(s)}ds\right) = \operatorname{Im}\left(e^{i\theta}\int_a^t \sqrt{\frac{s^4-1}{s^2}}ds\right) = 0, \tag{6.5}$$

where a is a turning point (i.e., a zero of Q). The level curves and their positive or negative components are defined similarly. See Figure 6 for the Stokes curve when $\theta = k\pi/8$ $(-4 \le k \le 4)$. As these figures show, the degeneration of the Stokes geometry for $\theta = 0$ is resolved for $-\pi/2 \le \theta < 0$ and for $0 < \theta \le \pi/2$ (actually the degeneration of Stokes geometry occurs only when $\theta = 0 \mod \pi$.).

For $\arg\eta \neq 0 \mod \pi$, i.e., when the degeneration is resolved, Theorem 4.8 gives only a partial answer to the Borel summability of the 0-parameter solutions (cf. Remark 4.10). To make the argument concrete, we consider the case when $\theta = -\pi/4$. In this case Stokes regions consist of 8 regions (Figure 7). All of the level curves passing through a point in Regions I, II, III and IV in Figure 7 flow into the infinity, and we can show the Borel summability of the 0-parameter solutions by the same argument which gives

Theorem 4.8. If we choose a point from Region V, VI, VII or VIII, however, one end of the level curve passing through it flow into the origin, as shown in 8. The analysis in this case will be studied in [KoS].

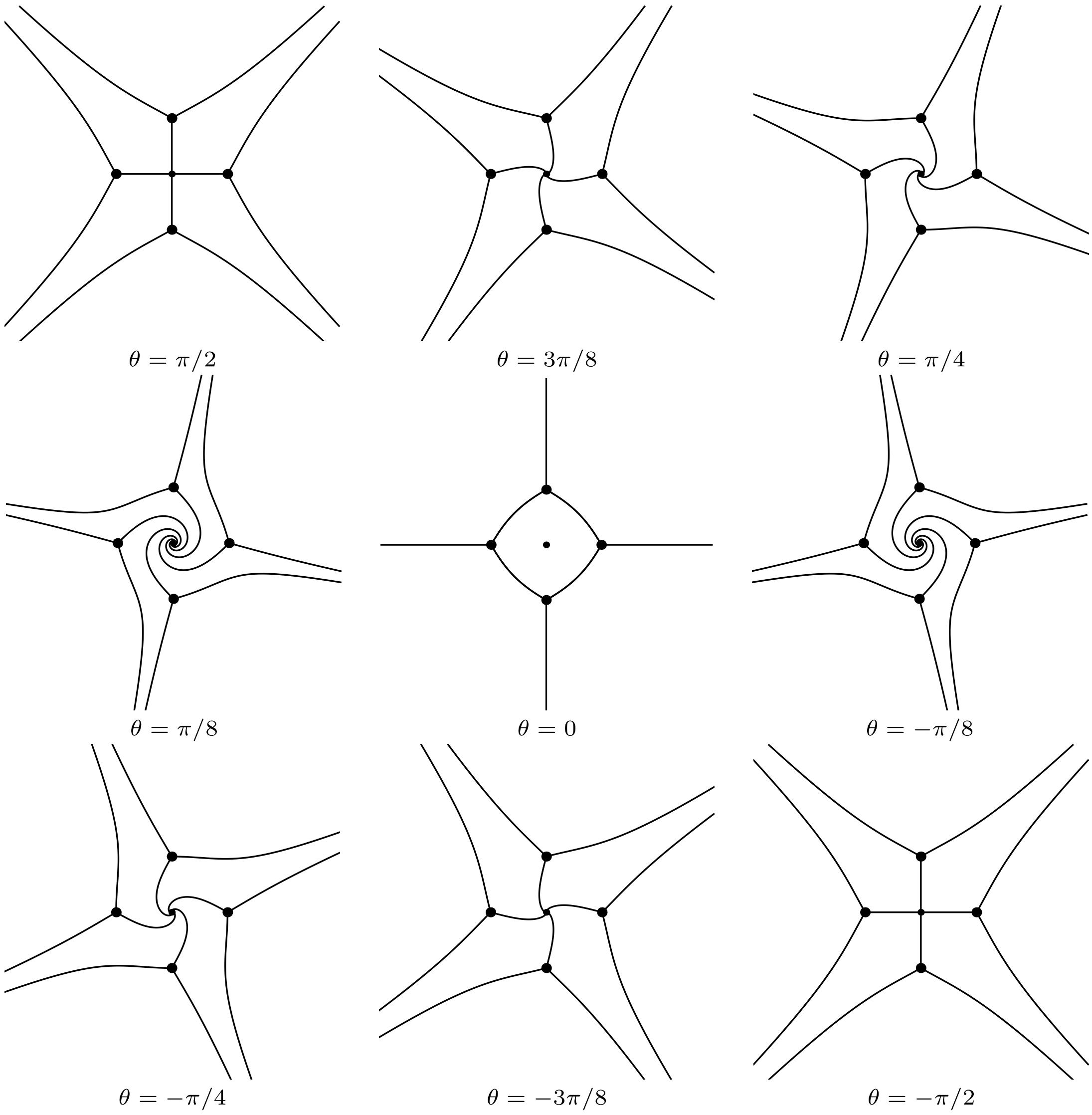

Figure 6. Stokes curves for $\theta = \pi/2, 3\pi/8, \pi/4, \pi/8, 0, -\pi/8, -\pi/4, -3\pi/8, -\pi/2$ ($\arg \eta = \theta$).

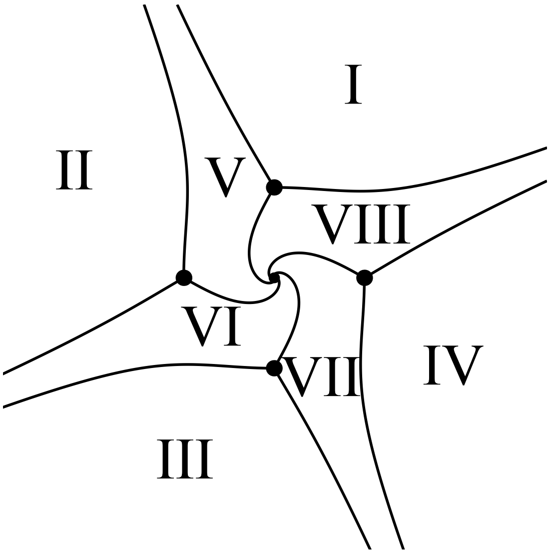

Figure 7. Stokes regions for $\theta = -\pi/4$.

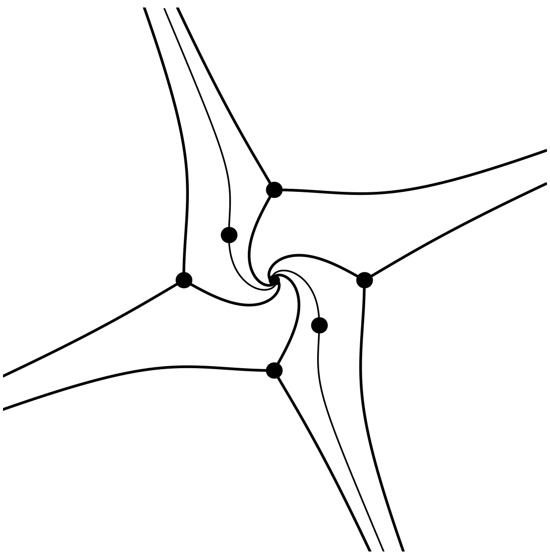

Figure 8. The level curve for $t_0 \in$V and VII.

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