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An example of a non 1-summable partial differential equation

By

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Abstract

We shall give a partial differential equation such that any divergent formal power series solution is not 1-summable in any direction. The motivation and the result of the present paper are closely related to those of the paper published by the second author, K. Kurogi recently. We refer the remark which follows Theorem 1.1 for the detail.

§ 1. Motivation and result

In this paper, we shall study the non summability of formal solutions of some linear partial differential equation. The summability for a partial differential equation has been studied by many mathematicians. Among these works, Lutz-Miyake-Schäfke, [3] studied the Borel summability of divergent solutions of the heat equation and they gave a necessary and sufficient condition for summability. Ōuchi [4] studied the multisummability of formal solutions of partial differential equations by using a Newton polygone. In [5], Tahara-Yamazawa studied the multisummability of a formal solution of an equation in a certain class of partial differential equations by extending the notion of the Newton polygone.

On the other hand, in [2] the second author gave an example of a non 1-summable partial differential equation, which is closely related to [5]. The method of the proof relies on constructing a singular solution with a natural boundary of the Borel transform

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of a given equation. The object of this paper is to simplify the proof of the construction of a singular solution and to extend the equation such that the non 1-summability in any direction of every formal solution holds.

Let $(t,x) \in \mathbb{C} \times \mathbb{C}$. For $r > 0$ we write $D_r = \{t \in \mathbb{C} \mid |t| < r\}$. For an open set W we denote by $\mathcal{O}(W)$ the set of all holomorphic functions on W. For a function $g(t) \in \mathcal{O}(D_r)$ we denote by $\mathrm{ord}_t(g)$ the order of the zero of $g(t)$ at $t = 0$.

We take the sequence of nonzero numbers $\{C_m\}_{m=1}^{\infty}$ such that $\sum_{m=1}^{\infty} |C_m| < \infty$. Let $\{h_m\}_{m=1}^{\infty}$ be a bounded sequence of complex numbers. Define $a(x) \in E^{\{1\}}(\mathbb{C})$ by

$$a(x) = \sum_{m=1}^{\infty} C_m e^{h_m x}. \tag{1.1}$$

Let n be a positive integer. For $j = 1, 2, \ldots, n$, we define a polynomial of ξ, $P_j(\xi)$ by

$$P_j(\xi) = \sum_{i=1}^{k} \alpha_{i,j} \xi^i \tag{1.2}$$

where $\alpha_{i,j} \in \mathbb{C}$. We consider the following Cauchy problem

$$\frac{\partial}{\partial t} u = a(x)t + \sum_{j=1}^{n} \left(t^2 \frac{\partial}{\partial t} \right)^j \frac{\partial}{\partial t} P_j(\partial_x) u, \quad u(0,x) = 0. \tag{1.3}$$

Then the unique formal solution $u = u(t,x)$ is given by

$$u(t,x) = \sum_{m=1}^{\infty} u_m(x) t^m, \tag{1.4}$$

where $\{u_m(x)\}_{m \geq 1}$ are entire functions by the assumption for $a(x)$. (cf. Proposition 3.1.) We have the main result of this paper

Theorem 1.1. *Assume that $\alpha_{k,1} \neq 0$. Then there exists a bounded sequence $\{h_m\}_{m=1}^{\infty}$ such that, for $a(x)$ in (1.1) any formal solution* (1.4) *of* (1.3) *is not 1-summable in any direction.*

Remark. (1.3) contains the equation studied in [2] in the sense that, by an appropriate change of the unknown function the equation is reduced to our case. Because the major motivation of the study in [2] is to construct an example which does not satisfy some of the conditions of [5] and whose formal solutions are not 1-summable in any direction, the present paper has a close relation with [5]. In [5] the following Cauchy problem is studied:

$$(E) \begin{cases} \partial_t^m u + \sum_{(j,\alpha) \in \Lambda} a_{j,\alpha}(t) \partial_t^j \partial_x^\alpha u = f(t,x) \\ \partial_t^j u|_{t=0} = \varphi_j(x) \quad (j = 0, 1, \ldots, m-1), \end{cases}$$

where $m \geq 1$ is an integer and Λ is a finite subset of $\mathbb{N} \times \mathbb{N}^N$, $N \geq 1$, $\mathbb{N} = \{0, 1, 2, \dots\}$. Here $(t, x) \in \mathbb{C} \times \mathbb{C}^N$, $a_{j,\alpha}(t) \in \mathcal{O}(D_r)$ $((j, \alpha) \in \Lambda)$, $f(t, x) \in \mathcal{O}(D_r \times \mathbb{C}^N)$, $\varphi_j(x) \in \mathcal{O}(\mathbb{C}^N)$ $(j = 0, 1, \dots, m-1)$, $\partial_t = \partial/\partial t$ and $\partial_x^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_N)^{\alpha_N}$, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N) \in \mathbb{N}^N$. They introduced a Newton polygone $N_t(E)$ with respect to t as follows. For $(a, b) \in \mathbb{R}^2$ we write $C(a, b) = \{(x, y) \,|\, x \leq a\,,\ y \geq b\}$. The Newton polygon $N_t(E)$ corresponding to (E) is defined by

$$N_t(E) := \text{the convex hull of } \left\{ C(m, -m) \cup \bigcup_{(j,\alpha)\in\Lambda} C(j, \mathrm{ord}_t(a_{j,\alpha}) - j) \right\}.$$

Then, under suitable conditions including $N_t(E)$ it was proved that the formal solution of (E) is multisummable in suitable directions.

We give two examples, where we assume that $a(x)$ satisfies the same condition as in (1.1).

Example 1. We set $\alpha_{i,j} = \delta_{i,j}$, where $\delta_{i,j}$ is Kronecker's delta. In this case, (1.2) reads $P_j(\xi) = \xi^j$, from which (1.3) is given by

$$\frac{\partial}{\partial t} u = a(x) t + \sum_{j=1}^{n} \left(t^2 \frac{\partial}{\partial t} \right)^j \frac{\partial}{\partial t} \left(\frac{\partial}{\partial x} \right)^j u, \quad u(0, x) = 0. \tag{1.5}$$

Example 2. We set $\alpha_{i,j} = \alpha_i$. Then, (1.2) reads $P_j(\xi) = \sum_{i=1}^{k} \alpha_i \xi^i$. (1.3) is given by

$$\frac{\partial}{\partial t} u = a(x) t + \sum_{j=1}^{n} \left(t^2 \frac{\partial}{\partial t} \right)^j \frac{\partial}{\partial t} \left\{ \sum_{i=1}^{k} \alpha_i \left(\frac{\partial}{\partial x} \right)^i \right\} u, \quad u(0, x) = 0. \tag{1.6}$$

This paper is organized as follows. In Section 2, we give definitions and symbols. In Section 3, we prove Gevrey estimate of the formal solution. The proof of Theorem 1.1 is given in Section 4.

§ 2. Definitions and notation

We give some definitions and introduce symbols necessary for the proof of main theorem, see [1].

Let $\mathbb{C}[[t]]$ be the ring of formal power series in t with coefficients in $\mathbb{C}$. We say that formal power series $\hat{f}(t) = \sum_{n=0}^{\infty} f_n t^n \in \mathbb{C}[[t]]$ has Gevrey order $1/k > 0$ if there exists $C > 0$ such that, for all $n \geq 0$

$$|f_n| \leq C^{n+1} \Gamma \left(1 + \frac{n}{k} \right)$$

where $\Gamma(x)$ is a Gamma function. We denote by $\mathbb{C}[[t]]_{1/k}$ the set of all formal power series with Gevrey order $1/k$.
A sector on the Riemann surface of the logarithm is the set of the form

$$S \equiv S(d,\alpha,\rho) := \left\{re^{i\theta} \middle| |\theta - d| < \frac{\alpha}{2}, 0 < r < \rho\right\}$$

where d is an arbitrary real number, α is a positive real and ρ is positive real or ∞. In case $\rho = \infty$, we mostly write $S(d,\alpha)$:

$$S(d,\alpha) \equiv S(d,\alpha,\infty) := \left\{re^{i\theta} \middle| |\theta - d| < \frac{\alpha}{2}, 0 < r\right\}.$$

A closed subsector $\bar{S}_1$ of S is the set of the form

$$\bar{S}_1 \equiv \bar{S}_1(d',\alpha',\rho') := \left\{re^{i\theta} \middle| |\theta - d'| \leq \frac{\alpha'}{2}, 0 < r \leq \rho'\right\}$$

where $0 < \alpha' < \alpha$, $0 < \rho' < \rho$, $|d - d'| < \alpha/2 - \alpha'/2$.
Given $k > 0$, we say that a function f being holomorphic in a sector S asymptotically equals $\hat{f}(t) \in \mathbb{C}[[t]]_{1/k}$ or $\hat{f}$ is the asymptotic expansion of Gevrey order $1/k$ of f if, for every closed subsector $\bar{S}_1$ of S there exists $C > 0$ such that for every $N \geq 0$ and every $t \in \bar{S}_1$

$$|r_f(t,N)| \leq C^{N+1}\Gamma\left(1 + \frac{N}{k}\right)$$

where $r_f(t,N) = t^{-N}\left(f(t) - \sum_{n=0}^{N-1} f_n t^n\right)$. In such a case, we write, for short, $f(t) \cong_k \hat{f}(t)$.
We denote by $A_k(S)$ the set of all holomorphic functions on S having an asymptotic expansion of Gevrey order $1/k$.

Definition 2.1. For a formal power series $\hat{f}(t) \in \mathbb{C}[[t]]_1$ without constant term, the formal Borel transform $\hat{B}_1\hat{f}$ is defined by

$$\hat{B}_1 : \hat{f}(t) = \sum_{n=1}^{\infty} f_n t^n \mapsto f(\xi) = \sum_{n=1}^{\infty} f_n \frac{\xi^{n-1}}{\Gamma(n)}. \tag{2.1}$$

Definition 2.2. For $\hat{f}(t) \in \mathbb{C}[[t]]_1$, we say that $\hat{f}(t)$ is 1-summable in the d-direction if there exists an $\epsilon > 0$ such that $\hat{B}_1\hat{f} \in A_1(S(d,\epsilon))$ with exponential size at most 1. We denote by $\mathbb{C}\{t\}_{1,d}$ the set of all formal power series that are 1-summable in the d-direction. We say that a function $f(t)$ on $S(d,\epsilon)$ has exponential size at most 1 on $S(d,\epsilon)$ if, for every subsector $\overline{S}_1$ in $S(d,\epsilon)$ there exist $C > 0$ and $h > 0$ such that

$$|f(t)| \leq Ce^{h|t|} \quad (t \in \overline{S}_1).$$

§ 3. Estimate of formal solution

In this section, we show the Gevrey estimate of formal solution.

Proposition 3.1. *Let $a(x)$ be given by (1.1), and let $h > 0$ satisfy that $|h_m| \leq h$ for any $m \in \mathbb{N}^*$. Then, for every $\ell \in \mathbb{N}^*$ we have $u_\ell(x)$ uniquely. Moreover, we have the estimate: there exist $K > 0$ and $M > 0$ such that, for any $\ell \in \mathbb{N}^*$ and $x \in \mathbb{C}$*

$$|u_\ell(x)| \leq \frac{(\ell-1)!}{\ell} M^\ell K e^{h|x|}. \tag{3.1}$$

Proof. Substituting (1.4) into (1.3) we get

$$\sum_{m=1}^{\infty} u_m(x) m t^{m-1} = a(x)t + \sum_{j=1}^{n} \left(\sum_{m=2}^{\infty} P_j(\partial_x) u_m(x) \frac{m(m+j-2)!}{(m-2)!} t^{m+j-1} \right).$$

Compare the coefficients of both sides about the power of t, to obtain

$$\begin{aligned} u_1(x) &= 0, \quad u_2(x) = a(x), \\ u_3(x) &= \frac{1}{3} P_1(\partial_x) u_2 = \frac{1}{3} P_1(\partial_x) a(x), \\ u_4(x) &= \frac{1}{4} \{6P_1(\partial_x) u_3(x) + 4P_2(\partial_x) u_2\} \\ &= \frac{1}{4} \left\{2P_1(\partial_x)^2 a(x) + 4P_2(\partial_x) a(x)\right\}, \\ &\vdots \end{aligned}$$

$$u_\ell(x) = \frac{1}{\ell} \left\{ \sum_{m+j=\ell, j\geq 1} P_j(\partial_x) u_m(x) \frac{m(\ell-2)!}{(m-2)!} \right\}. \tag{3.2}$$

Therefore, we have the formal solution uniquely.
By assumption, there exists $K > 0$ such that for every $j = 1, 2, \ldots, n$ and any $x \in \mathbb{C}$

$$|P_j(\partial_x) a(x)| \leq K e^{h|x|}.$$

We prove (3.1) by induction on l. The case $l = 1$ is trivial. Suppose now that (3.1) holds up to $l - 1$. Because the sum with respect to j in (3.2) is a finite sum and u_l is a function of $a(x)$, (3.1) is obtained. □

§ 4. Proof of Theorem 1.1.

Set $(\partial u/\partial t) =: v$. Then, (1.3) is written as follows

$$v = a(x)t + \sum_{j=1}^{n} \left(t^2 \frac{\partial}{\partial t} \right)^j P_j(\partial_x) v.$$

Applying the formal Borel transform to both sides and by setting $\hat{B}_1(v) = \hat{w}$ we have

$$\hat{w} = a(x) + \sum_{j=1}^{n} \tau^j P_j(\partial_x)\hat{w}, \tag{4.1}$$

where we used

$$\hat{B}_1 \left\{ \left(t^2 \frac{\partial}{\partial t} \right)^j P_j(\partial_x) v \right\} = \tau^j P_j(\partial_x) \hat{B}_1 (v) .$$

If we set

$$\hat{w} = \sum_{m=1}^{\infty} \hat{w}_m(\tau) e^{h_m x},$$

then we have

$$P_j(\partial_x)\hat{w} = \sum_{m\geq 1} \hat{w}_m(\tau) P_j(h_m) e^{h_m x}. \tag{4.2}$$

Substituting (4.2) and the expansion of $a(x)$, (1.1) into (4.1) we have

$$\begin{aligned} \hat{w}_m(\tau) &= C_m + \sum_{j=1}^{n} \tau^j P_j(h_m)\hat{w}_m(\tau) \\ &= C_m + \sum_{j=1}^{n} \tau^j \sum_{i=1}^{k} \alpha_{i,j} h_m^i \hat{w}_m(\tau) \end{aligned}$$

for every $m \in \mathbb{N}$. Therefore, we have

$$\hat{w}_m(\tau) = \frac{C_m}{1 - \sum_{j=1}^{n} \tau^j \sum_{i=1}^{k} \alpha_{i,j} h_m^i} \tag{4.3}$$

for every $m \in \mathbb{N}$.

Because $\alpha_{k,1} \neq 0$, we take $\epsilon > 0$ sufficiently small such that

$$\left| \sum_{j=1}^{n} \tau^j \alpha_{k,j} \right| > |\alpha_{k,1}|\epsilon/2 \tag{4.4}$$

on $|\tau| = \epsilon$. Define

$$F(\tau, h) := \sum_{j=1}^{n} \tau^j \sum_{i=1}^{k} \alpha_{i,j} h^i - 1. \tag{4.5}$$

Take a countable infinite set $T = \{T_m\}_{m=1}^{\infty}$ which is dense on $|\tau| = \epsilon$. For each $\tau_m \in T$, consider the equation $F(\tau_m, h) = 0$. Take one root arbitrarily and put it by h_m. Then $H := \{h_m\}_{m=1}^{\infty}$ is a bounded set. Indeed, every coefficient of h^i in $F(\tau_m, h)$ is uniformly bounded when $\tau_m \in T$. Moreover, the coefficient of the highest power, h^k, $\sum_{j=1}^{n} \tau_m^j \alpha_{k,j}$ is uniformly bounded from the below by $|\alpha_{k,1}|\epsilon/2$ by (4.4). Because the solution h of the algebraic equation $F(\tau_m, h) = 0$ is a continuous function of the coefficients of the equation, it follows that H is a bounded set.

We shall show that there exist constants $\delta_1 > 0$ and $0 < \epsilon_1 < \epsilon$ such that, for every $h_m \in H$

$$|F(\tau, h_m)| \geq \delta_1 \quad \text{on } |\tau| < \epsilon_1.$$

Suppose that this is not true. Then, for every $\nu = 1, 2, \ldots$ there exist a positive integer $m(\nu)$ and τ_ν such that

$$|F(\tau_\nu, h_{m(\nu)})| < 1/\nu, \quad |\tau_\nu| < 1/\nu.$$

On the other hand, by (4.5) we have

$$F(\tau_\nu, h_{m(\nu)}) = \sum_{j=1}^{n} \tau_\nu{}^j \sum_{i=1}^{k} \alpha_{i,j} (h_{m(\nu)})^i - 1.$$

Because τ_ν tends to 0 as ν tends to infinity and H is a bounded set, the right-hand side does not tends to zero. This is a contradiction. By the above definition of $T = \{\tau_m\}$ and $H = \{h_m\}$, the formal Borel transform $\hat{w}(\tau, x)$ has pole singularities at each point in $T = \{\tau_m\}$ which is dense on $|\tau| = \epsilon$. Therefore u is not 1-summable in any direction. This completes the proof.

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