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Mono-anabelian Reconstruction of Number Fields

By

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Abstract

The *Neukirch-Uchida theorem* asserts that every outer isomorphism between the absolute Galois groups of number fields arises from a uniquely determined isomorphism between the given number fields. In particular, the isomorphism class of a number field is completely determined by the isomorphism class of the absolute Galois group of the number field. On the other hand, *neither* the Neukirch-Uchida theorem *nor* the proof of this theorem *yields* an "explicit reconstruction of the given number field". In other words, the Neukirch-Uchida theorem only yields a *bi-anabelian reconstruction* of the given number field. In the present paper, we discuss a *mono-anabelian reconstruction* of the given number field. In particular, we give a *functorial "group-theoretic" algorithm* for reconstructing, from the absolute Galois group of a number field, the algebraic closure of the given number field [equipped with its natural Galois action] that gave rise to the given absolute Galois group. One important step of our reconstruction algorithm consists of the construction of a *global cyclotome* [i.e., a cyclotome constructed from a global Galois group] and a *local-global cyclotomic synchronization isomorphism* [i.e., a suitable isomorphism between a global cyclotome and a local cyclotome]. We also verify a certain *compatibility* between our reconstruction algorithm and the reconstruction algorithm given by *S. Mochizuki* concerning the étale fundamental groups of hyperbolic orbicurves of strictly Belyi type over number fields. Finally, we discuss a certain *global mono-anabelian log-Frobenius compatibility* property satisfied by the reconstruction algorithm obtained in the present paper.

Contents

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Introduction

The starting point of the present paper is the following *naive question*:

Can one reconstruct a number field [i.e., a finite extension of the field of rational numbers] from the absolute Galois group of the given number field?

Recall the following result, i.e., the *Neukirch-Uchida theorem* [cf., e.g., [11], Theorem 12.2.1]:

For $\square \in \{\circ, \bullet\}$, let $F_\square$ be a *number field* and $\overline{F}_\square$ an algebraic closure of $F_\square$. Write $G_\square \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{F}_\square/F_\square)$;

$$\mathrm{Isom}(\overline{F}_\bullet/F_\bullet, \overline{F}_\circ/F_\circ)$$

for the set of isomorphisms $\overline{F}_\bullet \xrightarrow{\sim} \overline{F}_\circ$ of fields which map $F_\bullet$ bijectively onto $F_\circ$;

$$\mathrm{Isom}(G_\circ, G_\bullet)$$

for the set of isomorphisms $G_\circ \xrightarrow{\sim} G_\bullet$ of profinite groups. Then the natural map

$$\mathrm{Isom}(\overline{F}_\bullet/F_\bullet, \overline{F}_\circ/F_\circ) \longrightarrow \mathrm{Isom}(G_\circ, G_\bullet)$$

is *bijective.*

That is to say, every outer isomorphism between the absolute Galois groups of number fields arises from a uniquely determined isomorphism between the given number fields. In other words, the functor given by "*forming the absolute Galois group*" from the category of number fields and field isomorphisms to the category of profinite groups and outer isomorphisms is *fully faithful.* It follows from the [surjectivity portion of the] Neukirch-Uchida theorem that the isomorphism class of a number field is *completely determined* by the isomorphism class of the absolute Galois group of the number field. From this point of view, one may regard the Neukirch-Uchida theorem as an *affirmative answer* to the above *naive question.*

On the other hand, let us observe that *neither* the statement of the Neukirch-Uchida theorem *nor* the proof of this theorem *yields* an "explicit reconstruction of the given number field". That is to say, although one may conclude from the Neukirch-Uchida theorem that the isomorphism class of a number field is completely determined by the isomorphism class of the associated absolute Galois group, the Neukirch-Uchida theorem does *not tell us how to reconstruct explicitly the given number field* from the associated absolute Galois group. In other words, the Neukirch-Uchida theorem yields only a *bi-anabelian reconstruction* — in the sense of [9], Introduction [cf. also [9], Remark 1.9.8] — of number fields.

In the present paper, we discuss a *mono-anabelian reconstruction* — in the sense of [9], Introduction [cf. also [9], Remark 1.9.8] — of number fields. In particular, we concentrate on the task of establishing "*group-theoretic software*" [i.e., a "*group-theoretic algorithm*"] whose

- *input data* consists of a *single abstract profinite group* [which is isomorphic to [a suitable quotient of] the absolute Galois group of a number field], and whose

- *output data* consists of a *field* [which is isomorphic to [a suitable subfield of] some algebraic closure, equipped with an action of the profinite group, of a number field].

We shall say that an algebraic extension of the field of rational numbers is *absolutely Galois* (respectively, *solvably closed*) if the extension field is Galois over the field of rational numbers (respectively, if the extension field does not admit any nontrivial finite abelian extensions) [cf. Definition 3.1]. We shall say that a profinite group G is *of AGSC-type* if there exist a number field F, a Galois extension $\widetilde{F}$ of F which is absolutely Galois and solvably closed, and an isomorphism of profinite groups $G \xrightarrow{\sim} \mathrm{Gal}(\widetilde{F}/F)$ [cf. Definition 3.2]. [In particular, if a profinite group is isomorphic to the absolute Galois group of a number field, then the profinite group is *of AGSC-type*.] Then the main result of the present paper may be summarized as follows [cf. Theorem 5.11]:

Theorem A. *There exists a* **functorial** [*cf. Remark* 5.11.4] **"group-theoretic" algorithm** [*cf.* [9]*, Remark* 1.9.8*, for more on the meaning the terminology "group-theoretic"*]

$$G \quad \mapsto \quad (G \curvearrowright \widetilde{F}(G))$$

for constructing, from a profinite group G **of AGSC-type** [*cf. Definition* 3.2]*, an* **absolutely Galois** *and* **solvably closed** *field $\widetilde{F}(G)$ equipped with an action of G such that the subfield $\widetilde{F}(G)^G$ of $\widetilde{F}(G)$ consisting of G-invariants is a* **number field***, and, moreover, the action of G on $\widetilde{F}(G)$ determines an* **isomorphism of profinite groups**

$$G \xrightarrow{\sim} \mathrm{Gal}(\widetilde{F}(G)/\widetilde{F}(G)^G).$$

We thus conclude from Theorem A that every profinite group which is isomorphic to the absolute Galois group of a number field admits a *ring-theoretic basepoint* [i.e., a *"ring-theoretic interpretation"* or *"ring-theoretic labeling"*] *group-theoretically* constructed from the given profinite group. Note that the Neukirch-Uchida theorem plays a crucial role in the establishment of our global reconstruction result. In particular, the proof of this global reconstruction result does *not yield* an alternative proof of the Neukirch-Uchida theorem.

In the present paper, we also verify a certain *compatibility* of the reconstruction algorithm of Theorem A with the reconstruction algorithm obtained in [9], Theorem 1.9, in the case where the "k" of [9], Theorem 1.9, is a number field. More precisely, we verify the following assertion [cf. Theorem 5.13]: Let Π be a profinite group which is isomorphic to the étale fundamental group of a hyperbolic orbicurve of strictly Belyi type over a number field [cf. [8], Definition 3.5]. Write

$$\Pi \curvearrowright \overline{F}(\Pi)$$

for the algebraically closed field equipped with an action of Π obtained by applying the *functorial "group-theoretic" algorithm* given in [9], Theorem 1.9, to Π [i.e., the field "$\overline{k}^{\times}_{\mathrm{NF}} \cup \{0\}$" of [9], Theorem 1.9, (e)] and

$$\Pi \twoheadrightarrow Q$$

for the *arithmetic quotient* of Π, i.e., the quotient of Π by the [uniquely determined — cf. [7], Theorem 2.6, (vi)] maximal topologically finitely generated normal closed subgroup of Π. [Thus, Q is a profinite group *of AGSC-type* — cf. [7], Theorem 2.6, (vi) — which thus implies that one may apply Theorem A to Q to construct a field $\widetilde{F}(Q)$ equipped with an action of Q.] Then the natural surjection $\Pi \twoheadrightarrow Q$ *group-theoretically determines an isomorphism of fields*

$$\widetilde{F}(Q) \xrightarrow{\sim} \overline{F}(\Pi)$$

which is *compatible* with the natural actions of Q and Π relative to the surjection $\Pi \twoheadrightarrow Q$.

Finally, we verify that the reconstruction algorithm of Theorem A satisfies a certain *global mono-anabelian log-Frobenius compatibility* property [cf. Theorem 6.10], i.e., a certain *compatibility* property with the NF-log-Frobenius functor $\mathfrak{log}$ [cf. Definition 6.8].

The present paper is organized as follows: In §1, we review mono-anabelian reconstructions of various objects which arise from a mixed characteristic local field [cf. Theorem 1.4]. In §2, we discuss the notion of an *NF-monoid* [cf. Definition 2.3]. In particular, we obtain a mono-anabelian reconstruction of the "*additive structure*" on an NF-monoid [cf. Theorem 2.9]. Note that the main result of §2 was already essentially proved in [3]; in [3], however, the author considered the issue of reconstruction of the additive structure not in a "*mono-anabelian" fashion* but rather in a "*bi-anabelian" fashion.* In §3, we define a *cyclotome* [cf. Proposition 3.7, (4)] associated to a profinite group *of GSC-type* [cf. Definition 3.2]. Moreover, we discuss a certain *local-global cyclotomic synchronization isomorphism* [cf. Theorem 3.8, (ii)], i.e., a certain natural isomorphism between global and local cyclotomes. We then apply this local-global cyclotomic synchronization isomorphism to construct *Kummer containers* associated to a profinite group of GSC-type [cf. Proposition 3.11]. In §4, we discuss the notion of a *GSC-Galois pair* [cf. Definition 4.1]. We then apply the main result of §2 to obtain a mono-anabelian reconstruction of the *additive structure* on a GSC-Galois pair [cf. Theorem 4.4]. In §5, we discuss the final portion of the *functorial "group-theoretic" algorithm* of Theorem A and prove a certain compatibility property of our reconstruction algorithm with the reconstruction algorithm obtained in [9], Theorem 1.9. In §6, we give an interpretation of the global reconstruction result obtained in the present paper in terms of a certain compatibility with the *NF-log-Frobenius functor* [cf. Theorem 6.10].

§ 0. Notations and Conventions

Numbers. The notation $\mathbb{N}$ will be used to denote the additive monoid of nonnegative rational integers. The notation $\mathbb{Z}$ will be used to denote the ring of rational integers. The notation $\mathbb{Q}$ will be used to denote the field of rational numbers. If $n \in \mathbb{Z}$, then we shall write $\mathbb{Z}_{\geq n} \subseteq \mathbb{Z}$ for the subset of $\mathbb{Z}$ consisting of $m \in \mathbb{Z}$ such that $m \geq n$. If p is a prime number, then we shall write $\mathbb{Q}_p$ for the field obtained by forming the p-adic completion of $\mathbb{Q}$ and $\mathbb{F}_p \stackrel{\text{def}}{=} \mathbb{Z}/p\mathbb{Z}$ for the finite field of cardinality p.

Sets. Let S be a finite set. Then we shall write $\sharp S$ for the *cardinality* of S. Let G be a group and T a G-set. Then we shall write $T^G \subseteq T$ for the subset of T consisting of G-invariants.

Monoids. In the present paper, every "monoid" is assumed to be commutative. Let M be a [multiplicative] monoid. Then we shall write $M^\times \subseteq M$ for the abelian group of invertible elements of M. We shall write M^{gp} for the *groupification* of M, i.e., the monoid [which is, in fact, an *abelian group*] given by the set of equivalence classes with respect to the relation "$\sim$" on $M \times M$ defined as follows: for (a_1, b_1), $(a_2, b_2) \in M \times M$, it holds that $(a_1, b_1) \sim (a_2, b_2)$ if and only if there exists an element $c \in M$ such that $ca_1b_2 = ca_2b_1$. We shall write M^{pf} for the *perfection* of M, i.e., the monoid given by the inductive limit of the inductive system I_* of monoids

$$\cdots \longrightarrow M \longrightarrow M \longrightarrow \cdots$$

given by assigning to each element of $n \in \mathbb{Z}_{\geq 1}$ a copy of M, which we denote by I_n, and to every two elements n, $m \in \mathbb{Z}_{\geq 1}$ such that n divides m the morphism $I_n = M \to I_m = M$ given by multiplication by m/n. We shall write $M^{\circledast} \stackrel{\text{def}}{=} M \cup \{*_M\}$; we regard $M^{\circledast}$ as a *monoid* [that contains M as a submonoid] by setting $a \cdot *_M \stackrel{\text{def}}{=} *_M$ and $*_M \cdot *_M \stackrel{\text{def}}{=} *_M$ for every $a \in M$.

MODULES. Let M be a module. If $n \in \mathbb{Z}$, then we shall write $M[n] \subseteq M$ for the submodule obtained by forming the kernel of the endomorphism of M given by multiplication by n. We shall write $M_{\text{tor}} \stackrel{\text{def}}{=} \bigcup_{n \in \mathbb{Z}_{\geq 1}} M[n] \subseteq M$ for the submodule of torsion elements of M,

$$M^{\wedge} \stackrel{\text{def}}{=} \varprojlim_{n} M/nM$$

— where the projective limit is taken over $n \in \mathbb{Z}_{\geq 1}$ [regarded as a multiplicative monoid] — and $\widehat{\mathbb{Z}} \stackrel{\text{def}}{=} \mathbb{Z}^{\wedge}$. Thus, if M is *finitely generated* [which implies that each M/nM in the above display is *finite*], then $M^{\wedge}$ is naturally isomorphic to the *profinite completion* of M.

GROUPS. Let G be a group and $H \subseteq G$ a subgroup of G. Then we shall write $Z_G(H) \subseteq G$ for the *centralizer* of H in G, i.e., the subgroup consisting of $g \in G$ such that $gh = hg$ for every $h \in H$. We shall write $N_G(H) \subseteq G$ for the *normalizer* of H in G, i.e., the subgroup consisting of $g \in G$ such that $H = gHg^{-1}$. We shall write $C_G(H) \subseteq G$ for the *commensurator* of H in G, i.e., the subgroup consisting of $g \in G$ such that $H \cap gHg^{-1}$ is of finite index in both H and gHg^{-1}. We shall say that H is *normally terminal* (respectively, *commensurably terminal*) in G if $N_G(H) = H$ (respectively, $C_G(H) = H$).

TOPOLOGICAL GROUPS. Let G be a topological group. Then we shall write G^{ab} for the *abelianization* of G [i.e., the quotient of G by the closure of the commutator subgroup of G], $G^{\text{ab/tor}}$ for the quotient of G^{ab} by the closure of $(G^{\text{ab}})_{\text{tor}} \subseteq G^{\text{ab}}$, and $\text{Aut}(G)$ for the group of [continuous] automorphisms of G. Let H be a profinite group and p a prime number. Then we shall write $H^{(p)}$ for the *maximal pro-p quotient* of H and $H^{(p')}$ for the *maximal pro-prime-to-p quotient* of H.

RINGS. In the present paper, every "ring" is assumed to be unital, associative, and commutative. If R is a ring, then we shall write $R^{\times} \subseteq R$ for the abelian group [hence, in particular, the multiplicative monoid] of invertible elements of R. If R is an integral domain, then we shall write $R^{\triangleright} \stackrel{\text{def}}{=} R \setminus \{0\} \subseteq R$ for the multiplicative monoid of nonzero elements of R; thus, we have a natural inclusion $R^{\times} \subseteq R^{\triangleright}$ of monoids.

Fields. We shall refer to a field which is isomorphic to a finite extension of $\mathbb{Q}$ as an *NF* [i.e., a *number field*]. We shall refer to a field which is isomorphic to a finite extension of $\mathbb{Q}_p$, for some prime number p, as an *MLF* [i.e., a *mixed characteristic local field*]. Here, we recall that, for a given MLF, by considering the additive subgroup generated by the elements $\in k$ that are l-divisible for some prime number l, one can recover the [usual "p-adic"] topology on the MLF. Let K be a field. Then we shall write $\boldsymbol{\mu}(K) \stackrel{\text{def}}{=} (K^\times)_{\text{tor}}$ for the group of roots of unity of K and $K_\times = K^\times \cup \{0\}$ for the multiplicative monoid obtained by forgetting the additive structure of K. Thus, we have a natural isomorphism $(K^\times)^\circledast \xrightarrow{\sim} K_\times$ of monoids that sends $*_{K^\times} \mapsto 0$. If, moreover, K is *algebraically closed* and *of characteristic zero*, then we shall write

$$\Lambda(K) \stackrel{\text{def}}{=} \varprojlim_n \boldsymbol{\mu}(K)[n] = \varprojlim_n K^\times[n]$$

— where the projective limits are taken over $n \in \mathbb{Z}_{\geq 1}$ [regarded as a multiplicative monoid] — and refer to $\Lambda(K)$ as the *cyclotome* associated to K. Thus, [the abstract module] $\Lambda(K)$ is [noncanonically] isomorphic to $\widehat{\mathbb{Z}}$; we have a natural identification $\boldsymbol{\mu}(K)[n] = \Lambda(K)/n\Lambda(K)$.

§ 1. Review of the Local Theory

In the present §1, let us review certain well-known *mono-anabelian reconstructions* of various objects which arise from an *MLF* [cf. Theorem 1.4 below].

In the present §1, let

$$k$$

be an *MLF*. We shall write

- $\mathcal{O}_k \subseteq k$ for the ring of integers of k,
- $\mathfrak{m}_k \subseteq \mathcal{O}_k$ for the maximal ideal of $\mathcal{O}_k$,
- $\underline{k} \stackrel{\text{def}}{=} \mathcal{O}_k/\mathfrak{m}_k$ for the residue field of $\mathcal{O}_k$,
- $p_k \stackrel{\text{def}}{=} \text{char}(\underline{k})$ for the characteristic of $\underline{k}$,
- d_k for the extension degree of k over the subfield of k obtained by forming the closure of the prime field contained in k [i.e., "$[k : \mathbb{Q}_{p_k}]$"],
- $\text{ord}_k \colon k^\times \twoheadrightarrow \mathbb{Z}$ for the [uniquely determined] surjective valuation on k,
- $e_k \stackrel{\text{def}}{=} \text{ord}_k(p_k)$ for the absolute ramification index of k, and
- f_k for the extension degree of $\underline{k}$ over the prime field contained in $\underline{k}$ [i.e., "$[\underline{k} : \mathbb{F}_{p_k}]$"].

Let

$$\overline{k}$$

be an algebraic closure of k. We shall write

- $G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$ for the absolute Galois group of k with respect to $\overline{k}/k$,
- $I_k \subseteq G_k$ for the inertia subgroup of G_k,
- $P_k \subseteq I_k$ for the wild inertia subgroup of G_k, and
- $\mathrm{Frob}_k \in G_k/I_k$ for the [$\sharp\underline{k}$-th power] Frobenius element of G_k/I_k.

Definition 1.1. Let G be a group. Then we shall refer to a collection of data

$$(K,\ \overline{K},\ \alpha\colon \mathrm{Gal}(\overline{K}/K) \stackrel{\sim}{\to} G)$$

consisting of an MLF K, an algebraic closure $\overline{K}$ of K, and an isomorphism $\alpha\colon \mathrm{Gal}(\overline{K}/K) \stackrel{\sim}{\to} G$ of groups as an *MLF-envelope* for G. We shall say that the group G is *of MLF-type* if there exists an MLF-envelope for G.

Proposition 1.2. *Let G be a group* **of MLF-type**. *Then the following hold:*

(i) *The natural homomorphism*

$$G \longrightarrow \varprojlim_{N} G/N$$

— where the projective limit is taken over the normal subgroups $N \subseteq G$ of G **of finite index** *— is an* **isomorphism** *of groups. In particular, any group* **of MLF-type** *admits a natural, group-theoretically determined* **profinite group** *structure.*

(ii) *Let*

$$(k,\ \overline{k},\ \alpha\colon G_k \stackrel{\sim}{\to} G)$$

be an **MLF-envelope** *for G. Then the isomorphism α is an* **isomorphism of profinite groups**.

PROOF. Assertion (i) follows from [12], Theorem 1.1, together with the fact that the absolute Galois group of an MLF is *topologically finitely generated* [cf., e.g., [11], Theorem 7.4.1]. Assertion (ii) follows from assertion (i). This completes the proof of Proposition 1.2. □

Remark 1.2.1. One verifies immediately that every open subgroup of a profinite group *of MLF-type* is *of MLF-type*.

Lemma 1.3. *The following hold:*

(i) *The reciprocity homomorphism* $k^\times \to G_k^{\mathrm{ab}}$ *in local class field theory determines a commutative diagram*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{O}_k^\times & \longrightarrow & k^\times & \xrightarrow{\mathrm{ord}_k} & \mathbb{Z} & \longrightarrow & 1 \\
 & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \\
1 & \longrightarrow & \mathrm{Im}(I_k \hookrightarrow G_k \twoheadrightarrow G_k^{\mathrm{ab}}) & \longrightarrow & G_k^{\mathrm{ab}} \times_{G_k/I_k} \mathrm{Frob}_k^{\mathbb{Z}} & \longrightarrow & \mathrm{Frob}_k^{\mathbb{Z}} & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Im}(I_k \hookrightarrow G_k \twoheadrightarrow G_k^{\mathrm{ab}}) & \longrightarrow & G_k^{\mathrm{ab}} & \longrightarrow & G_k/I_k & \longrightarrow & 1
\end{array}$$

— *where the horizontal sequences are* **exact**, *the upper vertical arrows are* **isomorphisms**, *the lower vertical arrows are the natural inclusions, the upper right-hand vertical arrow maps* $1 \in \mathbb{Z}$ *to* $\mathrm{Frob}_k \in \mathrm{Frob}_k^{\mathbb{Z}}$, *and we write* $\mathrm{Frob}_k^{\mathbb{Z}} \subseteq G_k/I_k$ *for the* [*discrete*] *subgroup of* G_k/I_k *generated by* Frob_k.

(ii) *The prime number* p_k *may be* **characterized** *as the unique prime number* l *such that* $\log_l(\sharp(G_k^{\mathrm{ab/tor}}/l \cdot G_k^{\mathrm{ab/tor}})) \geq 2$.

(iii) *It holds that* $d_k = \log_{p_k}(\sharp(G_k^{\mathrm{ab/tor}}/p_k \cdot G_k^{\mathrm{ab/tor}})) - 1$.

(iv) *It holds that* $f_k = \log_{p_k}(1 + \sharp((G_k^{\mathrm{ab}})_{\mathrm{tor}})^{(p'_k)})$.

(v) *It holds that* $e_k = d_k/f_k$.

(vi) *The closed subgroup* $I_k \subseteq G_k$ *may be* **characterized** *as the intersection of the normal open subgroups* $N \subseteq G_k$ *of* G_k *such that* $e_k = e_{k_N}$, *where we write* k_N *for the intermediate extension of* $\overline{k}/k$ *corresponding to* N.

(vii) *The closed subgroup* $P_k \subseteq G_k$ *may be* **characterized** *as the intersection of the normal open subgroups* $N \subseteq G_k$ *of* G_k *such that the integer* e_{k_N}/e_k *is prime to* p_k, *where we write* k_N *for the intermediate extension of* $\overline{k}/k$ *corresponding to* N.

(viii) *The element* $\mathrm{Frob}_k \in G_k/I_k$ *may be* **characterized** *as the unique element of* G_k/I_k *such that the action on* [*the abelian group*] I_k/P_k *by conjugation is given by multiplication by* $p_k^{f_k}$.

(ix) *The upper left-hand vertical arrow of the diagram of* (i) *determines an* **isomorphism** $\underline{k}^\times \xrightarrow{\sim} \mathrm{Im}(I_k \hookrightarrow G_k \twoheadrightarrow G_k^{\mathrm{ab}})^{(p'_k)}$ *of modules.*

(x) *The exact sequences of* G_k*-modules*

$$1 \longrightarrow \boldsymbol{\mu}(\overline{k})[n] \longrightarrow \overline{k}^\times \xrightarrow{n} \overline{k}^\times \longrightarrow 1$$

— *where* n *ranges over the positive integers — determine an* **injection**

$$\mathrm{Kmm}_k\colon\ k^\times \ \hookrightarrow\ (k^\times)^\wedge \ \stackrel{\sim}{\to}\ H^1(G_k, \Lambda(\overline{k})).$$

PROOF. Assertion (i) follows from *local class field theory* [cf., e.g., [10], Chapter V, §1]. Assertions (ii), (iii), (iv), (ix) follow immediately from assertion (i), together with the well-known explicit description of the topological module $k^\times$ [cf., e.g., [10], Chapter II, Proposition 5.3; also [10], Chapter II, Proposition 5.7, (i)]. Assertion (v) follows from [10], Chapter II, Proposition 6.8. Assertions (vi), (vii) follow immediately from the definitions of I_k, P_k, respectively. Assertion (viii) follows immediately from [11], Proposition 7.5.2, together with the easily verified *faithfulness* of the action of "Γ" [in *loc. cit.*] on "$\widehat{\mathbb{Z}}^{(p')}(1)$" [in *loc. cit.*]. Assertion (x) follows immediately from the fact that there is *no nontrivial divisible element* in $k^\times$ [cf., e.g., [10], Chapter II, Proposition 5.7, (i)]. This completes the proof of Lemma 1.3. □

Theorem 1.4. *In the notation introduced at the beginning of the present* §1*, let* G *be a profinite group* **of MLF-type** [*cf. Definition* 1.1*; Proposition* 1.2*,* (i)]*. We construct various objects associated to* G *as follows:*

(1) *It follows from Lemma* 1.3*,* (ii)*, that there exists a* **unique** *prime number* l *such that* $\log_l(\sharp(G^{\mathrm{ab/tor}}/l\cdot G^{\mathrm{ab/tor}}))\geq 2$*. We shall write*

$$p(G)$$

for this prime number.

(2) *We shall write*

$$d(G)\ \stackrel{\mathrm{def}}{=}\ \log_{p(G)}(\sharp(G^{\mathrm{ab/tor}}/p(G)\cdot G^{\mathrm{ab/tor}}))-1,$$

$$f(G)\ \stackrel{\mathrm{def}}{=}\ \log_{p(G)}(1+\sharp((G^{\mathrm{ab}})_{\mathrm{tor}})^{(p(G)')}),$$

$$e(G)\ \stackrel{\mathrm{def}}{=}\ d(G)/f(G).$$

Note that it follows from Lemma 1.3*,* (iii)*,* (iv)*,* (v)*, that* $d(G)$*,* $f(G)$*,* $e(G)$ *are* **positive integers**.

(3) *We shall write*

$$I(G)\ \subseteq\ G$$

for the normal closed subgroup obtained by forming the intersection of the normal open subgroups $N\subseteq G$ *of* G *such that* $e(N)=e(G)$ *and*

$$P(G)\ \subseteq\ G$$

for the normal closed subgroup obtained by forming the intersection of the normal open subgroups $N \subseteq G$ *of* G *such that the positive integer* $e(N)/e(G)$ *is prime to* $p(G)$ [*cf. Lemma* 1.3, (vi), (vii)].

(4) *It follows from Lemma* 1.3, (viii), *that there exists a* **unique** *element of* $G/I(G)$ *whose action on* [*the abelian group*] $I(G)/P(G)$ *by conjugation is given by multiplication by* $p(G)^{f(G)}$. *We shall write*

$$\mathrm{Frob}(G) \in G/I(G)$$

for this element.

(5) *We shall write*

$$\mathcal{O}^\times(G) \stackrel{\mathrm{def}}{=} \mathrm{Im}(I(G) \hookrightarrow G \twoheadrightarrow G^{\mathrm{ab}})$$

for the image of $I(G)$ *in* G^{ab} [*cf. Lemma* 1.3, (i)]. *By considering the topology induced by the topology of* $I(G)$, *we regard* $\mathcal{O}^\times(G)$ *as a* **profinite**, *hence also topological, module. We shall write*

$$\underline{k}^\times(G) \stackrel{\mathrm{def}}{=} \mathcal{O}^\times(G)^{(p(G)')}$$

for the module obtained by forming the maximal pro-prime-to-$p(G)$ *quotient of* $\mathcal{O}^\times(G)$ [*cf. Lemma* 1.3, (ix)].

(6) *We shall write*

$$k^\times(G) \stackrel{\mathrm{def}}{=} G^{\mathrm{ab}} \times_{G/I(G)} \mathrm{Frob}(G)^{\mathbb{Z}}$$

— *where we write* $\mathrm{Frob}(G)^{\mathbb{Z}}$ *for the* [*discrete*] *subgroup of* $G/I(G)$ *generated by* $\mathrm{Frob}(G)$ — *and*

$$\mathcal{O}^\triangleright(G) \stackrel{\mathrm{def}}{=} G^{\mathrm{ab}} \times_{G/I(G)} \mathrm{Frob}(G)^{\mathbb{N}}$$

— *where we write* $\mathrm{Frob}(G)^{\mathbb{N}}$ *for the* [*discrete*] *submonoid of* $G/I(G)$ *generated by* $\mathrm{Frob}(G)$ [*cf. Lemma* 1.3, (i)]. *Note that the topology of* $\mathcal{O}^\times(G)$ *discussed in* (5) *naturally determines respective structures of* **topological** *module, monoid on* $k^\times(G)$, $\mathcal{O}^\triangleright(G)$.

(7) *We shall write*

$$\mathrm{ord}(G)\colon\ k^\times(G) \twoheadrightarrow \mathrm{Frob}(G)^{\mathbb{Z}}$$

for the natural surjection [*cf. Lemma* 1.3, (i)]. *Thus, we have an exact sequence of topological modules*

$$1 \longrightarrow \mathcal{O}^\times(G) \longrightarrow k^\times(G) \stackrel{\mathrm{ord}(G)}{\longrightarrow} \mathrm{Frob}(G)^{\mathbb{Z}} \longrightarrow 1.$$

(8) *We shall write*

$$k_\times(G) \stackrel{\mathrm{def}}{=} k^\times(G)^{\circledast}, \quad \underline{k}_\times(G) \stackrel{\mathrm{def}}{=} \underline{k}^\times(G)^{\circledast}$$

[*cf. the discussion entitled "Monoids" in* §0]*.*

(9) *We shall write*

$$\overline{k}^{\times}(G) \stackrel{\mathrm{def}}{=} \varinjlim_{H} k^{\times}(H), \quad \overline{k}_{\times}(G) \stackrel{\mathrm{def}}{=} \varinjlim_{H} k_{\times}(H) = \overline{k}^{\times}(G)^{\circledast},$$

$$\overline{\mathcal{O}}^{\triangleright}(G) \stackrel{\mathrm{def}}{=} \varinjlim_{H} \mathcal{O}^{\triangleright}(H), \quad \boldsymbol{\mu}(G) \stackrel{\mathrm{def}}{=} \varinjlim_{H} (H^{\mathrm{ab}})_{\mathrm{tor}} = \overline{k}^{\times}(G)_{\mathrm{tor}}$$

— *where the injective limits are taken over the open subgroups $H \subseteq G$ of G, and the transition morphisms in the limits are given by the homomorphisms determined by the transfer maps — and*

$$\Lambda(G) \stackrel{\mathrm{def}}{=} \varprojlim_{n} \boldsymbol{\mu}(G)[n]$$

— *where the projective limit is taken over $n \in \mathbb{Z}_{\geq 1}$* [*cf. the discussion entitled "Fields" in* §0]. *Note that G acts on $\overline{k}^{\times}(G)$, $\overline{k}_{\times}(G)$, $\boldsymbol{\mu}(G)$, and $\Lambda(G)$ by conjugation. We shall refer to the G-module $\Lambda(G)$ as the* **cyclotome** *associated to G. Note that one verifies immediately from our construction that the cyclotome associated to G admits a natural structure of* **profinite** [*cf. also the above definition of $\Lambda(G)$*], *hence also topological, G-module; moreover, we have a natural identification $\boldsymbol{\mu}(G)[n] = \Lambda(G)/n\Lambda(G)$.*

(10) *It follows from Lemma* 1.3, (i), (x), *that the exact sequences of G-modules*

$$1 \longrightarrow \Lambda(G)/n\Lambda(G) \longrightarrow \overline{k}^{\times}(G) \stackrel{n}{\longrightarrow} \overline{k}^{\times}(G) \longrightarrow 1$$

— *where n ranges over the positive integers — determine an* **injection**

$$\mathrm{Kmm}(G)\colon\ k^{\times}(G) \hookrightarrow H^1(G, \Lambda(G)).$$

Let

$$(k,\ \overline{k},\ \alpha\colon G_k \xrightarrow{\sim} G)$$

be an **MLF-envelope** *for G* [*cf. Definition* 1.1]. *Then the following hold:*

(i) *It holds that*

$$p_k = p(G), \quad d_k = d(G), \quad f_k = f(G), \quad e_k = e(G).$$

(ii) *The isomorphism α determines* **isomorphisms**

$$I_k \xrightarrow{\sim} I(G), \quad P_k \xrightarrow{\sim} P(G).$$

Moreover, the resulting isomorphism $G_k/I_k \xrightarrow{\sim} G/I(G)$ **maps** Frob_k *to* $\mathrm{Frob}(G)$.

(iii) *The isomorphism* α*, together with the reciprocity homomorphism arising from the local class field theory of* k*, determines a* **commutative** *diagram of topological modules*

$$\begin{array}{ccccccc} \underline{k}^{\times} & \longleftarrow & \mathcal{O}_k^{\times} & \longrightarrow & \mathcal{O}_k^{\triangleright} & \longrightarrow & k^{\times} \\ \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow \\ \underline{k}^{\times}(G) & \longleftarrow & \mathcal{O}^{\times}(G) & \longrightarrow & \mathcal{O}^{\triangleright}(G) & \longrightarrow & k^{\times}(G) \end{array}$$

— where the horizontal arrows are the natural homomorphisms, and the vertical arrows are **isomorphisms***. Thus, the left-hand and right-hand vertical arrows of this diagram determine* **isomorphisms** *of monoids*

$$\underline{k}_{\times} \xrightarrow{\sim} \underline{k}_{\times}(G), \quad k_{\times} \xrightarrow{\sim} k_{\times}(G),$$

respectively.

(iv) *The isomorphism* α*, together with the reciprocity homomorphisms arising from the local class field theory of the various finite extensions of* k *in* $\overline{k}$*, determines* **isomorphisms** *of modules*

$$\overline{k}^{\times} \xrightarrow{\sim} \overline{k}^{\times}(G), \quad \boldsymbol{\mu}(\overline{k}) \xrightarrow{\sim} \boldsymbol{\mu}(G), \quad \Lambda(\overline{k}) \xrightarrow{\sim} \Lambda(G)$$

and an **isomorphism** *of monoids*

$$\overline{k}_{\times} \xrightarrow{\sim} \overline{k}_{\times}(G)$$

which are **compatible** *with the natural actions of* G_k *and* G *relative to* α*.*

(v) *The isomorphisms* $k^{\times} \xrightarrow{\sim} k^{\times}(G)$ *of* (iii) *and* $\Lambda(\overline{k}) \xrightarrow{\sim} \Lambda(G)$ *of* (iv) *fit into a* **commutative** *diagram*

$$\begin{array}{ccc} k^{\times} & \xrightarrow{\mathrm{Kmm}_k} & H^1(G_k, \Lambda(\overline{k})) \\ \wr\downarrow & & \wr\downarrow \\ k^{\times}(G) & \xrightarrow{\mathrm{Kmm}(G)} & H^1(G, \Lambda(G)). \end{array}$$

PROOF. These assertions follow immediately from Lemma 1.3, together with the various definitions involved. □

Remark 1.4.1.

(i) It is well-known [cf., e.g., [4], §1, Theorem; [4], §2] that there exist MLF's $k_\circ$ and $k_\bullet$ such that $k_\circ$ is *not isomorphic* to $k_\bullet$, but the absolute Galois group of $k_\circ$ [for some choice of an algebraic closure of $k_\circ$] is *isomorphic* to the absolute Galois group of $k_\bullet$ [for some choice of an algebraic closure of $k_\bullet$]. Moreover, it is known [cf., e.g., the final portion of [11], Chapter VII] that, for each MLF k such that p_k is *odd*, there exists an outer automorphism of the absolute Galois group of k which does *not arise* from an automorphism of k.

(ii) It follows immediately from the discussion of (i) that

> there is *no functorial "group-theoretic" algorithm* [as discussed in Theorem 1.4] for reconstructing, from the *absolute Galois group of an MLF*, [the field structure of] the *MLF*.

(iii) On the other hand, there are some results concerning the *geometricity* of an outer homomorphism between absolute Galois groups of MLF's. For instance, in [5], S. Mochizuki proved that, for an outer isomorphism between absolute Galois groups of MLF's, it holds that the outer isomorphism is *geometric* [i.e., arises from a — necessarily unique — isomorphism of MLF's] if and only if the outer isomorphism preserves the [positively indexed] higher ramification filtrations in the upper numbering. Mochizuki also gave, in [7], §3 [cf. [7], Theorem 3.5; [7], Corollary 3.7], other necessary and sufficient conditions for an outer open homomorphism between absolute Galois groups of MLF's to be *geometric* [i.e., arise from a — necessarily unique — embedding of MLF's]. Moreover, in [2], the author proved that, for an outer open homomorphism between absolute Galois groups of MLF's, it holds that the outer open homomorphism is *geometric* if and only if the outer open homomorphism is *Hodge-Tate-preserving* [i.e., the pull-back, via the outer open homomorphism under consideration, of a Hodge-Tate representation is still Hodge-Tate].

Remark 1.4.2.

(i) In the proof of the main result of [5] [cf. Remark 1.4.1, (iii)], Mochizuki essentially proved the following assertion:

> For $\square \in \{\circ, \bullet\}$, let $k_\square$ be an MLF. Write $G_\square$ for the absolute Galois group of $k_\square$ [which is well-defined up to conjugation]. Let $\alpha\colon G_\circ \xrightarrow{\sim} G_\bullet$ be an outer isomorphism of profinite groups. Then it holds that α is *geometric* if and only if, in the notation of Theorem 1.4, (6), the following condition is satisfied: For every open subgroup $G_\circ^\dagger \subseteq G_\circ$ of $G_\circ$, if we write $G_\bullet^\dagger \subseteq G_\bullet$ for the open subgroup of $G_\bullet$ corresponding to $G_\circ^\dagger \subseteq G_\circ$ via α, then the isomorphism $k^\times(G_\circ^\dagger) \xrightarrow{\sim} k^\times(G_\bullet^\dagger)$ induced by α maps, for each positive integer n, the submodule of $k^\times(G_\circ^\dagger)$ corresponding to "$1 + \mathfrak{m}_k^n$" bijectively onto the submodule of $k^\times(G_\bullet^\dagger)$ corresponding to "$1 + \mathfrak{m}_k^n$".

Here, we recall that, in the above notation, it follows from the *functorial "group-theoretic" algorithms* discussed in Theorem 1.4 that the induced isomorphism $k^\times(G_\circ^\dagger) \xrightarrow{\sim}$

$k^\times(G^\dagger_\bullet)$ maps the submodule of $k^\times(G^\dagger_\circ)$ corresponding to "$1+\mathfrak{m}_k$" [i.e., the kernel of the natural surjection $\mathcal{O}^\times(G^\dagger_\circ) \twoheadrightarrow \underline{k}^\times(G^\dagger_\circ)$ — cf. Theorem 1.4, (5)] bijectively onto the submodule of $k^\times(G^\dagger_\bullet)$ corresponding to "$1+\mathfrak{m}_k$" [i.e., the kernel of the natural surjection $\mathcal{O}^\times(G^\dagger_\bullet) \twoheadrightarrow \underline{k}^\times(G^\dagger_\bullet)$ — cf. Theorem 1.4, (5)].

(ii) In particular, we conclude form the discussion of (i) and Remark 1.4.1, (ii), that

> there is *no functorial "group-theoretic" algorithm* [as discussed in Theorem 1.4] for reconstructing, from a group G *of MLF-type*, the family of submodules of the module $k^\times(G)$ of Theorem 1.4, (6), corresponding to the family of submodules "$\{1+\mathfrak{m}_k^n\}_{n\geq 1}$" of "$k^\times$".

Remark 1.4.3.

(i) Write k_+, $(\mathcal{O}_k)_+$ for the modules obtained by forming the underlying *additive* modules of the rings k, $\mathcal{O}_k$, respectively. Then, by considering the p_k-adic logarithm on k, we obtain an *isomorphism* of modules $(\mathcal{O}_k^\times)^{\mathrm{pf}} \xrightarrow{\sim} k_+$ [cf. the discussion entitled "Monoids" in §0]. Thus, by assigning $G \mapsto \mathcal{O}^\times(G)^{\mathrm{pf}}$ [cf. Theorem 1.4, (5)], we obtain a *functorial "group-theoretic" algorithm* [as discussed in Theorem 1.4] for reconstructing, from a group G *of MLF-type*, the module corresponding to "k_+". Then one may give another interpretation of the assertion of Remark 1.4.2, (i), as follows:

> For $\Box \in \{\circ, \bullet\}$, let $k_\Box$ be an MLF. Write $G_\Box$ for the absolute Galois group of $k_\Box$ [which is well-defined up to conjugation]. Let $\alpha\colon G_\circ \xrightarrow{\sim} G_\bullet$ be an outer isomorphism of profinite groups. Then it holds that α is *geometric* if and only if, in the notation of Theorem 1.4, (5), the following condition is satisfied: For every open subgroup $G^\dagger_\circ \subseteq G_\circ$ of $G_\circ$, if we write $G^\dagger_\bullet \subseteq G_\bullet$ for the open subgroup of $G_\bullet$ corresponding to $G^\dagger_\circ \subseteq G_\circ$ via α, then the isomorphism $\mathcal{O}^\times(G^\dagger_\circ)^{\mathrm{pf}} \xrightarrow{\sim} \mathcal{O}^\times(G^\dagger_\bullet)^{\mathrm{pf}}$ induced by α maps the submodule of $\mathcal{O}^\times(G^\dagger_\circ)^{\mathrm{pf}}$ corresponding to "$(\mathcal{O}_k)_+ \subseteq k_+$" bijectively onto the submodule of $\mathcal{O}^\times(G^\dagger_\bullet)^{\mathrm{pf}}$ corresponding to "$(\mathcal{O}_k)_+ \subseteq k_+$".

(ii) In particular, we conclude from the discussion of (i) and Remark 1.4.1, (ii), that

> there is *no functorial "group-theoretic" algorithm* [as discussed in Theorem 1.4] for reconstructing, from a group G *of MLF-type*, the submodule of the module $\mathcal{O}^\times(G)^{\mathrm{pf}}$ corresponding to the submodule "$(\mathcal{O}_k)_+$" of "k_+".

Lemma 1.5. *The following hold:*

(i) *It holds that*

$$\mathcal{O}_k^\times \;=\; \mathrm{Ker}\big(k^\times \overset{\mathrm{Kmm}_k}{\hookrightarrow} H^1(G_k, \Lambda(\overline{k})) \;\to\; H^1(I_k, \Lambda(\overline{k})^{(p_k')})\big)$$

[*cf. Theorem* 1.4, (x)].

(ii) *The homomorphism*

$$\mathcal{O}_k^\times \longrightarrow H^1(G_k/I_k, \Lambda(\overline{k})^{(p'_k)})$$

determined by Kmm_k [*cf.* (i)] *induces an* **isomorphism**

$$\underline{k}^\times \xrightarrow{\sim} H^1(G_k/I_k, \Lambda(\overline{k})^{(p'_k)}).$$

PROOF. These assertions follow immediately from the well-known explicit description of the topological module $k^\times$ [cf., e.g., [10], Chapter II, Proposition 5.3; also [10], Chapter II, Proposition 5.7, (i)], together with the *Kummer theory* of k, $\underline{k}$. □

§ 2. Reconstruction of the Additive Structure on an NF-monoid

In the present §2, we introduce the notion of an *NF-monoid* [cf. Definition 2.3 below] and discuss a *mono-anabelian reconstruction* of the "*additive structure*" on an NF-monoid [cf. Theorem 2.9 below]. Note that the main result of the present §2 was already essentially proved in [3]; however, the discussion in [3] of the issue of reconstruction of the additive structure was presented in a "*bi-anabelian" fashion*, not in a "*mono-anabelian" fashion*, as is necessary in the present paper.

In the present §2, let

$$F$$

be an *NF*. We shall write

- $\mathcal{O}_F \subseteq F$ for the ring of integers of F,
- $\mathcal{V}_F$ for the set of nonarchimedean primes of F, and
- $F_{\mathrm{prm}} \subseteq F$ for the prime field contained in F [i.e., "$\mathbb{Q}$"].

If $v \in \mathcal{V}_F$, then we shall write

- $\mathrm{ord}_v \colon F^\times \twoheadrightarrow \mathbb{Z}$ for the [uniquely determined] surjective valuation associated to v,
- $\mathcal{O}_{(v)} \subseteq F$ for the subring of F obtained by forming the localization of $\mathcal{O}_F$ at the maximal ideal corresponding to v,
- $\mathfrak{m}_{(v)} \subseteq \mathcal{O}_{(v)}$ for the maximal ideal of $\mathcal{O}_{(v)}$,
- $\kappa_v \stackrel{\mathrm{def}}{=} \mathcal{O}_{(v)}/\mathfrak{m}_{(v)}$ for the residue field of $\mathcal{O}_{(v)}$,
- $\mathrm{char}(v) \stackrel{\mathrm{def}}{=} \mathrm{char}(\kappa_v)$ for the characteristic of κ_v, and
- $\mathcal{O}_{(v)}^{\prec} \stackrel{\mathrm{def}}{=} 1 + \mathfrak{m}_{(v)} \subseteq \mathcal{O}_{(v)}^\times$ for the kernel of the natural homomorphism $\mathcal{O}_{(v)}^\times \twoheadrightarrow \kappa_v^\times$.

Finally, for $a \in F^{\times}$, we shall write

- $\mathrm{Supp}(a) \stackrel{\mathrm{def}}{=} \{ v \in \mathcal{V}_F \mid \mathrm{ord}_v(a) \neq 0 \} \subseteq \mathcal{V}_F$.

Definition 2.1. We shall say that the NF F is *of PmF-type* [where "PmF" is to be understood as an abbreviation for "Prime Field"] if $F = F_{\mathrm{prm}}$.

Definition 2.2. We shall refer to the collection of data

$$(F_{\times},\ \mathcal{O}_F^{\triangleright} \subseteq F_{\times},\ \mathcal{V}_F,\ \{\mathcal{O}_{(v)}^{\prec} \subseteq F_{\times}\}_{v \in \mathcal{V}_F})$$

[consisting of the monoid $F_{\times}$, the submonoid $\mathcal{O}_F^{\triangleright} \subseteq F_{\times}$ of $F_{\times}$, the set $\mathcal{V}_F$, and, for each $v \in \mathcal{V}_F$, the submonoid $\mathcal{O}_{(v)}^{\prec} \subseteq F_{\times}$ of $F_{\times}$] as the *NF-monoid associated to F*.

Definition 2.3. Let

$$\mathcal{M} \ = \ (M,\ O^{\triangleright} \subseteq M,\ S,\ \{O_s^{\prec} \subseteq M\}_{s \in S})$$

be a collection of data consisting of a monoid M [the monoid operation of M will be written *multiplicatively*], a submonoid $O^{\triangleright} \subseteq M$ of M, a set S, and, for each $s \in S$, a submonoid $O_s^{\prec} \subseteq M$ of M. Then we shall refer to an isomorphism of the NF-monoid associated to an NF (respectively, an NF of PmF-type — cf. Definition 2.1) [cf. Definition 2.2] with $\mathcal{M}$ [in the evident sense, i.e., a pair consisting of an isomorphism of "$F_{\times}$" with M and a bijection of "$\mathcal{V}_F$" with S which satisfy suitable conditions] as an *NF-envelope* (respectively, *NF-envelope of PmF-type*) for $\mathcal{M}$. We shall say that $\mathcal{M}$ is an *NF-monoid* (respectively, *NF-monoid of PmF-type*) if there exists an NF-envelope (respectively, NF-envelope of PmF-type) for $\mathcal{M}$.

Lemma 2.4. *The following hold:*

(i) *The NF F is* **of PmF-type** *if and only if, for all but finitely many $v \in \mathcal{V}_F$, it holds that $\sharp\kappa_v$ is a* **prime** *number.*

(ii) *The element $0 \in F_{\times}$ of $F_{\times}$ may be* **characterized** *as the unique element of $F_{\times} \setminus F^{\times}$.*

(iii) *The element $1 \in F_{\times}$ of $F_{\times}$ may be* **characterized** *as the unique element $a \in F_{\times}$ such that $ax = x$ for every $x \in F_{\times}$.*

(iv) *The element $-1 \in F_{\times}$ of $F_{\times}$ may be* **characterized** *as the unique element $a \in F_{\times}$ such that $a \neq 1$ but $a^2 = 1$.*

(v) *Let $v \in \mathcal{V}_F$. Then the natural injection $\mathcal{O}_{(v)}^{\times} \hookrightarrow F^{\times}$ determines an* **isomorphism** $\kappa_v^{\times} \xrightarrow{\sim} (F^{\times}/\mathcal{O}_{(v)}^{\prec})_{\mathrm{tor}}$.

(vi) *Let $v \in \mathcal{V}_F$. Then the prime number $\mathrm{char}(v)$ may be* **characterized** *as the unique prime number that divides $\sharp\kappa_v$.*

(vii) *Let* $v \in \mathcal{V}_F$. *Then the* $\{\pm 1\}$*-orbit* [*with respect to the action of* $\{\pm 1\}$ *on* $\mathbb{Z}$] *of the valuation* $\mathrm{ord}_v \colon F^\times \to \mathbb{Z}$ *may be* **characterized** *as the* $\{\pm 1\}$*-orbit of the homomorphism* $F^\times \to \mathbb{Z}$ *obtained by forming the composite*

$$F^\times \twoheadrightarrow F^\times/\mathcal{O}^{\prec}_{(v)} \twoheadrightarrow (F^\times/\mathcal{O}^{\prec}_{(v)})^{\mathrm{ab/tor}} \xrightarrow{\sim} \mathbb{Z}$$

— *where we regard* $F^\times/\mathcal{O}^{\prec}_{(v)}$ *as a topological group by equipping it with the discrete topology, and the "*$\xrightarrow{\sim}$*" is an isomorphism of groups. Moreover, the valuation* $\mathrm{ord}_v \colon F^\times \to \mathbb{Z}$ *may be* **characterized** *as the unique element of this orbit which maps* $\mathcal{O}^{\triangleright}_F \subseteq F^\times$ *to* $\mathbb{Z}_{\geq 0} \subseteq \mathbb{Z}$.

(viii) *Let* $v \in \mathcal{V}_F$. *Then it holds that* $\mathcal{O}^\times_{(v)} = \mathrm{Ker}(\mathrm{ord}_v)$.

PROOF. Assertion (i) follows immediately from *Čebotarev's density theorem* [cf. also [10], Chapter VII, Corollary 13.7]. Assertions (ii), (iii), (iv), (vi), (viii) follow from the various definitions involved. Assertion (v) and the first portion of assertion (vii) follow immediately from the fact that $F^\times/\mathcal{O}^\times_{(v)}$ is [noncanonically] isomorphic to $\mathbb{Z}$, hence also *torsion-free* [cf. also the proof of [3], Lemma 1.5, (i)]. The final portion of assertion (vii) follows from the various definitions involved. This completes the proof of Lemma 2.4. □

Proposition 2.5. *Let*

$$\mathcal{M} = (M,\ O^{\triangleright} \subseteq M,\ S,\ \{O^{\prec}_s \subseteq M\}_{s \in S})$$

be an **NF-monoid**. *We construct various objects associated to* $\mathcal{M}$ *as follows:*

(1) *It follows from Lemma* 2.4, (ii), *that there exists a* **unique** *element of* $M \setminus M^\times$. *We shall write*

$$0_{\mathcal{M}} \in M$$

for this element.

(2) *It follows from Lemma* 2.4, (iii), *that there exists a* **unique** *element* $a \in M$ *of* M *such that* $ax = x$ *for any* $x \in M$. *We shall write*

$$1_{\mathcal{M}} \in M$$

for this element.

(3) *It follows from Lemma* 2.4, (iv), *that there exists a* **unique** *element* $a \in M$ *of* M *such that* $a \neq 1_{\mathcal{M}}$ *but* $a^2 = 1_{\mathcal{M}}$. *We shall write*

$$-1_{\mathcal{M}} \in M$$

for this element.

(4) *Let* $s \in S$. *Then we shall write*

$$\underline{O}_s^{\times} \stackrel{\text{def}}{=} (M^{\times}/O_s^{\prec})_{\text{tor}}, \quad (\underline{O}_s)_{\times} \stackrel{\text{def}}{=} (\underline{O}_s^{\times})^{\circledast}$$

[*cf. Lemma* 2.4, (v)].

(5) *Let* $s \in S$. *Then it follows from Lemma* 2.4, (v), (vi), *that there exists a* **unique** *prime number which divides* $\sharp(\underline{O}_s)_{\times}$. *We shall write*

$$\text{char}(s)$$

for this prime number.

(6) *Let* $s \in S$. *Then we shall write*

$$Z_s \stackrel{\text{def}}{=} (M^{\times}/O_s^{\prec})^{\text{ab/tor}}$$

— *where we regard* $M^{\times}/O_s^{\prec}$ *as a topological group by equipping it with the discrete topology — and*

$$\text{ord}_s^{\pm}\colon \ M^{\times} \twoheadrightarrow Z_s$$

for the natural surjection [*cf. Lemma* 2.4, (vii)].

(7) *Let* $s \in S$ *and* $a \in M^{\times}$. *Then we define an integer*

$$\text{ord}_s(a) \in \mathbb{Z}$$

as follows: Write $\text{ord}_s^{\pm}(a)^{\mathbb{N}} \subseteq \text{ord}_s^{\pm}(a)^{\mathbb{Z}} \subseteq Z_s$ *for the submonoids of* Z_s *generated, respectively, by* $\text{ord}_s^{\pm}(a) \in Z_s$, $\pm\text{ord}_s^{\pm}(a) \in Z_s$ [*where we write the monoid operation of* Z_s *additively*]; $i_{s,a} \stackrel{\text{def}}{=} [Z_s : \text{ord}_s^{\pm}(a)^{\mathbb{Z}}] \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$. *Then*

$$\text{ord}_s(a) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } i_{s,a} = \infty, \\ i_{s,a} & \text{if } i_{s,a} < \infty \text{ and } \sharp\big(\text{ord}_s^{\pm}(a)^{\mathbb{N}} \cap \text{ord}_s^{\pm}(O^{\rhd})\big) \neq 1, \\ -i_{s,a} & \text{if } i_{s,a} < \infty \text{ and } \sharp\big(\text{ord}_s^{\pm}(a)^{\mathbb{N}} \cap \text{ord}_s^{\pm}(O^{\rhd})\big) = 1 \end{cases}$$

[*cf. Lemma* 2.4, (vii)].

(8) *Let* $a \in M^{\times}$. *Then we shall write*

$$\text{Supp}(a) \stackrel{\text{def}}{=} \{\, s \in S \mid \text{ord}_s(a) \neq 0 \,\} \subseteq S.$$

(9) *Let* $s \in S$. *Then we shall write*

$$O_s^{\times} \stackrel{\text{def}}{=} \text{Ker}(\text{ord}_s) \subseteq M^{\times}$$

[*cf.* *Lemma* 2.4, (viii)].

Let

$$(\phi\colon F_\times \xrightarrow{\sim} M,\ \tau\colon \mathcal{V}_F \xrightarrow{\sim} S)$$

be an **NF-envelope** *for* $\mathcal{M}$. *Then the following hold:*

(i) *The NF-monoid* $\mathcal{M}$ *is* **of PmF-type** *if and only if, for all but finitely many* $s \in S$, *it holds that* $\sharp(\underline{O}_s)_\times$ *is a* **prime** *number* [*cf. Lemma* 2.4, (i)].

(ii) *The isomorphism* $\phi\colon F_\times \xrightarrow{\sim} M$ *of monoids maps* 0, 1, −1 *to* $0_\mathcal{M}$, $1_\mathcal{M}$, $-1_\mathcal{M}$, *respectively.*

(iii) *Let* $v \in \mathcal{V}_F$. *Write* $s \stackrel{\mathrm{def}}{=} \tau(v)$. *Then it holds that*

$$\mathrm{char}(v) \ = \ \mathrm{char}(s), \quad \mathrm{ord}_v \ = \ \mathrm{ord}_s \circ (\phi|_{F^\times}).$$

Moreover, the isomorphism $\phi\colon F_\times \xrightarrow{\sim} M$ *of monoids determines* **isomorphisms** *of monoids*

$$\kappa_v^\times \ \xrightarrow{\sim}\ \underline{O}_s^\times, \quad (\kappa_v)_\times \ \xrightarrow{\sim}\ (\underline{O}_s)_\times, \quad \mathcal{O}_{(v)}^\times \ \xrightarrow{\sim}\ O_s^\times.$$

(iv) *Let* $a \in F^\times$. *Then the bijection* $\tau\colon \mathcal{V}_F \xrightarrow{\sim} S$ *determines a* **bijection**

$$\mathrm{Supp}(a) \ \xrightarrow{\sim}\ \mathrm{Supp}(\phi(a)).$$

(v) *Let* $s \in S$. *Then the composite* $O_s^\times \hookrightarrow M^\times \twoheadrightarrow M^\times/O_s^\prec$ *determines a* **surjection**

$$O_s^\times \ \twoheadrightarrow\ \underline{O}_s^\times$$

which fits into a **commutative** *diagram*

$$\begin{array}{ccc}
\mathcal{O}_{(v)}^\times & \longrightarrow & \kappa_v^\times \\
\wr\downarrow & & \wr\downarrow \\
O_s^\times & \longrightarrow & \underline{O}_s^\times
\end{array}$$

— *where the upper horizontal arrow is the natural surjection, and the vertical arrows are the isomorphisms of* (iii).

PROOF. These assertions follow immediately from Lemma 2.4, together with the various definitions involved. □

Lemma 2.6. *Suppose that* F *is* **of PmF-type**. *Write* $(\mathcal{O}_F)_+ \subseteq \mathcal{O}_F$ *for the complement of* $\{0\} \subseteq \mathcal{O}_F$ *in the submonoid of the underlying additive module of* $\mathcal{O}_F$ *generated by the identity element of the multiplicative group* $F^\times$, *i.e., the subset "*$\mathbb{Z}_{\geq 1} \subseteq$

$\mathbb{Z}$". For each prime number p, write $v_p \in \mathcal{V}_F$ for the nonarchimedean prime of F corresponding to the maximal ideal $p\mathcal{O}_F \subseteq \mathcal{O}_F$ of $\mathcal{O}_F$. Then the following hold:

(i) *The nonarchimedean prime v_2 (respectively, v_3; v_5) of F may be* **characterized** *as the unique nonarchimedean prime v of F such that* $\mathrm{char}(v) = 2$ *(respectively,* 3*;* 5*).*

(ii) *The element $2 \in \mathcal{O}_F^{\triangleright}$ of $\mathcal{O}_F^{\triangleright}$ may be* **characterized** *as the unique element $a \in \mathcal{O}_F^{\triangleright}$ such that* $\mathrm{Supp}(a) = \{v_2\}$, $\mathrm{ord}_{v_2}(a) = 1$, *and* $a \notin \mathcal{O}_{(v_3)}^{\prec}$.

(iii) *The element $3 \in \mathcal{O}_F^{\triangleright}$ of $\mathcal{O}_F^{\triangleright}$ may be* **characterized** *as the unique element $a \in \mathcal{O}_F^{\triangleright}$ such that* $\mathrm{Supp}(a) = \{v_3\}$, $\mathrm{ord}_{v_3}(a) = 1$, *and* $2a \in \mathcal{O}_{(v_5)}^{\prec}$.

(iv) *Let $a \in \mathcal{O}_F^{\triangleright}$ be such that $a \notin \{-1, 1\}$. Then it holds that*

$$\begin{aligned}\{a-1, a+1\} &= \{\, b \in \mathcal{O}_F^{\triangleright} \setminus \{a\} \mid \mathrm{Supp}(a-b) = \emptyset \,\} \\ &= \{\, b \in \mathcal{O}_F^{\triangleright} \mid \mathrm{Supp}(a) \cap \mathrm{Supp}(b) = \emptyset,\ a \cdot b^{-1} \notin \mathcal{O}_{(v)}^{\prec} \text{ for all } v \in \mathcal{V}_F \,\}.\end{aligned}$$

(v) *Let $a \in \mathcal{O}_F^{\triangleright}$ be such that $\mathrm{Supp}(a) \not\subseteq \{v_2\}$. Then it holds that*

$$\{a+1\} \;=\; \{a-1, a+1\} \;\cap \bigcap_{v \in \mathrm{Supp}(a)} \mathcal{O}_{(v)}^{\prec}.$$

(vi) *Let $a \in \mathcal{O}_F^{\triangleright}$ be such that $a \notin \{-2, -1, 1, 2\}$, and, moreover,* $\mathrm{Supp}(a) \subseteq \{v_2\}$. *Then, for every $b \in \{a-1, a+1\}$, it holds that $\mathrm{Supp}(b) \not\subseteq \{v_2\}$, hence also that $b \notin \{-2, -1, 1, 2\}$.*

(vii) *The map $\mathcal{O}_F \to \mathcal{O}_F$ given by mapping a to $a+1$ is* **bijective**.

(viii) *The subset $(\mathcal{O}_F)_+ \subseteq \mathcal{O}_F$ may be characterized uniquely as the* **minimal** *subset of $\mathcal{O}_F$ which contains $1 \in \mathcal{O}_F$ and, moreover, is mapped into itself by the bijection discussed in* (vii).

(ix) *Let $v \in \mathcal{V}_F$. Then the composite $(\mathcal{O}_F)_+ \cap \mathcal{O}_{(v)}^{\times} \hookrightarrow \mathcal{O}_{(v)}^{\times} \twoheadrightarrow \kappa_v^{\times}$ is* **surjective**.

PROOF. These assertions follow from the various definitions involved. □

Proposition 2.7. *Let*

$$\mathcal{M} \;=\; (M,\ O^{\triangleright} \subseteq M,\ S,\ \{O_s^{\prec} \subseteq M\}_{s \in S})$$

be an **NF-monoid of PmF-type**. *We construct various objects associated to $\mathcal{M}$ as follows:*

(1) *It follows from Proposition* 2.5, (iii), *and Lemma* 2.6, (i), *that there exists a* **unique** *element $s \in S$ such that* $\mathrm{char}(s) = 2$ *(respectively,* 3*;* 5*) [cf. Proposition* 2.5, (5)]. *We shall write*

$$(2)_{\mathcal{M}} \;\; (\textit{respectively},\;\; (3)_{\mathcal{M}};\;\; (5)_{\mathcal{M}}) \;\in\; S$$

for this element.

(2) *It follows from Proposition* 2.5, (iii), (iv), *and Lemma* 2.6, (ii), *that there exists a* **unique** *element* $a \in O^{\rhd}$ *of* $O^{\rhd}$ *such that* $\mathrm{Supp}(a) = \{(2)_{\mathcal{M}}\}$ [*cf. Proposition* 2.5, (8)], $\mathrm{ord}_{(2)_{\mathcal{M}}}(a) = 1$ [*cf. Proposition* 2.5, (7)], *and* $a \notin O^{\prec}_{(3)_{\mathcal{M}}}$. *We shall write*

$$2_{\mathcal{M}} \ \in \ O^{\rhd}$$

for this element $a \in O^{\rhd}$ *and*

$$-2_{\mathcal{M}} \ \stackrel{\mathrm{def}}{=} \ -1_{\mathcal{M}} \cdot 2_{\mathcal{M}} \ \in \ O^{\rhd}$$

[*cf. Proposition* 2.5, (3); *Proposition* 2.5, (ii)].

(3) *It follows from Proposition* 2.5, (iii), (iv), *and Lemma* 2.6, (iii), *that there exists a* **unique** *element* $a \in O^{\rhd}$ *of* $O^{\rhd}$ *such that* $\mathrm{Supp}(a) = \{(3)_{\mathcal{M}}\}$, $\mathrm{ord}_{(3)_{\mathcal{M}}}(a) = 1$, *and* $2_{\mathcal{M}} \cdot a \in O^{\prec}_{(5)_{\mathcal{M}}}$. *We shall write*

$$3_{\mathcal{M}} \ \in \ O^{\rhd}$$

for this element $a \in O^{\rhd}$.

(4) *Let* $a \in O^{\rhd} \setminus \{-1_{\mathcal{M}}, 1_{\mathcal{M}}\}$ [*cf. Proposition* 2.5, (2)]. *Then we shall write*

$$\mathrm{adj}_{\mathcal{M}}(a) \ \stackrel{\mathrm{def}}{=} \ \{\, b \in O^{\rhd} \mid \mathrm{Supp}(a) \cap \mathrm{Supp}(b) = \emptyset,\ a \cdot b^{-1} \notin O^{\prec}_{s} \textit{ for all } s \in S \,\} \ \subseteq \ O^{\rhd}$$

[*cf. Proposition* 2.5, (ii), (iv); *Lemma* 2.6, (iv)].

(5) *Let* $a \in O^{\rhd}$ *be such that* $\mathrm{Supp}(a) \not\subseteq \{(2)_{\mathcal{M}}\}$. *Then it follows from Proposition* 2.5, (iv), *and Lemma* 2.6, (iv), (v), *that the intersection*

$$\mathrm{adj}_{\mathcal{M}}(a) \ \cap \bigcap_{s \in \mathrm{Supp}(a)} O^{\prec}_{s}$$

is **of cardinality one**. *We shall write*

$$\mathrm{next}_{\mathcal{M}}(a) \ \in \ O^{\rhd}$$

for the unique element of this intersection.

(6) *Let* $a \in O^{\rhd} \setminus \{-2_{\mathcal{M}}, -1_{\mathcal{M}}, 1_{\mathcal{M}}, 2_{\mathcal{M}}\}$ *be such that* $\mathrm{Supp}(a) \subseteq \{(2)_{\mathcal{M}}\}$. *Then it follows from Proposition* 2.5, (ii), (iv), *and Lemma* 2.6, (iv), (v), (vi), *that there exists a* **unique** *element* $b \in \mathrm{adj}_{\mathcal{M}}(a)$ *of* $\mathrm{adj}_{\mathcal{M}}(a)$ *such that* $\mathrm{Supp}(b) \not\subseteq \{(2)_{\mathcal{M}}\}$, *and, moreover,* $a \neq \mathrm{next}_{\mathcal{M}}(b)$. *We shall write*

$$\mathrm{next}_{\mathcal{M}}(a) \ \in \ O^{\rhd}$$

for this element $b \in \mathrm{adj}_{\mathcal{M}}(a)$.

(7) *We shall write*

$$\mathrm{next}_{\mathcal{M}}(-2_{\mathcal{M}}) \stackrel{\mathrm{def}}{=} -1_{\mathcal{M}}, \ \mathrm{next}_{\mathcal{M}}(-1_{\mathcal{M}}) \stackrel{\mathrm{def}}{=} 0_{\mathcal{M}}, \ \mathrm{next}_{\mathcal{M}}(0_{\mathcal{M}}) \stackrel{\mathrm{def}}{=} 1_{\mathcal{M}},$$

$$\mathrm{next}_{\mathcal{M}}(1_{\mathcal{M}}) \stackrel{\mathrm{def}}{=} 2_{\mathcal{M}}, \ \mathrm{next}_{\mathcal{M}}(2_{\mathcal{M}}) \stackrel{\mathrm{def}}{=} 3_{\mathcal{M}}$$

[*cf. Proposition* 2.5, (1)*; Proposition* 2.5, (ii)]*. Then, by Lemma* 2.6, (vii)*, together with our construction, we have a* **bijection**

$$\mathrm{next}_{\mathcal{M}} \colon O^{\triangleright} \cup \{0_{\mathcal{M}}\} \xrightarrow{\sim} O^{\triangleright} \cup \{0_{\mathcal{M}}\}.$$

(8) *It follows from Lemma* 2.6, (viii)*, that there exists a* **unique** *subset of* $O^{\triangleright} \cup \{0_{\mathcal{M}}\}$ *which is minimal among those subsets that contain* $1_{\mathcal{M}}$ *and, moreover, are mapped into themselves by the map* $\mathrm{next}_{\mathcal{M}}$*. We shall write*

$$O_{+} \subseteq O^{\triangleright} \cup \{0_{\mathcal{M}}\}$$

for this subset.

(9) *Let* $s \in S$*;* $a, b \in (\underline{O}_s)_{\times}$ [*cf. Proposition* 2.5, (4)]*. Then we define an element of* $(\underline{O}_s)_{\times}$

$$a \boxplus_s b \in (\underline{O}_s)_{\times}$$

as follows: Write $\underline{0}_s \in (\underline{O}_s)_{\times}$ *for the unique element of* $(\underline{O}_s)_{\times} \setminus \underline{O}_s^{\times}$ [*cf. Proposition* 2.5, (4)]*. If* $a = \underline{0}_s$*, then* $a \boxplus_s b \stackrel{\mathrm{def}}{=} b$*. If* $b = \underline{0}_s$*, then* $a \boxplus_s b \stackrel{\mathrm{def}}{=} a$*. In the following, suppose that* $a, b \in \underline{O}_s^{\times}$*. Then it follows from Lemma* 2.6, (ix)*, that there exist respective liftings* $\widetilde{a}, \widetilde{b} \in O_{+} \cap O_s^{\times}$ [*cf. Proposition* 2.5, (9)] *of* $a, b \in \underline{O}_s^{\times}$ [*relative to the surjection* $O_s^{\times} \twoheadrightarrow \underline{O}_s^{\times}$ *of Proposition* 2.5, (v)]*. Write* $n_{\widetilde{b}} \in \mathbb{Z}$ *for the positive integer defined by* $\prod_{s \in S} \mathrm{char}(s)^{\mathrm{ord}_s(\widetilde{b})}$ [*cf. Proposition* 2.5, (iii)] *and*

$$c \stackrel{\mathrm{def}}{=} \overbrace{\mathrm{next}_{\mathcal{M}} \circ \cdots \circ \mathrm{next}_{\mathcal{M}}}^{n_{\widetilde{b}}}(\widetilde{a}) \in O_{+}.$$

Then

$$a \boxplus_s b \stackrel{\mathrm{def}}{=} \begin{cases} \underline{0}_s & \text{if } c \notin O_s^{\times}, \\ \text{the image of } c \text{ in } (\underline{O}_s)_{\times} & \text{if } c \in O_s^{\times}. \end{cases}$$

Note that one verifies immediately from our construction that "$a \boxplus_s b$*" does* **not depend** *on the choice of the respective liftings* $\widetilde{a}, \widetilde{b} \in O_{+} \cap O_s^{\times}$ *of* $a, b \in \underline{O}_s^{\times}$*.*

(10) *Let* $s \in S$*. Then it follows immediately from our construction that the "*$\boxplus_s$*" of* (9)*, together with the monoid structure of* $(\underline{O}_s)_{\times}$*, determines a* **structure of field** *on* $(\underline{O}_s)_{\times}$*. We shall write*

$$\underline{O}_s$$

for the resulting field.

Let

$$(\phi\colon F_{\times} \overset{\sim}{\to} M,\ \tau\colon \mathcal{V}_F \overset{\sim}{\to} S)$$

be an **NF-envelope** [*necessarily* **of PmF-type** — *cf. Lemma* 2.4, (i)*; Proposition* 2.5, (i), (iii)] *for* $\mathcal{M}$. *Let* $v \in \mathcal{V}_F$*; write* $s \overset{\mathrm{def}}{=} \tau(v)$. *Then the isomorphism of monoids*

$$(\kappa_v)_{\times} \overset{\sim}{\longrightarrow} (\underline{O}_s)_{\times}$$

of Proposition 2.5, (iii), *determines an* **isomorphism of fields**

$$\kappa_v \overset{\sim}{\longrightarrow} \underline{O}_s.$$

PROOF. This follows immediately from Lemma 2.6, together with the various definitions involved. □

Lemma 2.8. *The following hold:*

(i) *For* $a \in F^{\times}$, *it holds that* $a \in F^{\times}_{\mathrm{prm}}$ *if and only if, for all but finitely many* $v \in \mathcal{V}_F$, *it holds that* $a^{\mathrm{char}(v)-1} \in \mathcal{O}^{\prec}_{(v)}$.

(ii) *Let* $v \in \mathcal{V}_F$. *Then the intersection* $F^{\times}_{\mathrm{prm}} \cap \mathcal{O}^{\triangleright}_F$ (*respectively,* $F^{\times}_{\mathrm{prm}} \cap \mathcal{O}^{\prec}_{(v)}$) **coincides** *with "*$\mathcal{O}^{\triangleright}_F$*" (respectively, "*$\mathcal{O}^{\prec}_{(v)}$*") in the case where we take "*(F, v)*" to be* $(F_{\mathrm{prm}}, v_{\mathrm{char}(v)})$ [*cf. the notation introduced in Lemma* 2.6].

(iii) *Write* $\mathcal{V}^{f=1}_F \subseteq \mathcal{V}_F$ *for the subset of* $\mathcal{V}_F$ *consisting of* $v \in \mathcal{V}_F$ *such that* $\sharp\kappa_v = \mathrm{char}(v)$. *Then* $\mathcal{V}^{f=1}_F$ *is* **infinite**.

(iv) *Let* $a, b \in F^{\times}$ *be such that* $0 \notin \{a, b, a+b\}$. *Then the element* $a+b \in F^{\times}$ *may be* **characterized** *as the unique element* $c \in F^{\times}$ *which satisfies the following condition: For infinitely many* $v \in \mathcal{V}_F$ *such that* $\{a, b, c\} \subseteq \mathcal{O}^{\times}_{(v)}$, *if we write* $\overline{a}, \overline{b}, \overline{c} \in \kappa^{\times}_v$ *for the respective images of* $a, b, c \in \mathcal{O}^{\times}_{(v)}$, *then it holds that* $\overline{a} + \overline{b} = \overline{c}$.

PROOF. Assertion (i) follows from [3], Lemma 2.3. Assertions (ii) and (iv) follow from the various definitions involved. Assertion (iii) follows from *Čebotarev's density theorem* [cf., e.g., [10], Chapter VII, Theorem 13.4]. This completes the proof of Lemma 2.8. □

Theorem 2.9. *Let*

$$\mathcal{M} = (M,\ O^{\triangleright} \subseteq M,\ S,\ \{O^{\prec}_s \subseteq M\}_{s \in S})$$

be an **NF-monoid** [*cf. Definition* 2.3]. *We construct various objects associated to* $\mathcal{M}$ *as follows:*

(1) *We shall write*

$$M^{\times}_{\mathrm{prm}} \subseteq M^{\times}$$

for the submodule consisting of $a \in M^{\times}$ *such that, for all but finitely many* $s \in S$, *it holds that* $a^{\mathrm{char}(s)-1} \in O^{\prec}_{s}$ [*cf. Proposition* 2.5, (7)*; Proposition* 2.5, (iii)*; Lemma* 2.8, (i)]*;*

$$M_{\mathrm{prm}} \stackrel{\mathrm{def}}{=} M^{\times}_{\mathrm{prm}} \cup \{0_{\mathcal{M}}\} \subseteq M; \quad O^{\rhd}_{\mathrm{prm}} \stackrel{\mathrm{def}}{=} M_{\mathrm{prm}} \cap O^{\rhd}$$

[*cf. Proposition* 2.5, (1)*; Proposition* 2.5, (ii)*; Lemma* 2.8, (ii)].

(2) *We shall write*

$$S_{\mathrm{prm}} \stackrel{\mathrm{def}}{=} S/\sim_{\mathrm{prm}}$$

for the set of equivalence classes with respect to the relation "$\sim_{\mathrm{prm}}$*" on* S *defined as follows: For* s_1, $s_2 \in S$, *it holds that* $s_1 \sim_{\mathrm{prm}} s_2$ *if and only if* $\mathrm{char}(s_1) = \mathrm{char}(s_2)$ [*cf. Proposition* 2.5, (iii)].

(3) *Let* $s_{\mathrm{prm}} \in S_{\mathrm{prm}}$. *Then it follows from Lemma* 2.8, (ii), *that the intersection* $M_{\mathrm{prm}} \cap O^{\prec}_{s}$ *does* **not depend** *on the choice of a lifting* $s \in S$ *of* s_{prm}. *We shall write*

$$O^{\prec}_{s_{\mathrm{prm}}} \subseteq M_{\mathrm{prm}}$$

for this intersection.

(4) *It follows from Lemma* 2.8, (i), (ii), *that the collection of data*

$$\mathcal{M}_{\mathrm{prm}} \stackrel{\mathrm{def}}{=} (M_{\mathrm{prm}},\ O^{\rhd}_{\mathrm{prm}} \subseteq M_{\mathrm{prm}},\ S_{\mathrm{prm}},\ \{O^{\prec}_{s_{\mathrm{prm}}} \subseteq M_{\mathrm{prm}}\}_{s_{\mathrm{prm}} \in S_{\mathrm{prm}}})$$

forms an **NF-monoid of PmF-type** [*cf. Definition* 2.3].

(5) *We shall write*

$$S^{f=1} \stackrel{\mathrm{def}}{=} \{\, s \in S \mid \sharp(\underline{O}_s)_{\times} = \mathrm{char}(s) \,\}$$

[*cf. Proposition* 2.5, (4)]. *Then it follows from Proposition* 2.5, (iii), *and Lemma* 2.8, (iii), *that* $S^{f=1}$ *is* **infinite**.

(6) *Let* $s \in S^{f=1}$. *Write* $s_{\mathrm{prm}} \in S_{\mathrm{prm}}$ *for the element of* S_{prm} *determined by* $s \in S^{f=1}$. *Then one verifies immediately that the homomorphism* $(\underline{O}_{s_{\mathrm{prm}}})_{\times} \to (\underline{O}_s)_{\times}$ [*cf. Proposition* 2.5, (iii)] *of monoids induced by the natural inclusion* $M_{\mathrm{prm}} \hookrightarrow M$ *is an* **isomorphism**. *Thus, it follows from Proposition* 2.7, (10), *that the "*$\boxplus_s$*" of Proposition* 2.7, (9), *in the case where we take the "*$(\mathcal{M}, s)$*" of Proposition* 2.7, (9), *to be* $(\mathcal{M}_{\mathrm{prm}}, s_{\mathrm{prm}})$, *together with the monoid structure of* $(\underline{O}_s)_{\times}$, *determines a* **structure of field** *on* $(\underline{O}_s)_{\times}$. *We shall write*

$$\underline{O}_s$$

for the resulting field.

(7) *Let* $a, b \in M$. *Then we define an element of* M

$$a \boxplus_{\mathcal{M}} b \;\in\; M$$

as follows [*cf. Proposition* 2.5, (ii)]*: If* $a = 0_{\mathcal{M}}$, *then* $a \boxplus_{\mathcal{M}} b \stackrel{\mathrm{def}}{=} b$. *If* $b = 0_{\mathcal{M}}$, *then* $a \boxplus_{\mathcal{M}} b \stackrel{\mathrm{def}}{=} a$. *If* $a = -1_{\mathcal{M}} \cdot b$, [*cf. Proposition* 2.5, (3)], *then* $a \boxplus_{\mathcal{M}} b \stackrel{\mathrm{def}}{=} 0_{\mathcal{M}}$. *Suppose that* $a, b \in M^{\times}$, *and that* $a \neq -1_{\mathcal{M}} \cdot b$. *Then* $a \boxplus_{\mathcal{M}} b$ *is defined to be the* **uniquely determined** [*cf. Lemma* 2.8, (iv)] *element* $c \in M^{\times}$ *of* $M^{\times}$ *which satisfies the following condition: For infinitely many* $s \in S^{f=1}$ *such that* $\{a, b, c\} \subseteq O_s^{\times}$ [*cf. Proposition* 2.5, (9)*; Proposition* 2.5, (iii)], *if we write* $\overline{a}, \overline{b}, \overline{c} \in \underline{O}_s^{\times}$ [*cf. Proposition* 2.5, (4)*; Proposition* 2.5, (iii)] *for the respective images of* a, b, $c \in O_s^{\times}$ [*cf. Proposition* 2.5, (v)], *then it holds that* $\overline{a} \boxplus_s \overline{b} = \overline{c}$, *where we write* $\boxplus_s$ *for the addition operation of the field* $\underline{O}_s$ *defined in* (6).

(8) *It follows immediately from our construction that the operation "*$\boxplus_{\mathcal{M}}$*" of* (7), *together with the monoid structure of* M, *determines a* **structure of field** *on* M. *We shall write*

$$M^{\mathrm{fld}}$$

for the resulting field.

In the notation introduced at the beginning of the present §2, *let*

$$(\phi\colon F_{\times} \xrightarrow{\sim} M,\ \tau\colon \mathcal{V}_F \xrightarrow{\sim} S)$$

be an **NF-envelope** *for* $\mathcal{M}$ [*cf. Definition* 2.3]. *Then the isomorphism of monoids*

$$\phi\colon\ F_{\times} \ \xrightarrow{\sim}\ M$$

determines an **isomorphism of fields**

$$F \ \xrightarrow{\sim}\ M^{\mathrm{fld}}.$$

In particular, the field M^{fld} *of* (8) *is an* **NF**.

PROOF. This follows immediately from Lemma 2.8, together with the various definitions involved. □

§ 3. Local-global Cyclotomic Synchronization

In the present §3, we construct a *global cyclotome* [cf. Proposition 3.7, (4), below] associated to a profinite group *of GSC-type* [cf. Definition 3.2 below] and discuss a closely related *local-global cyclotomic synchronization isomorphism* [cf. Theorem 3.8, (ii), below], i.e., a certain natural isomorphism between this *global cyclotome* and various *local cyclotomes*. Finally, we apply this local-global cyclotomic synchronization isomorphism to construct *Kummer containers* associated to a profinite group *of GSC-type* [cf. Proposition 3.11 below].

In the present §3, we maintain the notation introduced at the beginning of the preceding §2. In particular, we assume that we have been given an *NF* F. Let

$$\overline{F}$$

be an algebraic closure of F. We shall write

- $d_F \stackrel{\mathrm{def}}{=} [F : F_{\mathrm{prm}}]$ for the extension degree of F over F_{prm},
- $\mathbb{I}_F^{\mathrm{fin}}$ for the group of finite idèles of F, and
- $\mathbb{I}_F$ for the group of idèles of F.

If $v \in \mathcal{V}_F$, then we shall write

- F_v for the MLF obtained by forming the completion of F at v.

We shall write

- $\mathcal{V}_F^{d=1} \subseteq \mathcal{V}_F$ for the subset consisting of $v \in \mathcal{V}_F$ such that $d_{F_v} = 1$ [cf. the notation introduced at the beginning of the §1].

Definition 3.1. Let E be a field of characteristic zero which is algebraic over the prime field contained in E [i.e., "$\mathbb{Q}$"]. Then we shall say that E is *absolutely Galois* if E is Galois over the prime field contained in E [i.e., "$\mathbb{Q}$"]. We shall say that E is *solvably closed* if there is no nontrivial finite abelian extension of E.

Definition 3.2. Let G be a profinite group. Then we shall refer to a collection of data

$$(K,\ \widetilde{K},\ \alpha\colon \mathrm{Gal}(\widetilde{K}/K) \stackrel{\sim}{\to} G)$$

consisting of an NF K, a Galois extension $\widetilde{K}$ of K which is solvably closed (respectively, absolutely Galois and solvably closed; algebraically closed), and an isomorphism of profinite groups $\alpha\colon \mathrm{Gal}(\widetilde{K}/K) \stackrel{\sim}{\to} G$ as a *GSC-envelope* (respectively, an *AGSC-envelope*; an *NF-envelope*) [where "GSC" (respectively, "AGSC"; "NF") is to be understood as an abbreviation for "Global Solvably Closed" (respectively, "Absolutely Galois and Global Solvably Closed"; "Number Field"] for G. We shall say that the profinite group G is *of GSC-type* (respectively, *of AGSC-type*; *of NF-type*) if there exists a GSC-envelope (respectively, an AGSC-envelope; an NF-envelope) for G.

Remark 3.2.1.

(i) One verifies immediately that every open subgroup of a profinite group *of GSC-type* (respectively, *of AGSC-type*; *of NF-type*) is *of GSC-type* (respectively, *of AGSC-type*; *of NF-type*).

(ii) It follows from the definitions that

$$\text{NF-type} \quad \Longrightarrow \quad \text{AGSC-type} \quad \Longrightarrow \quad \text{GSC-type}.$$

Note that these two implications are *strict*. Indeed, let us recall the well-known fact that there exists a finite Galois extension K of $\mathbb{Q}$ whose Galois group is isomorphic to the *symmetric group on* 6 *letters* [hence, in particular, *not solvable*]. This fact already implies that the absolute Galois group $G_{\mathbb{Q}}$ of $\mathbb{Q}$ is *not solvable* and hence that the first implication is *strict*. Next, let $L \subseteq K$ be an intermediate field of the Galois extension $K/\mathbb{Q}$ such that the subgroup $\mathrm{Gal}(K/L) \subseteq \mathrm{Gal}(K/\mathbb{Q})$ is isomorphic to the *symmetric group on* 5 *letters* [hence, in particular, *not solvable*] and $\widetilde{L}$ a solvable closure of L. Then observe that the assumption that the extension $\widetilde{L}/\mathbb{Q}$ is Galois implies [since, as is easily verified, K is a Galois closure of L over $\mathbb{Q}$] that there exists a *surjection* of Galois groups $\mathrm{Gal}(\widetilde{L}/L) \twoheadrightarrow \mathrm{Gal}(K/L)$, in contradiction to the fact that $\mathrm{Gal}(K/L)$ is *not solvable*. Thus, the field $\widetilde{L}$ is solvably closed, but *not absolutely Galois*. In particular, the second implication is also *strict*.

(iii) A typical example of a field which is *absolutely Galois* and *solvably closed* is a solvable closure of an absolutely Galois NF.

Now let us recall the famous *Neukirch-Uchida theorem*:

Theorem 3.3 (Neukirch-Uchida). *For $\square \in \{\circ, \bullet\}$, let $F_\square$ be an* **NF** *and $\widetilde{F}_\square$ a Galois extension of $F_\square$ which is* **solvably closed**. *Write $Q_\square \overset{\mathrm{def}}{=} \mathrm{Gal}(\widetilde{F}_\square/F_\square)$;*

$$\mathrm{Isom}(\widetilde{F}_\bullet/F_\bullet, \widetilde{F}_\circ/F_\circ)$$

for the set of isomorphisms $\widetilde{F}_\bullet \overset{\sim}{\to} \widetilde{F}_\circ$ of fields which map $F_\bullet$ bijectively onto $F_\circ$;

$$\mathrm{Isom}(Q_\circ, Q_\bullet)$$

for the set of isomorphisms $Q_\circ \overset{\sim}{\to} Q_\bullet$ of profinite groups. Then the natural map

$$\mathrm{Isom}(\widetilde{F}_\bullet/F_\bullet, \widetilde{F}_\circ/F_\circ) \longrightarrow \mathrm{Isom}(Q_\circ, Q_\bullet)$$

is **bijective**.

PROOF. This follows from [13], Theorem. $\square$

In the remainder of the present §3, let

$$\widetilde{F}$$

be a Galois extension of F which is *solvably closed* and contained in $\overline{F}$. We shall write

- $\mathcal{V}_{\widetilde{F}}$ for the set of nonarchimedean primes of $\widetilde{F}$ and
- $Q_F \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\widetilde{F}/F)$ for the Galois group of $\widetilde{F}/F$.

Note that, for $\widetilde{v} \in \mathcal{V}_{\widetilde{F}}$, if we write $v \in \mathcal{V}_F$ for the nonarchimedean prime of F determined by $\widetilde{v}$, then since $\widetilde{F}$ is *solvably closed*, it follows immediately from [6], Proposition 2.3, (iii) [i.e., the *Grunwald-Wang Theorem* — cf., e.g., [11], Theorem 9.2.8], that the pair $(\widetilde{F}, \widetilde{v})$ determines an *algebraic closure* $\overline{F}_{\widetilde{v}}$ of F_v, together with an inclusion $\widetilde{F} \hookrightarrow \overline{F}_{\widetilde{v}}$ of fields.

Lemma 3.4. *The following hold:*

(i) *The map given by assigning to $\widetilde{v} \in \mathcal{V}_{\widetilde{F}}$ the decomposition subgroup of Q_F associated to $\widetilde{v}$ determines a* **bijection** *of $\mathcal{V}_{\widetilde{F}}$ with the set of* **maximal** *closed subgroups of Q_F* **of MLF-type**. *Moreover, the natural map $\mathcal{V}_{\widetilde{F}} \to \mathcal{V}_F$ and the natural action of Q_F on $\mathcal{V}_{\widetilde{F}}$ determines a* **bijection**

$$\mathcal{V}_{\widetilde{F}}/Q_F \xrightarrow{\sim} \mathcal{V}_F$$

from the quotient $\mathcal{V}_{\widetilde{F}}/Q_F$ of $\mathcal{V}_{\widetilde{F}}$ by the action of Q_F onto $\mathcal{V}_F$.

(ii) *Let p be a prime number. Then it holds that*

$$d_F = \sum_{v \in \mathcal{V}_F;\ \mathrm{char}(v)=p} d_{F_v}.$$

Proof. Assertion (i) follows immediately, in light of [6], Proposition 2.3, (iii), (iv), from a similar argument to the argument applied in the proof of [11], Corollary 12.1.11. Assertion (ii) follows from [10], Chapter II, Corollary 8.4. This completes the proof of Lemma 3.4. □

Proposition 3.5. *Let G be a profinite group* **of GSC-type**. *We construct various objects associated to G as follows:*

(1) *We shall write*

$$\widetilde{\mathcal{V}}(G)$$

for the set of **maximal** *closed subgroups of G* **of MLF-type** *and*

$$\mathcal{V}(G) \stackrel{\mathrm{def}}{=} \widetilde{\mathcal{V}}(G)/G$$

for the quotient of $\widetilde{\mathcal{V}}(G)$ by the action of G by conjugation [*cf. Lemma* 3.4, (i)].

(2) *Let $v \in \mathcal{V}(G)$. Then we shall write*

$$p(v) \stackrel{\mathrm{def}}{=} p(D), \quad d(v) \stackrel{\mathrm{def}}{=} d(D), \quad f(v) \stackrel{\mathrm{def}}{=} f(D), \quad e(v) \stackrel{\mathrm{def}}{=} e(D)$$

for any $D \in v$ [*cf. Theorem* 1.4, (1), (2)]. [*One verifies immediately that the quantities of the above display do* **not depend** *on the choice of* $D \in v$.] *We shall write*

$$\mathcal{V}^{d=1}(G) \subseteq \mathcal{V}(G)$$

for the subset of $\mathcal{V}(G)$ *consisting of* $v \in \mathcal{V}(G)$ *such that* $d(v) = 1$.

(3) *Let* $v_0 \in \mathcal{V}(G)$. *Then since the sum*

$$\sum_{v \in \mathcal{V}(G);\ p(v)=p(v_0)} d(v)$$

does **not depend** *on the choice of* $v_0 \in \mathcal{V}(G)$ [*cf. Lemma* 3.4, (i), (ii)], *we shall write*

$$d(G)$$

for this sum.

Let

$$(F,\ \widetilde{F},\ \alpha\colon Q_F \stackrel{\sim}{\to} G)$$

be a **GSC-envelope** *for* G. *Then the following hold:*

(i) *The isomorphism* α *determines* [*cf. the first bijection of Lemma* 3.4, (i)] *a* **bijection**

$$\mathcal{V}_{\widetilde{F}} \stackrel{\sim}{\longrightarrow} \widetilde{\mathcal{V}}(G).$$

This bijection is **compatible** *with the natural actions of* Q_F *and* G *relative to* α, *which thus induces a* **bijection** [*cf. the second bijection of Lemma* 3.4, (i)]

$$\mathcal{V}_F \stackrel{\sim}{\longrightarrow} \mathcal{V}(G).$$

Let us **identify** $\mathcal{V}_{\widetilde{F}}$, $\mathcal{V}_F$ *with* $\widetilde{\mathcal{V}}(G)$, $\mathcal{V}(G)$, *respectively, by means of these bijections.*

(ii) *Let* $v \in \mathcal{V}(G)$. *Then it holds that*

$$p_{F_v} = p(v), \quad d_{F_v} = d(v), \quad f_{F_v} = f(v), \quad e_{F_v} = e(v)$$

[*cf. Theorem* 1.4, (i)].

(iii) *It holds that* $d_F = d(G)$ [*cf.* (ii)*; Lemma* 3.4, (ii)].

(iv) *Let* $H \subseteq G$ *be an open subgroup of* G. *Then we have a* **bijection**

$$\begin{array}{ccc} \widetilde{\mathcal{V}}(G) & \stackrel{\sim}{\longrightarrow} & \widetilde{\mathcal{V}}(H) \\ D & \mapsto & D \cap H \end{array}$$

whose inverse is given by

$$\begin{array}{ccc} \widetilde{\mathcal{V}}(H) & \stackrel{\sim}{\longrightarrow} & \widetilde{\mathcal{V}}(G) \\ D & \mapsto & C_G(D). \end{array}$$

Moreover, this inverse determines a **surjection**

$$\mathcal{V}(H) \twoheadrightarrow \mathcal{V}(G).$$

PROOF. Assertions (i), (ii), (iii) follow immediately from the references quoted in the statements of these assertions, together with the various definitions involved. Assertion (iv) follows immediately from assertion (i), together with the *commensurable terminality* in G [cf. [6], Proposition 2.3, (v)] of a closed subgroup of G which is contained in $\widetilde{\mathcal{V}}(G)$. This completes the proof of Proposition 3.5. □

Lemma 3.6. *The following hold:*

(i) *We have a natural* **injection** *of groups*

$$F^\times \hookrightarrow \mathbb{I}_F^{\mathrm{fin}}.$$

We regard $F^\times$ as a subgroup of $\mathbb{I}_F^{\mathrm{fin}}$ by means of this **injection**.

(ii) *By considering the reciprocity homomorphism $\mathbb{I}_F \twoheadrightarrow (\mathrm{Gal}(\overline{F}/F)^{\mathrm{ab}} \stackrel{\sim}{\to})\ Q_F^{\mathrm{ab}}$ in global class field theory, together with the natural inclusion $\mathbb{I}_F^{\mathrm{fin}} \hookrightarrow \mathbb{I}_F$, we obtain* [*cf. also* (i)] *homomorphisms of groups*

$$F^\times \hookrightarrow \mathbb{I}_F^{\mathrm{fin}} \to Q_F^{\mathrm{ab}}.$$

[*Note that, in general, this composite is* **nontrivial**. *For instance, one verifies easily that if F is* **of PmF-type**, *then the image of $-1 \in F^\times$ via this composite is* **nontrivial**.]

(iii) *Relative to the arrows of the display of* (ii), *we have*

$$\mathrm{Ker}(\mathbb{I}_F^{\mathrm{fin}} \to Q_F^{\mathrm{ab}})_{\mathrm{tor}} \subseteq \boldsymbol{\mu}(F) \quad (\subseteq F^\times).$$

If, moreover, F is **totally imaginary**, *then, relative to the arrows of the display of* (ii), *we have*

$$\mathrm{Ker}(\mathbb{I}_F^{\mathrm{fin}} \to Q_F^{\mathrm{ab}})_{\mathrm{tor}} = \boldsymbol{\mu}(F) \quad (\subseteq F^\times).$$

(iv) *Let n be a positive integer and $\zeta_n \in \overline{F}$ a primitive n-th root of unity. Then it holds that $\zeta_n \in \widetilde{F}$. Moreover, the subfield of $\widetilde{F}$ corresponding to the kernel of the natural action of Q_F on*

$$\big(\varinjlim_E \mathrm{Ker}(\mathbb{I}_E^{\mathrm{fin}} \to Q_E^{\mathrm{ab}})_{\mathrm{tor}}\big)[n]$$

— where the injective limit is taken over the finite extensions E of F contained in $\widetilde{F}$, and we write $Q_E \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\widetilde{F}/E)$ — is **equal** *to $F(\zeta_n)$.*

PROOF. Assertions (i), (ii) follow from the various definitions involved. Next, we verify assertion (iii). Write $F_{\mathbb{I}}^{\times} \subseteq \mathbb{I}_F$ for the image of the natural injection $F^{\times} \hookrightarrow \mathbb{I}_F$ and $D_F \subseteq \mathbb{I}_F/F_{\mathbb{I}}^{\times}$ for the kernel of the *reciprocity homomorphism* $\mathbb{I}_F/F_{\mathbb{I}}^{\times} \twoheadrightarrow Q_F^{\mathrm{ab}}$ in global class field theory, i.e., the connected component of $\mathbb{I}_F/F_{\mathbb{I}}^{\times}$ containing the identity element [cf. [11], Corollary 8.2.2]. [Let us recall that the subgroup $F_{\mathbb{I}}^{\times} \subseteq \mathbb{I}_F$ does *not coincide* with the image of the composite $F^{\times} \hookrightarrow \mathbb{I}_F^{\mathrm{fin}} \hookrightarrow \mathbb{I}_F$.] First, we verify the inclusion

$$\mathrm{Ker}(\mathbb{I}_F^{\mathrm{fin}} \to Q_F^{\mathrm{ab}})_{\mathrm{tor}} \subseteq F^{\times}.$$

Let $\alpha \in \mathbb{I}_F^{\mathrm{fin}} \subseteq \mathbb{I}_F$ be a *torsion* finite idèle whose image in Q_F^{ab} is *trivial*. Then one verifies immediately that the image of α via the composite

$$\mathbb{I}_F^{\mathrm{fin}} \hookrightarrow \mathbb{I}_F \twoheadrightarrow \mathbb{I}_F/F_{\mathbb{I}}^{\times}$$

is a *torsion* element *contained* in D_F. In particular, it follows immediately from [11], Theorem 8.2.5, together with the fact that the objects "$\overline{\mathbb{Z}}/\mathbb{Z}$" and "$\mathbb{R}$" in *loc. cit.* are *torsion-free*, that there exists an *infinite idèle* $\beta \in \mathbb{I}_F$ such that the image of α in $\mathbb{I}_F/F_{\mathbb{I}}^{\times}$ *coincides* with the image of β in $\mathbb{I}_F/F_{\mathbb{I}}^{\times}$, i.e., that $\alpha \cdot \beta^{-1} \in F_{\mathbb{I}}^{\times}$ in $\mathbb{I}_F$. On the other hand, it follows immediately from the various definitions involved that this implies that $\alpha \in F^{\times}$ [i.e., the image of $F^{\times}$ in $\mathbb{I}_F^{\mathrm{fin}}$]. This completes the proof of the desired inclusion.

Next, we verify the inclusion

$$\boldsymbol{\mu}(F) \subseteq \mathrm{Ker}(\mathbb{I}_F^{\mathrm{fin}} \to Q_F^{\mathrm{ab}})$$

under the assumption that F is *totally imaginary*. Let $\alpha \in F^{\times}$ be a *torsion* element. Then it follows immediately from the various definitions involved that, to complete the verification of the desired inclusion, it suffices to verify that the image in $\mathbb{I}_F/F_{\mathbb{I}}^{\times}$ of the *infinite idèle* determined by $\alpha \in F^{\times}$ is *contained* in D_F. On the other hand, since F is *totally imaginary*, this follows immediately from [11], Theorem 8.2.5. This completes the proof of the desired inclusion, hence also of assertion (iii). Assertion (iv) follows immediately from assertion (iii), together with our assumption that $\widetilde{F}$ is *solvably closed.* This completes the proof of Lemma 3.6. ☐

Proposition 3.7. *Let G be a profinite group* **of GSC-type**. *We construct various objects associated to G as follows:*

(1) *Let $v \in \mathcal{V}(G)$ [cf. Proposition 3.5, (1)]. Then one verifies immediately from the* **commensurable terminality** *in G [cf. [6], Proposition 2.3, (v)] of any closed subgroup of G that belongs to $\widetilde{\mathcal{V}}(G)$ [cf. Proposition 3.5, (1)] that there exists a* **uniquely determined** *submodule (respectively, submonoid)*

$$k^{\times}(v) \subseteq \prod_{D \in v} k^{\times}(D) \quad (\subseteq \prod_{D \in v} D^{\mathrm{ab}})$$

$$(\textit{respectively, } \mathcal{O}^{\rhd}(v) \subseteq \prod_{D \in v} \mathcal{O}^{\rhd}(D) \quad (\subseteq \prod_{D\in v} D^{\mathrm{ab}}))$$

[*cf.* *Theorem* 1.4, (6)] *which satisfies the following two conditions:*

(a) *The action of* G *on* $\prod_{D\in v} k^{\times}(D)$ (*respectively,* $\prod_{D\in v} \mathcal{O}^{\rhd}(D)$) *by conjugation* [*preserves and*] *induces the* **identity automorphism** *on the submodule* $k^{\times}(v)$ (*respectively,* $\mathcal{O}^{\rhd}(v)$)*.*

(b) *For every* $D_0 \in v$, *the composite*

$$k^{\times}(v) \hookrightarrow \prod_{D\in v} k^{\times}(D) \twoheadrightarrow k^{\times}(D_0)$$

$$(\textit{respectively, } \mathcal{O}^{\rhd}(v) \hookrightarrow \prod_{D\in v} \mathcal{O}^{\rhd}(D) \twoheadrightarrow \mathcal{O}^{\rhd}(D_0))$$

is an **isomorphism** *of modules* (*respectively, monoids*)*.*

The isomorphism of (b), *together with the topology on* $k^{\times}(D_0)$ (*respectively,* $\mathcal{O}^{\rhd}(D_0)$), *determines a* **topology** *on* $k^{\times}(v)$ (*respectively,* $\mathcal{O}^{\rhd}(v)$). [*Note that one verifies immediately that this topology on* $k^{\times}(v)$ (*respectively,* $\mathcal{O}^{\rhd}(v)$) *does* **not depend** *on the choice of* $D_0 \in v$.]

(2) *We shall write*

$$k_{\times}(v) \stackrel{\mathrm{def}}{=} k^{\times}(v)^{\circledast} \subseteq \prod_{D\in v} k_{\times}(D)$$

[*cf.* *Theorem* 1.4, (8)].

(3) *Let* $\Sigma \subseteq \mathcal{V}(G)$ *be a finite subset of* $\mathcal{V}(G)$. *Then we shall write*

$$\mathbb{I}^{\mathrm{fin}}_{\Sigma}(G) \stackrel{\mathrm{def}}{=} \big(\prod_{v\in\Sigma} k^{\times}(v)\big) \times \big(\prod_{v\notin\Sigma} \mathcal{O}^{\times}(v)\big) \quad (\subseteq \prod_{D\in\widetilde{\mathcal{V}}(G)} D^{\mathrm{ab}})$$

— *where we write* $\mathcal{O}^{\times}(v) \stackrel{\mathrm{def}}{=} \mathcal{O}^{\rhd}(v)^{\times}$ — *and*

$$\mathbb{I}^{\mathrm{fin}}(G) \stackrel{\mathrm{def}}{=} \bigcup_{\Sigma^{\dagger}} \mathbb{I}^{\mathrm{fin}}_{\Sigma^{\dagger}}(G) \quad (\subseteq \prod_{D\in\widetilde{\mathcal{V}}(G)} D^{\mathrm{ab}})$$

— *where the union is taken over the finite subsets* $\Sigma^{\dagger} \subseteq \mathcal{V}(G)$ *of* $\mathcal{V}(G)$.

(4) *It follows from our construction in* (3) *that the inclusions* $D \hookrightarrow G$, *where* D *ranges over the elements of* $\widetilde{\mathcal{V}}(G)$, *determine a homomorphism of groups*

$$\mathbb{I}^{\mathrm{fin}}(G) \longrightarrow G^{\mathrm{ab}}.$$

We shall write

$$\boldsymbol{\mu}(G) \stackrel{\mathrm{def}}{=} \varinjlim_{H} \mathrm{Ker}(\mathbb{I}^{\mathrm{fin}}(H) \to H^{\mathrm{ab}})_{\mathrm{tor}}$$

— *where the injective limit is taken over the open subgroups* $H \subseteq G$ *of* G*, and the transition morphisms in the limit are given by the homomorphisms determined by the transfer maps* [*cf. Lemma* 3.6, (iii)]*;*

$$\Lambda(G) \stackrel{\mathrm{def}}{=} \varprojlim_{n} \boldsymbol{\mu}(G)[n]$$

— *where the projective limit is taken over* $n \in \mathbb{Z}_{\geq 1}$*. Note that* G *acts on* $\boldsymbol{\mu}(G)$*,* $\Lambda(G)$ *by conjugation. We shall refer to the* G*-module* $\Lambda(G)$ *as the* **cyclotome** *associated to* G*. Note that one verifies immediately from our construction of* $\Lambda(G)$ *that* $\Lambda(G)$ *has a natural structure of* **profinite** [*cf. also the above definition of* $\Lambda(G)$]*, hence also topological,* G*-module; moreover, we have a natural identification* $\boldsymbol{\mu}(G)[n] = \Lambda(G)/n\Lambda(G)$*.*

(5) *Let* n *be a positive integer. Then we shall write*

$$^{\boldsymbol{\mu}_n}G \subseteq G$$

for the open subgroup of G *obtained by forming the kernel of the action*

$$G \longrightarrow \mathrm{Aut}(\Lambda(G)/n\Lambda(G))$$

[*cf. Lemma* 3.6, (iv)].

Let

$$(F,\ \widetilde{F},\ \alpha\colon Q_F \xrightarrow{\sim} G)$$

be a **GSC-envelope** *for* G*. Then the following hold:*

(i) *Let* $v \in \mathcal{V}(G)$*. Then the isomorphism* α *determines a* **commutative** *diagram of topological monoids*

$$\begin{array}{ccc} \mathcal{O}^{\rhd}_{F_v} & \longrightarrow & F_v^{\times} \\ \wr\downarrow & & \wr\downarrow \\ \mathcal{O}^{\rhd}(v) & \longrightarrow & k^{\times}(v) \end{array}$$

— *where the horizontal arrows are the natural inclusions, and the vertical arrows are* **isomorphisms***. Thus, the right-hand vertical arrow of this diagram determines an* **isomorphism** *of monoids*

$$(F_v)_{\times} \xrightarrow{\sim} k_{\times}(v).$$

(ii) *The diagram of groups*

$$\begin{array}{ccc} \mathbb{I}^{\mathrm{fin}}_F & \longrightarrow & Q_F^{\mathrm{ab}} \\ \wr\downarrow & & \wr\downarrow \\ \mathbb{I}^{\mathrm{fin}}(G) & \longrightarrow & G^{\mathrm{ab}} \end{array}$$

— *where the upper horizontal arrow is the homomorphism of Lemma* 3.6, (ii)*; the lower horizontal arrow is the homomorphism of* (4)*; the left-hand vertical arrow is the isomorphism induced by the various isomorphisms "*$F_v^\times \xrightarrow{\sim} k^\times(v)$*" of* (i)*; the right-hand vertical arrow is the isomorphism induced by* α — **commutes**.

(iii) *The commutative diagram of* (ii) *determines* **isomorphisms**

$$\boldsymbol{\mu}(\overline{F}) \xrightarrow{\sim} \boldsymbol{\mu}(G), \quad \Lambda(\overline{F}) \xrightarrow{\sim} \Lambda(G)$$

which are **compatible** *with the natural actions of* Q_F *and* G *relative to* α [*cf. Lemma* 3.6, (iii)].

(iv) *Let* n *be a positive integer and* $\zeta_n \in \overline{F}$ *a primitive* n*-th root of unity. Then the isomorphism* α *determines an isomorphism of profinite groups*

$$\mathrm{Gal}(\widetilde{F}/F(\zeta_n)) \xrightarrow{\sim} {}_{\boldsymbol{\mu}_n}G$$

[*cf. Lemma* 3.6, (iv)].

PROOF. These assertions follow immediately from Lemma 3.6, together with the various definitions involved. ☐

Theorem 3.8. *In the notation introduced at the beginning of the present* §3 *and the discussion following Theorem* 3.3, *let* G *be a profinite group* **of GSC-type** [*cf. Definition* 3.2] *and* $D \in \widetilde{\mathcal{V}}(G)$. *Then the following hold:*

(i) *Let* $H \subseteq G$ *be an open subgroup of* G. *Then we have natural* **identifications**

$$\boldsymbol{\mu}(G) \xrightarrow{\sim} \boldsymbol{\mu}(H), \quad \Lambda(G) \xrightarrow{\sim} \Lambda(H)$$

[*arising from the definitions of "*$\boldsymbol{\mu}(-)$*" and "*$\Lambda(-)$*" — cf. Proposition* 3.7, (4)] *which are* ***H*-equivariant**.

(ii) *The natural homomorphism* $\mathbb{I}^{\mathrm{fin}}(G) \to k^\times(D)$ [*arising from the definition of* $\mathbb{I}^{\mathrm{fin}}(G)$ — *cf. Proposition* 3.7, (3)] *determines* ***D*-equivariant isomorphisms**

$$\boldsymbol{\mu}(G) \xrightarrow{\sim} \boldsymbol{\mu}(D), \quad \Lambda(G) \xrightarrow{\sim} \Lambda(D)$$

[*cf. Theorem* 1.4, (9)]. *We shall refer to the isomorphism of cyclotomes* $\Lambda(G) \xrightarrow{\sim} \Lambda(D)$ *as the* **local-global cyclotomic synchronization isomorphism** *with respect to* $D \in \widetilde{\mathcal{V}}(G)$.

(iii) *Let*

$$(F,\ \widetilde{F},\ \alpha\colon Q_F \xrightarrow{\sim} G)$$

be a **GSC-envelope** *for* G [*cf. Definition* 3.2]. *Write* $\widetilde{v} \in \mathcal{V}_{\widetilde{F}}$ *for the nonarchimedean prime of* $\widetilde{F}$ *which corresponds to* $D \in \widetilde{\mathcal{V}}(G)$ [*cf. Proposition* 3.5, (i)] *and* $v \in \mathcal{V}_F$ *for the nonarchimedean prime of* F *determined by* $\widetilde{v}$. *Thus, by the discussion following Theorem* 3.3, *we have an algebraic closure* $\overline{F}_{\widetilde{v}}$ *of* F_v, *together with an inclusion* $\widetilde{F} \hookrightarrow \overline{F}_{\widetilde{v}}$ *of fields. Then the diagram*

$$\begin{array}{ccc} \Lambda(\overline{F}) & \xrightarrow{\sim} & \Lambda(\overline{F}_{\widetilde{v}}) \\ \wr\downarrow & & \wr\downarrow \\ \Lambda(G) & \xrightarrow{\sim} & \Lambda(D) \end{array}$$

— *where the upper horizontal arrow is the isomorphism induced by the inclusion* $\widetilde{F} \hookrightarrow \overline{F}_{\widetilde{v}}$ *of fields* [*cf. the first assertion of Lemma* 3.6, (iv)]; *the lower horizontal arrow is the local-global cyclotomic synchronization isomorphism; the left-hand vertical arrow is the isomorphism of Proposition* 3.7, (iii); *the right-hand vertical arrow is the isomorphism of Theorem* 1.4, (iv) — **commutes**.

PROOF. These assertions follow immediately from the various definitions involved. □

Definition 3.9. We shall write

$$\mathcal{H}^{\times}(F)$$

for the module obtained by forming the fiber product of the diagram of natural injections

$$\begin{array}{ccc} & & \mathbb{I}_F^{\mathrm{fin}} \\ & & \downarrow \\ (F^{\times})^{\wedge} & \longrightarrow & \prod_{v \in \mathcal{V}_F} (F_v^{\times})^{\wedge} \end{array}$$

[cf. the discussion entitled "Modules" in §0; [11], Theorem 9.1.11, (i)] and

$$\mathcal{H}_{\times}(F) \stackrel{\mathrm{def}}{=} \mathcal{H}^{\times}(F)^{\circledast}.$$

Thus, we have natural injections of monoids

$$F_{\times} \hookrightarrow \mathcal{H}_{\times}(F) \hookrightarrow \prod_{v \in \mathcal{V}_F} (F_v)_{\times}.$$

We shall refer to $\mathcal{H}^{\times}(F)$, $\mathcal{H}_{\times}(F)$ as the *Kummer containers* associated to F.

Lemma 3.10. *The following hold:*

(i) *We have a natural commutative diagram of modules*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{O}_F^\times & \longrightarrow & F^\times & \longrightarrow & F^\times/\mathcal{O}_F^\times & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & (\mathcal{O}_F^\times)^\wedge & \longrightarrow & \mathcal{H}^\times(F) & \longrightarrow & F^\times/\mathcal{O}_F^\times & \longrightarrow & 1 \end{array}$$

— *where the horizontal sequences are* **exact**, *and the vertical arrows are* **injective**.

(ii) *If, moreover,* $\mathcal{O}_F^\times$ *is* **finite** [*or, equivalently,* F *is* **contained** *in an* **imaginary quadratic** *field — cf., e.g.,* [10], *Chapter* I, *Theorem* 7.4], *then the natural homomorphism* $F^\times \to \mathcal{H}^\times(F)$, *hence also* $F_\times \to \mathcal{H}_\times(F)$, *is an* **isomorphism** *of monoids.*

(iii) *The natural inclusion* $F^\times \hookrightarrow \mathcal{H}^\times(F)$ *determines an* **isomorphism of finite groups**

$$\boldsymbol{\mu}(F) \xrightarrow{\sim} \mathcal{H}^\times(F)_{\mathrm{tor}}.$$

(iv) *The module* $\mathcal{H}^\times(F)$ *is* **generated** *by the images* [*cf.* (i)] *of* $(\mathcal{O}_F^\times)^\wedge$ *and* $F^\times$.

(v) *The composite of natural homomorphisms*

$$\mathcal{H}_\times(F) \to \prod_{v\in\mathcal{V}_F}(F_v)_\times \twoheadrightarrow \prod_{v\in\mathcal{V}_F^{d=1}}(F_v)_\times$$

is **injective**.

(vi) *Let* n *be a positive integer. Then the sequence of* Q_F*-modules*

$$1 \longrightarrow \boldsymbol{\mu}(\overline{F})[n] \longrightarrow \widetilde{F}^\times \xrightarrow{n} \widetilde{F}^\times \longrightarrow 1$$

[*cf. the first assertion of Lemma* 3.6, (iv)] *is* **exact**. *Moreover, these sequences — where* n *ranges over the positive integers — determine an* **injection**, *together with an* **isomorphism**,

$$\mathrm{Kmm}_{\widetilde{F}/F}\colon F^\times \hookrightarrow (F^\times)^\wedge \xrightarrow{\sim} H^1(Q_F, \Lambda(\overline{F})).$$

PROOF. First, we verify assertion (i). The [existence and] exactness of the lower horizontal sequence of the diagram of (i) follows immediately from [1], Lemma 5.29, (i), together with the various definitions involved. The injectivity of the left-hand, hence also middle, vertical arrow follows immediately from the fact that $\mathcal{O}_F^\times$ is a *finitely generated module* [cf., e.g., [10], Chapter I, Theorem 7.4]. This completes the proof of assertion (i). Assertions (ii), (iv) follow immediately from assertion (i). Assertion (iii) follows immediately from assertion (i), together with the [easily verified] fact that $F^\times/\mathcal{O}_F^\times$ is *torsion-free*. Next, we verify assertion (v). Let us first observe that the subset $\mathcal{V}_F^{d=1} \subseteq \mathcal{V}_F$ of $\mathcal{V}_F$ is *of density* 1 [cf., e.g., the discussion preceding [10], Chapter

VII, Theorem 13.2]. Thus, it follows immediately from [11], Theorem 9.1.11 [cf. also [10], Chapter I, Theorem 7.4], that the composite

$$(F^{\times})^{\wedge} \longrightarrow \prod_{v \in \mathcal{V}_F} (F_v^{\times})^{\wedge} \longrightarrow \prod_{v \in \mathcal{V}_F^{d=1}} (F_v^{\times})^{\wedge},$$

hence [cf. the easily verified *injectivity* of the natural homomorphism $F_v^{\times} \to (F_v^{\times})^{\wedge}$] also the composite discussed in assertion (v), is *injective*. This completes the proof of assertion (v). Finally, we verify assertion (vi). The first portion of assertion (vi) follows from our assumption that $\widetilde{F}$ is *solvably closed*. The final portion of assertion (vi) follows from *Kummer theory*, together with the fact that the module $F^{\times}$ has *no nontrivial divisible elements* [cf., e.g., assertion (i), together with [10], Chapter I, Theorem 7.4]. This completes the proof of assertion (vi), hence also of Lemma 3.10. ☐

Proposition 3.11. *Let G be a profinite group* **of GSC-type**. *Then, as D ranges over the elements of $\widetilde{\mathcal{V}}(G)$, the inclusions $D \hookrightarrow G$ and local-global cyclotomic synchronization isomorphisms $\Lambda(G) \xrightarrow{\sim} \Lambda(D)$ [cf. Theorem 3.8, (ii)] determine an* **injective** *[cf. Lemma 1.3, (x); Definition 3.9; Lemma 3.10, (vi)]* **homomorphism**

$$H^1(G, \Lambda(G)) \hookrightarrow \prod_{D \in \widetilde{\mathcal{V}}(G)} H^1(D, \Lambda(D)).$$

We shall write

$$\mathcal{H}^{\times}(G) \subseteq H^1(G, \Lambda(G))$$

for the inverse image, via the above injective homomorphism, of the image of the composite of injections

$$\mathbb{I}^{\mathrm{fin}}(G) \hookrightarrow \prod_{v \in \mathcal{V}(G)} k^{\times}(v) \hookrightarrow \prod_{D \in \widetilde{\mathcal{V}}(G)} k^{\times}(D) \overset{\prod_{D \in \widetilde{\mathcal{V}}(G)} \mathrm{Kmm}(D)}{\hookrightarrow} \prod_{D \in \widetilde{\mathcal{V}}(G)} H^1(D, \Lambda(D))$$

[cf. Theorem 1.4, (10); Proposition 3.7, (1), (3)]. Thus, the injective homomorphism in the first display determines an injective homomorphism $\mathcal{H}^{\times}(G) \hookrightarrow \prod_{v \in \mathcal{V}(G)} k^{\times}(v)$, which we shall apply to regard $\mathcal{H}^{\times}(G)$ [by abuse of notation] as a submodule of the product $\prod_{v \in \mathcal{V}(G)} k^{\times}(v)$:

$$\mathcal{H}^{\times}(G) \subseteq \prod_{v \in \mathcal{V}(G)} k^{\times}(v).$$

We shall write

$$\mathcal{H}_{\times}(G) \overset{\mathrm{def}}{=} \mathcal{H}^{\times}(G)^{\circledast} \subseteq \prod_{v \in \mathcal{V}(G)} k_{\times}(v)$$

[cf. Proposition 3.7, (2)]. We shall refer to $\mathcal{H}^{\times}(G)$, $\mathcal{H}_{\times}(G)$ as the **Kummer containers** *associated to G.*

Let

$$(F,\ \widetilde{F},\ \alpha\colon Q_F \overset{\sim}{\to} G)$$

be a **GSC-envelope** *for* G. *Then the following hold:*

(i) *The isomorphism* α *determines a* **commutative** *diagram of monoids*

$$\begin{array}{ccc} \mathcal{H}_\times(F) & \longrightarrow & \prod_{v\in\mathcal{V}_F}(F_v)_\times \\ \wr\downarrow & & \wr\downarrow \\ \mathcal{H}_\times(G) & \longrightarrow & \prod_{v\in\mathcal{V}(G)} k_\times(v) \end{array}$$

— *where the horizontal arrows are the natural inclusions* [*cf. Definition* 3.9*; the above discussion*]*, and the right-hand vertical arrow is the isomorphism determined by the various isomorphisms of monoids of Proposition* 3.7*,* (i)*.*

(ii) *The composite*

$$\mathcal{H}_\times(G) \hookrightarrow \prod_{v\in\mathcal{V}(G)} k_\times(v) \twoheadrightarrow \prod_{v\in\mathcal{V}^{d=1}(G)} k_\times(v)$$

[*cf. Proposition* 3.5*,* (2)] *is* **injective***.*

(iii) *Let* $H \subseteq G$ *be an open subgroup of* G. *Then the various restriction maps of cohomology groups involved determine a* **commutative** *diagram of* **inclusions** *of monoids*

$$\begin{array}{ccc} \mathcal{H}_\times(G) & \longrightarrow & \prod_{v\in\mathcal{V}(G)} k_\times(v) \\ \downarrow & & \downarrow \\ \mathcal{H}_\times(H) & \longrightarrow & \prod_{w\in\mathcal{V}(H)} k_\times(w). \end{array}$$

Proof. These assertions follow immediately from Lemma 3.10, (v), (vi), together with the various definitions involved. □

§ 4. Reconstruction of the Additive Structure on a GSC-Galois Pair

In the present §4, we discuss the notion of a *GSC-Galois pair* [cf. Definition 4.1 below]. In particular, we apply the main result of §2 to obtain a *mono-anabelian reconstruction* of the "*additive structure*" on a GSC-Galois pair [cf. Theorem 4.4 below].

In the present §4, we maintain the notation introduced at the beginning of the preceding §3. Let

$$\widetilde{F}$$

be a Galois extension of F which is *solvably closed* and contained in $\overline{F}$. We shall write

- $\mathcal{V}_{\widetilde{F}}$ for the set of nonarchimedean primes of $\widetilde{F}$,
- $\mathcal{O}_{\widetilde{F}} \subseteq \widetilde{F}$ for the ring of integers of $\widetilde{F}$, and
- $Q_F \stackrel{\text{def}}{=} \text{Gal}(\widetilde{F}/F)$ for the Galois group of $\widetilde{F}/F$.

Definition 4.1. Let

$$(G \curvearrowright M)$$

be a collection of data consisting of a group G and a G-monoid M. Then we shall refer to a collection of data

$$(K,\ \widetilde{K},\ \alpha\colon \text{Gal}(\widetilde{K}/K) \xrightarrow{\sim} G,\ \beta\colon \mathcal{O}^{\rhd}_{\widetilde{K}} \xrightarrow{\sim} M)$$

consisting of an NF K, a Galois extension $\widetilde{K}$ of K which is solvably closed (respectively, absolutely Galois and solvably closed; algebraically closed), an isomorphism of groups $\alpha\colon \text{Gal}(\widetilde{K}/K) \xrightarrow{\sim} G$, and an isomorphism of monoids $\beta\colon \mathcal{O}^{\rhd}_{\widetilde{K}} \xrightarrow{\sim} M$ [where we write $\mathcal{O}_{\widetilde{K}}$ for the ring of integers of $\widetilde{K}$] which is compatible with the actions of $\text{Gal}(\widetilde{K}/K)$ and G relative to α as a *GSC-envelope* (respectively, an *AGSC-envelope*; an *NF-envelope*) for $(G \curvearrowright M)$. We shall say that the collection of data $(G \curvearrowright M)$ is a *GSC-Galois pair* (respectively, an *AGSC-Galois pair*; an *NF-Galois pair*) if there exists a GSC-envelope (respectively, an AGSC-envelope; an NF-envelope) for $(G \curvearrowright M)$.

Lemma 4.2. *Let $H \subseteq Q_F$ be a subgroup of Q_F. Then H is an* **open subgroup** *of Q_F if and only if H coincides with the* **stabilizer** *of some element of $\widetilde{F}^{\times}$ [with respect to the natural action of Q_F on $\widetilde{F}^{\times}$].*

PROOF. This follows from elementary field theory. $\square$

Proposition 4.3. *Let $(G \curvearrowright M)$ be a* **GSC-Galois pair**. *Then the following hold:*

(i) *The natural homomorphism*

$$G \longrightarrow \varprojlim_{N} G/N$$

— where the projective limit is taken over the normal subgroups $N \subseteq G$ of G such that N coincides with the **stabilizer** *[with respect to the natural action of G on M^{gp}] of some element of M^{gp} [so N is necessarily* **of finite index** *— cf. Lemma 4.2] — is an* **isomorphism** *of groups. In particular, the group "G" of any* **GSC-Galois pair** *"$(G \curvearrowright M)$" admits a natural, group-theoretically determined* **profinite group** *structure.*

(ii) *Let*

$$(F,\ \widetilde{F},\ \alpha\colon Q_F \xrightarrow{\sim} G,\ \beta\colon \mathcal{O}^{\rhd}_{\widetilde{F}} \xrightarrow{\sim} M)$$

be a **GSC-envelope** *for* $(G \curvearrowright M)$. *Then the isomorphism* α *is an* **isomorphism of profinite groups** [*cf.* (i)]. *In particular, the collection of data*

$$(F,\ \widetilde{F},\ \alpha\colon Q_F \xrightarrow{\sim} G)$$

forms a **GSC-envelope** *for the profinite group* G.

(iii) *The profinite group* G *is* **of GSC-type**. *If, moreover, the GSC-Galois pair* $(G \curvearrowright M)$ *is an* **AGSC-Galois pair** (*respectively,* **NF-Galois pair**), *then the profinite group* G *is* **of AGSC-type** (*respectively,* **of NF-type**).

Proof. These assertions follow immediately from Lemma 4.2, together with the various definitions involved. □

Theorem 4.4. *In the notation introduced at the beginning of the present* §4, *let* $(G \curvearrowright M)$ *be a* **GSC-Galois pair** [*cf. Definition* 4.1]. *We construct various objects associated to* $(G \curvearrowright M)$ *as follows:*

(1) *We shall write*

$$\boldsymbol{\mu}(M) \ \stackrel{\text{def}}{=}\ (M^{\times})_{\text{tor}}$$

and

$$\Lambda(M) \ \stackrel{\text{def}}{=}\ \varprojlim_{n}\ \boldsymbol{\mu}(M)[n]$$

— where the projective limit is taken over $n \in \mathbb{Z}_{\geq 1}$. *Note that* G *acts on* $\boldsymbol{\mu}(M)$ *and* $\Lambda(M)$. *We shall refer to the* G*-module* $\Lambda(M)$ *as the* **cyclotome** *associated to* $(G \curvearrowright M)$. *Note that one verifies immediately from our construction of* $\Lambda(M)$ *that* $\Lambda(M)$ *has a natural structure of* **profinite** [*cf. also the above definition of* $\Lambda(M)$], *hence also topological,* G*-module; moreover, we have a natural identification* $\boldsymbol{\mu}(M)[n] = \Lambda(M)/n\Lambda(M)$.

(2) *It follows from Lemma* 3.10, (vi), *that the exact sequences of* G*-modules*

$$1 \longrightarrow \Lambda(M)/n\Lambda(M) \longrightarrow M^{\text{gp}} \stackrel{n}{\longrightarrow} M^{\text{gp}} \longrightarrow 1$$

— where n *ranges over the positive integers — determine an* **injection**

$$(M^{\text{gp}})^G \hookrightarrow H^1(G, \Lambda(M)).$$

(3) *Let* $D \in v \in \mathcal{V}(G)$ [*cf. Proposition* 3.5, (1); *Proposition* 4.3, (iii)]. *Then it follows immediately, by considering the conjugation action of* G, *that the kernel of the composite*

$$(M^{\text{gp}})^G \hookrightarrow H^1(G, \Lambda(M)) \to H^1(I(D), \Lambda(M)^{(p(D)')})$$

[*cf.* *Theorem* 1.4, (1), (3)*; Lemma* 1.5, (i)] **depends** *only on* v [*i.e., does* **not depend** *on the choice of* $D \in v$]*. We shall write*

$$(M^{\mathrm{gp}})^G|_v^{\times} \ \subseteq \ (M^{\mathrm{gp}})^G$$

for this kernel. Thus, it follows from the definition of $(M^{\mathrm{gp}})^G|_v^{\times} \subseteq (M^{\mathrm{gp}})^G$ *that the composite*

$$(M^{\mathrm{gp}})^G \ \hookrightarrow \ H^1(G, \Lambda(M)) \ \to \ H^1(D, \Lambda(M)^{(p(D)')})$$

determines a homomorphism

$$(M^{\mathrm{gp}})^G|_v^{\times} \ \longrightarrow \ H^1(D/I(D), \Lambda(M)^{(p(D)')});$$

moreover, it follows immediately, by considering the conjugation action of G*, that the kernel of this homomorphism*

$$(M^{\mathrm{gp}})^G|_v^{\prec} \ \subseteq \ (M^{\mathrm{gp}})^G|_v^{\times}$$

[*cf.* *Lemma* 1.5, (ii)] **depends** *only on* v [*i.e., does* **not depend** *on the choice of* $D \in v$]*.*

(4) *It follows from the construction of* (3)*, together with Lemma* 1.5*, that the collection of data*

$$\mathcal{M}(G \curvearrowright M) \ \stackrel{\mathrm{def}}{=} \ (((M^{\mathrm{gp}})^G)^{\circledast}, \ M^G \subseteq ((M^{\mathrm{gp}})^G)^{\circledast}, \ \mathcal{V}(G), \ \{(M^{\mathrm{gp}})^G|_v^{\prec} \subseteq ((M^{\mathrm{gp}})^G)^{\circledast}\}_{v \in \mathcal{V}(G)})$$

forms an **NF-monoid** [*cf. Definition* 2.3]*. Thus, by Theorem* 2.9*,* (7)*,* (8)*, we have a map*

$$\boxplus_{F(G \curvearrowright M)} \ \stackrel{\mathrm{def}}{=} \ \boxplus_{\mathcal{M}(G \curvearrowright M)} \colon \ ((M^{\mathrm{gp}})^G)^{\circledast} \times ((M^{\mathrm{gp}})^G)^{\circledast} \ \longrightarrow \ ((M^{\mathrm{gp}})^G)^{\circledast}$$

such that the map $\boxplus_{F(G \curvearrowright M)}$*, together with the monoid structure of* $((M^{\mathrm{gp}})^G)^{\circledast}$*, determines a* **structure of field** *on* $((M^{\mathrm{gp}})^G)^{\circledast}$*. We shall write*

$$F(G \curvearrowright M)$$

for the resulting field.

(5) *If* $H \subseteq G$ *is an open subgroup of* G*, then we shall write* $(H \curvearrowright M)$ *for the* **GSC-Galois pair** *obtained by forming the collection of data consisting of* H*,* M*, and the action of* H *on* M *induced by the action of* G *on* M*. Write*

$$\widetilde{F}(G \curvearrowright M) \ \stackrel{\mathrm{def}}{=} \ \varinjlim_{H} F(H \curvearrowright M)$$

— where the injective limit is taken over the open subgroups $H \subseteq G$ *of* G*. Thus,* G *acts naturally on* $\widetilde{F}(G \curvearrowright M)$*.*

Let

$$(F,\ \widetilde{F},\ \alpha\colon Q_F \xrightarrow{\sim} G,\ \beta\colon \mathcal{O}^{\triangleright}_{\widetilde{F}} \xrightarrow{\sim} M)$$

be a **GSC-envelope** *for* $(G \curvearrowright M)$ [*cf. Definition* 4.1]. *Then the isomorphism* β *determines a* **commutative diagram of fields**

$$\begin{array}{ccc} F & \longrightarrow & \widetilde{F} \\ \wr\downarrow & & \wr\downarrow \\ F(G \curvearrowright M) & \longrightarrow & \widetilde{F}(G \curvearrowright M) \end{array}$$

— where the horizontal arrows are the natural inclusions, the vertical arrows are **isomorphisms***, and the right-hand vertical arrow is* **compatible** *with the natural actions of* Q_F *and* G *relative to* α.

Proof. This follows immediately from the various definitions involved. ☐

Remark 4.4.1. One verifies immediately from the various definitions involved that we have a natural identification

$$\widetilde{F}(G \curvearrowright M)_\times \ = \ (M^{\mathrm{gp}})^{\circledast}.$$

Corollary 4.5. *Let* $(G \curvearrowright M)$ *be a* **GSC-Galois pair** [*cf. Definition* 4.1]. *Write*

$$\mathrm{Aut}(M)$$

for the group of automorphisms of the monoid M *and*

$$\mathrm{Aut}^{\mathrm{fld}}(M) \ \subseteq \ \mathrm{Aut}(M)$$

for the subgroup of $\mathrm{Aut}(M)$ *consisting of those automorphisms* α *of* M *such that the automorphism of* $(M^{\mathrm{gp}})^{\circledast}$ *induced by* α *is* **compatible** *with the field structure of* $\widetilde{F}(G \curvearrowright M)$ [*cf. Remark* 4.4.1]. [*Thus, the image of the faithful action*

$$G \ \hookrightarrow \ \mathrm{Aut}(M)$$

is **contained** *in* $\mathrm{Aut}^{\mathrm{fld}}(M) \subseteq \mathrm{Aut}(M)$.] *Then it holds that*

$$N_{\mathrm{Aut}(M)}(G) \ \subseteq \ \mathrm{Aut}^{\mathrm{fld}}(M).$$

Proof. This follows immediately from Theorem 4.4. ☐

Corollary 4.6. *Let* $\overline{\mathbb{Q}}$ *be an algebraic closure of* $\mathbb{Q}$. *Write* $\mathcal{O}_{\overline{\mathbb{Q}}} \subseteq \overline{\mathbb{Q}}$ *for the ring of integers of* $\overline{\mathbb{Q}}$ *and* $\mathrm{Aut}(\mathcal{O}^{\triangleright}_{\overline{\mathbb{Q}}})$ *for the group of automorphisms of the monoid* $\mathcal{O}^{\triangleright}_{\overline{\mathbb{Q}}}$. *Thus, we have a natural injection*

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \ \hookrightarrow \ \mathrm{Aut}(\mathcal{O}^{\triangleright}_{\overline{\mathbb{Q}}}).$$

Let us regard $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *as a subgroup of* $\mathrm{Aut}(\mathcal{O}^{\triangleright}_{\overline{\mathbb{Q}}})$ *by means of this injection. Then the following hold:*

(i) *The subgroup* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *is* **normally terminal** *in* $\mathrm{Aut}(\mathcal{O}^{\triangleright}_{\overline{\mathbb{Q}}})$*, i.e., it holds that*

$$N_{\mathrm{Aut}(\mathcal{O}^{\triangleright}_{\overline{\mathbb{Q}}})}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) \;=\; \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}).$$

(ii) *The* **centralizer** *of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *in* $\mathrm{Aut}(\mathcal{O}^{\triangleright}_{\overline{\mathbb{Q}}})$ *is* **trivial***, i.e., it holds that*

$$Z_{\mathrm{Aut}(\mathcal{O}^{\triangleright}_{\overline{\mathbb{Q}}})}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) \;=\; \{1\}.$$

(iii) *The group* $\mathrm{Aut}(\mathcal{O}^{\triangleright}_{\overline{\mathbb{Q}}})$ *is* **center-free***.*

PROOF. Assertion (i) follows from Corollary 4.5. Assertion (ii) follows from assertion (i), together with the well-known fact that $Z_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) = \{1\}$ [cf., e.g., [11], Corollary 12.1.6]. Assertion (iii) follows from assertion (ii). This completes the proof of Corollary 4.6. ☐

§ 5. Mono-anabelian Reconstruction of Number Fields

In the present §5, we finish establishing a *functorial "group-theoretic" algorithm* for reconstructing, from [a suitable quotient of] the absolute Galois group of an NF, the given NF [cf. Theorem 5.11 below].

In the present §5, we maintain the notation introduced at the beginning the preceding §4. Suppose that $\widetilde{F}$ is *absolutely Galois*. We shall write

- $Q_{F_{\mathrm{prm}}} \overset{\mathrm{def}}{=} \mathrm{Gal}(\widetilde{F}/F_{\mathrm{prm}})$ for the Galois group of $\widetilde{F}/F_{\mathrm{prm}}$.

Lemma 5.1. *The following hold:*

(i) *It holds that F is* **absolutely Galois** *if and only if the following condition is satisfied: For any v, $w \in \mathcal{V}_F$ such that $p_{F_v} = p_{F_w}$ and $f_{F_v} = 1$, it holds that $f_{F_w} = 1$.*

(ii) *There exists a uniquely determined minimal intermediate extension of $\widetilde{F}/F$ among the intermediate extensions of $\widetilde{F}/F$ which are* **absolutely Galois** *and* **finite** *over F.*

(iii) *Suppose that F is* **absolutely Galois***. Then the action*

$$Q_{F_{\mathrm{prm}}} \longrightarrow \mathrm{Aut}(Q_F)$$

by conjugation is an **isomorphism** *of groups.*

PROOF. Assertion (i) follows from [10], Chapter VII, Corollary 13.8. Assertion (ii) follows immediately from our assumption that $\widetilde{F}$ is *absolutely Galois*, together with elementary field theory. Assertion (iii) follows from Theorem 3.3. This completes the proof of Lemma 5.1. □

Proposition 5.2. *Let G be a profinite group* **of AGSC-type** [*cf. Definition* 3.2]*. We construct various objects associated to G as follows:*

(1) *We shall say that G is* **absolutely Galois** *if the following condition is satisfied: For any v, $w \in \mathcal{V}(G)$* [*cf. Proposition* 3.5, (1)] *such that $p(v) = p(w)$* [*cf. Proposition* 3.5, (2)] *and $f(v) = 1$* [*cf. Proposition* 3.5, (2)]*, it holds that $f(w) = 1$* [*cf. Lemma* 5.1, (i)].

(2) *It follows from* (1) *and Lemma* 5.1, (ii)*, that there exists a uniquely determined maximal open subgroup of G which is* **absolutely Galois***. We shall refer to this open subgroup as the* **Galois closure-subgroup** *of G.*

(3) *We shall write*

$$G_{\mathfrak{C}} \stackrel{\text{def}}{=} \mathrm{Aut}(H)$$

for the group obtained by forming the group of automorphisms of the Galois closure-subgroup $H \subseteq G$ of G [*cf. Lemma* 5.1, (iii)]*. Thus, since H is* **normal** *in G* [*cf. Lemma* 5.1, (ii)]*, by considering the action of G on H by conjugation, we obtain a homomorphism of groups*

$$G \longrightarrow G_{\mathfrak{C}}.$$

(4) *It follows from Lemma* 5.1, (iii)*, that the homomorphism $G \to G_{\mathfrak{C}}$ of* (3) *is an* **injection** *whose image is* **of finite index***. Thus, the structure of profinite group on G determines a structure of profinite group on $G_{\mathfrak{C}}$. We always regard $G_{\mathfrak{C}}$ as a* **profinite group** *by means of this structure of profinite group.*

Let

$$(F,\ \widetilde{F},\ \alpha\colon Q_F \stackrel{\sim}{\to} G)$$

be an **AGSC-envelope** *for G* [*cf. Definition* 3.2]*. Then the following hold:*

(i) *It holds that F is* **absolutely Galois** *if and only if G is* **absolutely Galois**.

(ii) *The isomorphism α determines a* **commutative** *diagram of profinite groups*

$$\begin{array}{ccc} Q_F & \longrightarrow & Q_{F_{\mathrm{prm}}} \\ \wr\downarrow\alpha & & \wr\downarrow \\ G & \longrightarrow & G_{\mathfrak{C}} \end{array}$$

— where the horizontal arrows are the natural open injections [*cf.* (4)]*, and the vertical arrows are* **isomorphisms**.

PROOF. These assertions follow immediately from Lemma 5.1, together with the various definitions involved. □

Proposition 5.3. *Let G be a profinite group* **of AGSC-type**. *We construct various objects associated to G as follows:*

(1) *We shall write*

$$^{\dagger}F^{\times}(G_{\mathfrak{C}}) \stackrel{\text{def}}{=} \mathcal{H}^{\times}(G_{\mathfrak{C}}) \subseteq {}^{\dagger}F_{\times}(G_{\mathfrak{C}}) \stackrel{\text{def}}{=} \mathcal{H}_{\times}(G_{\mathfrak{C}}) \subseteq \prod_{v \in \mathcal{V}(G_{\mathfrak{C}})} k_{\times}(v)$$

[*cf. Proposition* 3.5, (1)*; Proposition* 3.7, (2)*; Proposition* 3.11*; Proposition* 5.2, (ii)].

(2) *We shall write*

$$^{\dagger}\mathcal{O}^{\triangleright}(G_{\mathfrak{C}}) \subseteq {}^{\dagger}F_{\times}(G_{\mathfrak{C}})$$

for the submonoid of $^{\dagger}F_{\times}(G_{\mathfrak{C}})$ *consisting of* $a \in {}^{\dagger}F_{\times}(G_{\mathfrak{C}})$ *such that, for every* $v \in \mathcal{V}(G_{\mathfrak{C}})$*, the image in* $k_{\times}(v)$ *is contained in* $\mathcal{O}^{\triangleright}(v) \subseteq k_{\times}(v)$ [*cf. Proposition* 3.7, (1)].

(3) *For* $v \in \mathcal{V}(G_{\mathfrak{C}})$*, we shall write*

$$\mathcal{O}^{\times}(v) \stackrel{\text{def}}{=} \mathcal{O}^{\triangleright}(v)^{\times},$$

$$\underline{k}^{\times}(v) \stackrel{\text{def}}{=} \mathcal{O}^{\times}(v)^{(p(v)')},$$

$$^{\dagger}F^{\times}(G_{\mathfrak{C}})|_{v}^{\times} \stackrel{\text{def}}{=} \operatorname{Ker}\big({}^{\dagger}F^{\times}(G_{\mathfrak{C}}) \to k^{\times}(v) \twoheadrightarrow k^{\times}(v)/\mathcal{O}^{\times}(v)\big),$$

$$^{\dagger}F^{\times}(G_{\mathfrak{C}})|_{v}^{\prec} \stackrel{\text{def}}{=} \operatorname{Ker}\big({}^{\dagger}F^{\times}(G_{\mathfrak{C}})|_{v}^{\times} \to \mathcal{O}^{\times}(v) \twoheadrightarrow \underline{k}^{\times}(v)\big)$$

[*cf. Proposition* 3.5, (2)*; Proposition* 3.7, (1)].

(4) *It follows from Lemma* 3.10, (ii)*, together with the constructions of* (1), (2)*, and* (3)*, that the collection of data*

$$\mathcal{M}(G_{\mathfrak{C}}) \stackrel{\text{def}}{=} ({}^{\dagger}F_{\times}(G_{\mathfrak{C}}),\ {}^{\dagger}\mathcal{O}^{\triangleright}(G_{\mathfrak{C}}) \subseteq {}^{\dagger}F_{\times}(G_{\mathfrak{C}}),\ \mathcal{V}(G_{\mathfrak{C}}),\ \{{}^{\dagger}F^{\times}(G_{\mathfrak{C}})|_{v}^{\prec} \subseteq {}^{\dagger}F_{\times}(G_{\mathfrak{C}})\}_{v \in \mathcal{V}(G_{\mathfrak{C}})})$$

forms an **NF-monoid of PmF-type** [*cf. Definition* 2.3]. *Thus, by Theorem* 2.9, (7)*,* (8)*, we have a map*

$$\boxplus_{^{\dagger}F(G_{\mathfrak{C}})} \stackrel{\text{def}}{=} \boxplus_{\mathcal{M}(G_{\mathfrak{C}})} \colon {}^{\dagger}F_{\times}(G_{\mathfrak{C}}) \times {}^{\dagger}F_{\times}(G_{\mathfrak{C}}) \longrightarrow {}^{\dagger}F_{\times}(G_{\mathfrak{C}})$$

such that the map $\boxplus_{^{\dagger}F(G_{\mathfrak{C}})}$*, together with the monoid structure of* $^{\dagger}F_{\times}(G_{\mathfrak{C}})$*, determines a* **structure of field** *on* $^{\dagger}F_{\times}(G_{\mathfrak{C}})$*. We shall write*

$$^{\dagger}F(G_{\mathfrak{C}})$$

for the resulting field.

Let

$$(F,\ \widetilde{F},\ \alpha\colon Q_F \xrightarrow{\sim} G)$$

be an **AGSC-envelope** *for* G. *Then the arrows in the second display of Lemma* 3.10, (vi), *and the inclusion in the fourth display of Proposition* 3.11, *together with the isomorphism* α, *determine* **isomorphisms of fields**

$$F_{\mathrm{prm}} \ \xrightarrow{\sim}\ {}^{\dagger}F((Q_F)_{\mathfrak{C}}) \ \xrightarrow{\sim}\ {}^{\dagger}F(G_{\mathfrak{C}}).$$

Proof. This follows immediately from the various definitions involved. □

Lemma 5.4. *Let* $v \in \mathcal{V}_F$. *Write*

$$(F_v^{\times} \times F_v^{\times})^{\neq 0} \ \subseteq\ F_v^{\times} \times F_v^{\times}$$

for the subset of $F_v^{\times} \times F_v^{\times}$ *consisting of* $(a, b) \in F_v^{\times} \times F_v^{\times}$ *such that* $a + b \neq 0$. *Then the subset*

$$(F^{\times} \times F^{\times}) \ \cap\ (F_v^{\times} \times F_v^{\times})^{\neq 0} \ \subseteq\ (F_v^{\times} \times F_v^{\times})^{\neq 0}$$

is **dense** *in* $(F_v^{\times} \times F_v^{\times})^{\neq 0}$.

Proof. This follows immediately from the various definitions involved. □

Proposition 5.5. *Let* G *be a profinite group* **of AGSC-type**. *We construct various objects associated to* G *as follows:*

(1) *Let* $v \in \mathcal{V}(G)$ [*cf. Proposition* 3.5, (1)]. *Then we shall write*

$$(k^{\times}(v) \times k^{\times}(v))^{=0} \ \subseteq\ k^{\times}(v) \times k^{\times}(v)$$

[*cf. Proposition* 3.7, (1)] *for the subset of the topological space* $k^{\times}(v) \times k^{\times}(v)$ *consisting of* $(a, b) \in k^{\times}(v) \times k^{\times}(v)$ *such that* $ab^{-1} \neq 1$ *but* $(ab^{-1})^2 = 1$;

$$(k^{\times}(v) \times k^{\times}(v))^{\neq 0} \ \stackrel{\mathrm{def}}{=}\ \big(k^{\times}(v) \times k^{\times}(v)\big) \setminus \big((k^{\times}(v) \times k^{\times}(v))^{=0}\big).$$

(2) *Let* $v_{\mathfrak{C}} \in \mathcal{V}(G_{\mathfrak{C}})$ [*cf. Proposition* 5.2, (3); *Proposition* 5.2, (ii)]. *Note that* [*it follows from the various definitions involved that*] *the natural homomorphism* ${}^{\dagger}F^{\times}(G_{\mathfrak{C}}) \to k^{\times}(v_{\mathfrak{C}})$ [*cf. Proposition* 5.3, (1)] *is* **injective**. *Let us regard* ${}^{\dagger}F^{\times}(G_{\mathfrak{C}})$ *as a submodule of* $k^{\times}(v_{\mathfrak{C}})$ *by means of this injection. Write* $* \in k_{\times}(v_{\mathfrak{C}})$ [*cf. Proposition* 3.7, (2)] *for the unique element of the set* $k_{\times}(v_{\mathfrak{C}}) \setminus k^{\times}(v_{\mathfrak{C}})$. *Then we define a map*

$$\boxplus_{k(v_{\mathfrak{C}})}\colon\ k_{\times}(v_{\mathfrak{C}}) \times k_{\times}(v_{\mathfrak{C}}) \ \longrightarrow\ k_{\times}(v_{\mathfrak{C}})$$

as follows:

(a) *It holds that* $\boxplus_{k(v_{\mathfrak{C}})}(*, a) = \boxplus_{k(v_{\mathfrak{C}})}(a, *) = a$ *for every* $a \in k_{\times}(v_{\mathfrak{C}})$.

(b) *The image of* $(k^{\times}(v_{\mathfrak{C}}) \times k^{\times}(v_{\mathfrak{C}}))^{=0}$ *via* $\boxplus_{k(v_{\mathfrak{C}})}$ *is* $\{*\}$.

(c) *Let* $(a, b) \in (k^{\times}(v_{\mathfrak{C}}) \times k^{\times}(v_{\mathfrak{C}}))^{\neq 0}$. *Now it follows immediately from Lemma* 5.4 *that there exists a sequence* $(a_i, b_i)_{i \geq 1}$ *consisting of elements of* $({}^{\dagger}F^{\times}(G_{\mathfrak{C}}) \times {}^{\dagger}F^{\times}(G_{\mathfrak{C}})) \cap (k^{\times}(v_{\mathfrak{C}}) \times k^{\times}(v_{\mathfrak{C}}))^{\neq 0}$ *such that* $\lim_{i \to \infty} (a_i, b_i) = (a, b)$ [*with respect to the topology of the topological module* $k^{\times}(v_{\mathfrak{C}}) \times k^{\times}(v_{\mathfrak{C}})$]. *Then write* $\boxplus_{k(v_{\mathfrak{C}})}(a, b) \stackrel{\text{def}}{=} \lim_{i \to \infty} \boxplus_{{}^{\dagger}F(G_{\mathfrak{C}})}(a_i, b_i)$ [*cf. Proposition* 5.3, (4)]. *Note that it follows from Proposition* 3.11, (i), *and Proposition* 5.3 *that this limit "*$\boxplus_{k(v_{\mathfrak{C}})}(a, b)$*"* **exists** *and does* **not depend** *on the choice of the sequence* $(a_i, b_i)_{i \geq 1}$.

If $D_{\mathfrak{C}} \in v_{\mathfrak{C}}$, *then we shall write*

$$\boxplus_{k(D_{\mathfrak{C}})} \colon\ k_{\times}(D_{\mathfrak{C}}) \times k_{\times}(D_{\mathfrak{C}}) \longrightarrow k_{\times}(D_{\mathfrak{C}})$$

[*cf. Theorem* 1.4, (8)] *for the map determined by* $\boxplus_{k(v_{\mathfrak{C}})}$ *and the isomorphism of* $k_{\times}(v_{\mathfrak{C}})$ *with* $k_{\times}(D_{\mathfrak{C}})$ *of Proposition* 3.7, (1), (b). *Then one verifies immediately* [*cf. Proposition* 3.11, (i); *Proposition* 5.3] *that the map* $\boxplus_{k(v_{\mathfrak{C}})}$ (*respectively,* $\boxplus_{k(D_{\mathfrak{C}})}$), *together with the monoid structure of* $k_{\times}(v_{\mathfrak{C}})$ (*respectively,* $k_{\times}(D_{\mathfrak{C}})$), *determines a* **structure of field** *on* $k_{\times}(v_{\mathfrak{C}})$ (*respectively,* $k_{\times}(D_{\mathfrak{C}})$). *We shall write*

$$k(v_{\mathfrak{C}}) \quad (\textit{respectively, } k(D_{\mathfrak{C}}))$$

for the resulting field.

(3) *Let* $v \in \mathcal{V}^{d=1}(G)$ [*cf. Proposition* 3.5, (2)] *and* $D \in v$. *Write* $v_{\mathfrak{C}} \in \mathcal{V}(G_{\mathfrak{C}})$ *for the element determined by* v, *i.e., the* $G_{\mathfrak{C}}$*-conjugacy class of* $C_{G_{\mathfrak{C}}}(D) \subseteq G_{\mathfrak{C}}$ [*cf. Proposition* 3.5, (iv)]. *Then since* $d(D) = 1$, *it follows immediately from the various definitions involved that the natural inclusion* $D \hookrightarrow C_{G_{\mathfrak{C}}}(D)$ *determines an* **isomorphism** *of monoids*

$$(k_{\times}(v_{\mathfrak{C}}) \xrightarrow{\sim})\quad k_{\times}(C_{G_{\mathfrak{C}}}(D)) \xrightarrow{\sim} k_{\times}(D) \quad (\xleftarrow{\sim} k_{\times}(v)).$$

We shall write

$$\boxplus_{k(v)} \colon\ k_{\times}(v) \times k_{\times}(v) \longrightarrow k_{\times}(v), \quad \boxplus_{k(D)} \colon\ k_{\times}(D) \times k_{\times}(D) \longrightarrow k_{\times}(D)$$

for the maps determined by this isomorphism and the map $\boxplus_{k(v_{\mathfrak{C}})}$. *Then one verifies immediately that the map* $\boxplus_{k(v)}$ (*respectively,* $\boxplus_{k(D)}$), *together with the monoid structure of* $k_{\times}(v)$ (*respectively,* $k_{\times}(D)$), *determines a* **structure of field** *on* $k_{\times}(v)$ (*respectively,* $k_{\times}(D)$). *We shall write*

$$k(v) \quad (\textit{respectively, } k(D))$$

for the resulting field.

(4) *It follows from Proposition* 3.11, (ii), *that the composite of homomorphisms of monoids*

$$\mathcal{H}_\times(G) \hookrightarrow \prod_{v\in\mathcal{V}(G)} k_\times(v) \twoheadrightarrow \prod_{v\in\mathcal{V}^{d=1}(G)} k_\times(v)$$

is **injective**. *Observe that the field structures on the* $k_\times(v)$*'s of* (3), *where* v *ranges over the elements of* $\mathcal{V}^{d=1}(G)$, *determine a* **ring structure** *on*

$$\prod_{v\in\mathcal{V}^{d=1}(G)} k_\times(v).$$

Let

$$(F,\ \widetilde{F},\ \alpha\colon Q_F \xrightarrow{\sim} G)$$

be an **AGSC-envelope** *for* G *and* $v \in \mathcal{V}^{d=1}(G)$. *Then the isomorphism of monoids* $(F_v)_\times \xrightarrow{\sim} k_\times(v)$ *of Proposition* 3.7, (i), *determines an* **isomorphism of fields**

$$F_v \xrightarrow{\sim} k(v).$$

Proof. This follows immediately from the various definitions involved. □

Lemma 5.6. *Let* E *be a finite Galois extension of* F *contained in* $\widetilde{F}$ *and* d *a positive integer. Thus, we have a commutative diagram of monoids*

$$\begin{array}{ccccc} F_\times & \longrightarrow & \mathcal{H}_\times(F) & & \\ \downarrow & & \downarrow & & \\ E_\times & \longrightarrow & \mathcal{H}_\times(E) & \longrightarrow & \prod_{w\in\mathcal{V}_E^{d=1}} (E_w)_\times \end{array}$$

[*cf. Definition* 3.9] — *where the arrows are* **injective** [*cf. Lemma* 3.10, (i), (v)]. *For a positive integer* r, *we use the notation* $\zeta_r \in \overline{F}$ *to denote a primitive* r*-th root of unity in* $\overline{F}$. *Then the following hold:*

(i) *Suppose that* $E = F(\zeta_d)$. *Then there exists a* **torsion** *element of* $\mathcal{H}^\times(E)$ [*cf. Definition* 3.9] **of order** $\boldsymbol{d}$. *Moreover, for every* **torsion** *element* $\zeta \in \mathcal{H}^\times(E)$ **of order** $\boldsymbol{d}$, *the monoid* $E_\times$ **maps isomorphically** *onto the submonoid of* $\prod_{w\in\mathcal{V}_E^{d=1}} (E_w)_\times$ *obtained by forming the underlying* [*multiplicative*] *monoid of the* **subring** *of* $\prod_{w\in\mathcal{V}_E^{d=1}} E_w$ *generated by* F *and* ζ.

(ii) *Suppose that* d *is a* **prime** *number, that* $\zeta_d \in F$, *and that* $\mathrm{Gal}(E/F)$ *is* **of order** $\boldsymbol{d}$. *Then there exists an element* $x \in \mathcal{H}_\times(E)$ *such that* $x \notin F_\times$ *but* $x^d \in F_\times$. *Moreover, for every such an "*x*", the monoid* $E_\times$ **maps isomorphically** *onto the submonoid of* $\prod_{w\in\mathcal{V}_E^{d=1}} (E_w)_\times$ *obtained by forming the underlying* [*multiplicative*] *monoid of the* **subring** *of* $\prod_{w\in\mathcal{V}_E^{d=1}} E_w$ *generated by* F *and* x.

(iii) *Suppose that E is* **contained** *in a finite* **solvable** *extension of F_{prm}. Then, after possibly replacing E by a finite extension of E which is contained in a finite solvable extension of F_{prm}, there exists a finite sequence of finite extensions of F_{prm} contained in E*

$$F_{\mathrm{prm}} = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{n-1} \subseteq F_n \stackrel{\mathrm{def}}{=} E$$

such that, for each $i \in \{1, \ldots, n\}$, the extension F_i/F_{i-1} is **Galois**, *and, moreover,* **one** *of the following two conditions is satisfied:*

(a) *It holds that $F_i = F_{i-1}(\zeta_{r_i})$ for some positive integer r_i.*

(b) *It holds that $d_i \stackrel{\mathrm{def}}{=} \sharp\mathrm{Gal}(F_i/F_{i-1})$ is a* **prime** *number, and, moreover, $\zeta_{d_i} \in F_{i-1}$.*

PROOF. Assertion (i) follows immediately from Lemma 3.10, (iii), together with the various definitions involved. Next, we verify assertion (ii). Since [we have assumed] that $\mathrm{Gal}(E/F)$ is [necessarily *cyclic* and] *of order d*, the existence of such an x follows immediately from *Kummer theory*, together with our assumption that $\zeta_d \in F$. In order to verify the final portion of assertion (ii), let $u \in (\mathcal{O}_E^\times)^\wedge$, $a \in E^\times$ be such that $x = u \cdot a$ [cf. Lemma 3.10, (iv)]. [Here, we regard $(\mathcal{O}_E^\times)^\wedge$ and $E^\times$ as submonoids of $\mathcal{H}_\times(E)$.] Since $x^d \in F^\times$, it follows from Lemma 3.10, (i), that $u^d \in \mathcal{O}_E^\times$. Thus, since [one verifies immediately that] the cokernel of the natural homomorphism $\mathcal{O}_E^\times \hookrightarrow (\mathcal{O}_E^\times)^\wedge$ is *torsion-free* [cf. also [1], Lemma 5.29, (ii)], it holds that $u \in \mathcal{O}_E^\times$, hence that $x \in E^\times$. In particular, the *subring* of $\prod_{w \in \mathcal{V}_E^{d=1}} E_w$ generated by F and x determines an intermediate extension of the finite extension E/F. On the other hand, since d is a *prime* number, and $\mathrm{Gal}(E/F)$ is of order d, the assumption that $x \notin F$ implies that this intermediate extension *coincides* with E. This completes the proof of assertion (ii). Assertion (iii) follows immediately from elementary field theory. This completes the proof of Lemma 5.6. □

Proposition 5.7. *Let G be a profinite group* **of AGSC-type**. *We construct various objects associated to G as follows:*

(1) *Let $H \subseteq G_{\mathfrak{C}}$ [cf. Proposition 5.2, (3); Proposition 5.2, (ii)] be an open subgroup of $G_{\mathfrak{C}}$. Suppose that we are given a finite sequence of open subgroups of $G_{\mathfrak{C}}$*

$$H \stackrel{\mathrm{def}}{=} G_n \subseteq G_{n-1} \subseteq \cdots \subseteq G_1 \subseteq G_0 \stackrel{\mathrm{def}}{=} G_{\mathfrak{C}}$$

such that, for each $i \in \{1, \ldots, n\}$, G_i is **normal** *in G_{i-1}, and, moreover,* **one** *of the following two conditions is satisfied:*

(a) *It holds that $G_i = {}^{\boldsymbol{\mu}_{r_i}}G_{i-1}$ [cf. Proposition 3.7, (5)] for some positive integer r_i.*

(b) *It holds that* $d_i \stackrel{\text{def}}{=} \sharp(G_{i-1}/G_i)$ *is a* **prime** *number, and, moreover,* $G_{i-1} \subseteq {}^{\boldsymbol{\mu}_{d_i}}G_{\mathfrak{C}}$.

Then we shall inductively define submonoids ${}^{\dagger}F_{\times}(G_i)$*'s of the* $\mathcal{H}_{\times}(G_i)$*'s [cf. Proposition* 3.11*]*

$$\begin{array}{ccccccc} {}^{\dagger}F_{\times}(G_0) = {}^{\dagger}F_{\times}(G_{\mathfrak{C}}) & \subseteq & {}^{\dagger}F_{\times}(G_1) \subseteq \cdots \subseteq & {}^{\dagger}F_{\times}(G_{n-1}) & \subseteq & {}^{\dagger}F_{\times}(G_n) = {}^{\dagger}F_{\times}(H) \\ \cap & & \cap & \cap & & \cap \\ \mathcal{H}_{\times}(G_0) = \mathcal{H}_{\times}(G_{\mathfrak{C}}) & \subseteq & \mathcal{H}_{\times}(G_1) \subseteq \cdots \subseteq & \mathcal{H}_{\times}(G_{n-1}) & \subseteq & \mathcal{H}_{\times}(G_n) = \mathcal{H}_{\times}(H) \end{array}$$

as follows: Let $i \in \{1, \ldots, n\}$. *Suppose that we are given a submonoid* ${}^{\dagger}F_{\times}(G_{i-1}) \subseteq \mathcal{H}_{\times}(G_{i-1})$. *[Note that the submonoid* ${}^{\dagger}F_{\times}(G_0) = {}^{\dagger}F_{\times}(G_{\mathfrak{C}})$ *of* $\mathcal{H}_{\times}(G_0) = \mathcal{H}_{\times}(G_{\mathfrak{C}})$ *was already defined in Proposition* 5.3, (1).*]*

- *Suppose that* $G_i = {}^{\boldsymbol{\mu}_{r_i}}G_{i-1}$ *for some positive integer* r_i *[cf. condition* (a)*]. Let* $\zeta \in \mathcal{H}_{\times}(G_i)$ *be a* **torsion** *element* **of order** $\boldsymbol{r_i}$ *[cf. Lemma* 5.6, (i)*]. We shall write*

$${}^{\dagger}F_{\times}(G_i) \subseteq \prod_{v \in \mathcal{V}^{d=1}(G_i)} k_{\times}(v)$$

[cf. Proposition 3.5, (2)*; Proposition* 3.7, (2)*] for the underlying [multiplicative] monoid of the* **subring** *of the ring* $\prod_{v \in \mathcal{V}^{d=1}(G_i)} k_{\times}(v)$ *[cf. Proposition* 5.5, (4)*] generated by the images of* ${}^{\dagger}F_{\times}(G_{i-1})$ *and* ζ. *Then it follows from our construction [cf. also Lemma* 5.6, (i)*] that* ${}^{\dagger}F_{\times}(G_i)$ *is* **contained** *in the image of* $\mathcal{H}_{\times}(G_i)$ *[relative to the injection discussed in Proposition* 5.5, (4)*] and, moreover,* **independent** *of the choice of* ζ. *In particular, it makes sense to regard* ${}^{\dagger}F_{\times}(G_i)$ *as a submonoid of* $\mathcal{H}_{\times}(G_i)$.

- *Suppose that* $d_i \stackrel{\text{def}}{=} \sharp(G_{i-1}/G_i)$ *is a* **prime** *number, and, moreover,* $G_{i-1} \subseteq {}^{\boldsymbol{\mu}_{d_i}}G_{\mathfrak{C}}$ *[cf. condition* (b)*]. Let* $x \in \mathcal{H}_{\times}(G_i)$ *be such that* $x \notin {}^{\dagger}F_{\times}(G_{i-1})$ *but* $x^{d_i} \in {}^{\dagger}F_{\times}(G_{i-1})$ *[cf. Lemma* 5.6, (ii)*]. We shall write*

$${}^{\dagger}F_{\times}(G_i) \subseteq \prod_{v \in \mathcal{V}^{d=1}(G_i)} k_{\times}(v)$$

[cf. Proposition 3.5, (2)*; Proposition* 3.7, (2)*] for the underlying [multiplicative] monoid of the* **subring** *of the ring* $\prod_{v \in \mathcal{V}^{d=1}(G_i)} k_{\times}(v)$ *[cf. Proposition* 5.5, (4)*] generated by the images of* ${}^{\dagger}F_{\times}(G_{i-1})$ *and* x. *Then it follows from our construction [cf. also Lemma* 5.6, (ii)*] that* ${}^{\dagger}F_{\times}(G_i)$ *is* **contained** *in the image of* $\mathcal{H}_{\times}(G_i)$ *[relative to the injection discussed in Proposition* 5.5, (4)*] and, moreover,* **independent** *of the choice of* x. *In particular, it makes sense to regard* ${}^{\dagger}F_{\times}(G_i)$ *as a submonoid of* $\mathcal{H}_{\times}(G_i)$.

Next, let us observe that it follows immediately from our construction [cf. also Lemma 5.6, (i), (ii)*] that the maps* $\boxplus_{k(v)}$ *[cf. Proposition* 5.5, (3)*], where* v *ranges over the elements* *of* $\mathcal{V}^{d=1}(H)$*, determine* [*cf. the injection discussed in Proposition* 5.5, (4)] *a map*

$$\boxplus_{{}^\dagger F(H)} \colon \ {}^\dagger F_\times(H) \times {}^\dagger F_\times(H) \ \longrightarrow \ {}^\dagger F_\times(H).$$

Moreover, one verifies immediately that this map $\boxplus_{{}^\dagger F(H)}$*, together with the monoid structure of* ${}^\dagger F_\times(H)$*, determines a* **structure of field** *on* ${}^\dagger F_\times(H)$*. We shall write*

$${}^\dagger F(H)$$

for the resulting field. Note that it follows from the various definitions involved [*cf. also Proposition* 3.11, (i)*; Proposition* 5.3*; Lemma* 5.6, (i), (ii)] *that the submonoid* ${}^\dagger F_\times(H) \subseteq \mathcal{H}_\times(H)$ *and the map* $\boxplus_{{}^\dagger F(H)}$*, hence also the field structure of* ${}^\dagger F(H)$*, do* **not depend** *on the choice of the sequence*

$$H \ \stackrel{\text{def}}{=} \ G_n \ \subseteq \ G_{n-1} \ \subseteq \ \cdots \ \subseteq \ G_1 \ \subseteq \ G_0 \ \stackrel{\text{def}}{=} \ G_{\mathfrak{C}}.$$

(2) *Write* $G_{\mathfrak{C}} \twoheadrightarrow G_{\mathfrak{C}}^{\text{slv}}$ *for the maximal prosolvable quotient of* $G_{\mathfrak{C}}$*. Then it follows from Lemma* 5.6, (iii)*, that every open subgroup of* $G_{\mathfrak{C}}$ *which arises from an open subgroup of* $G_{\mathfrak{C}}^{\text{slv}}$ **contains** *an open subgroup of* $G_{\mathfrak{C}}$ *which satisfies the conditions imposed on "H" in* (1)*, i.e., the conditions to the effect that there exists a suitable sequence of open subgroups of* $G_{\mathfrak{C}}$*. Thus, we have a submonoid*

$${}^\dagger F_\times^{\text{slv}}(G_{\mathfrak{C}}) \ \stackrel{\text{def}}{=} \ \varinjlim_{H} {}^\dagger F_\times(H) \ \subseteq \ \varinjlim_{H} \mathcal{H}_\times(H)$$

— where the injective limits are taken over the open subgroups $H \subseteq G_{\mathfrak{C}}$ *of* $G_{\mathfrak{C}}$ *which satisfy the conditions imposed on "H" in* (1) [*cf. Proposition* 3.11, (iii)] *— equipped with a map* [*determined by the various* $\boxplus_{{}^\dagger F(H)}$*'s — where H ranges over the open subgroups of* $G_{\mathfrak{C}}$ *which satisfy the conditions imposed on "H" in* (1)]

$$\boxplus_{{}^\dagger F^{\text{slv}}(G_{\mathfrak{C}})} \colon \ {}^\dagger F_\times^{\text{slv}}(G_{\mathfrak{C}}) \times {}^\dagger F_\times^{\text{slv}}(G_{\mathfrak{C}}) \ \longrightarrow \ {}^\dagger F_\times^{\text{slv}}(G_{\mathfrak{C}}).$$

Moreover, one verifies immediately that the map $\boxplus_{{}^\dagger F^{\text{slv}}(G_{\mathfrak{C}})}$*, together with the monoid structure of* ${}^\dagger F_\times^{\text{slv}}(G_{\mathfrak{C}})$*, determines a* **structure of field** *on* ${}^\dagger F_\times^{\text{slv}}(G_{\mathfrak{C}})$*. We shall write*

$${}^\dagger F^{\text{slv}}(G_{\mathfrak{C}})$$

for the resulting field.

Let

$$(F, \ \widetilde{F}, \ \alpha \colon Q_F \stackrel{\sim}{\to} G)$$

be an **AGSC-envelope** *for* G*. Write* $F_{\text{prm}}^{\text{slv}}$ *for the solvable closure of* F_{prm} *in* $\widetilde{F}$*. Then the arrows in the second display of Lemma* 3.10, (vi)*, and the inclusion in the fourth display of Proposition* 3.11*, together with the isomorphism* α*, determine* **isomorphisms of fields**

$$F_{\text{prm}}^{\text{slv}} \ \stackrel{\sim}{\longrightarrow} \ {}^\dagger F^{\text{slv}}((Q_F)_{\mathfrak{C}}) \ \stackrel{\sim}{\longrightarrow} \ {}^\dagger F^{\text{slv}}(G_{\mathfrak{C}}).$$

Proof. This follows from Lemma 5.6, together with the various definitions involved. □

Proposition 5.8. *Let G be a profinite group* **of AGSC-type**. *We construct various objects associated to G as follows:*

(1) *Let $D_{\mathfrak{C}} \in \widetilde{\mathcal{V}}(G_{\mathfrak{C}})$ [cf. Proposition 3.5, (1); Proposition 5.2, (3); Proposition 5.2, (ii)]. Write $G_{\mathfrak{C}} \twoheadrightarrow G_{\mathfrak{C}}^{\mathrm{slv}}$ for the maximal prosolvable quotient of $G_{\mathfrak{C}}$. Then since the composite $D_{\mathfrak{C}} \hookrightarrow G_{\mathfrak{C}} \twoheadrightarrow G_{\mathfrak{C}}^{\mathrm{slv}}$ is* **injective** *[cf. [6], Proposition 2.3, (iii)], there exists a sequence of normal open subgroups of $G_{\mathfrak{C}}$ which* **arise from open subgroups** *of $G_{\mathfrak{C}}^{\mathrm{slv}}$*

$$\cdots \subseteq G_{n+1} \subseteq G_n \subseteq \cdots \subseteq G_1 \subseteq G_0 = G_{\mathfrak{C}}$$

such that if we write $(D_{\mathfrak{C}})_n \stackrel{\mathrm{def}}{=} G_n \cap D_{\mathfrak{C}}$ for each n, then

$$\bigcap_{n \geq 0} (D_{\mathfrak{C}})_n = \{1\}.$$

Write ${}^{\ddagger}F^{\times}(G_n) \stackrel{\mathrm{def}}{=} ({}^{\dagger}F^{\mathrm{slv}}(G_{\mathfrak{C}})^{G_n})^{\times}$ [cf. Proposition 5.7, (2)]. Note that one verifies immediately [cf. Proposition 3.11, (i); Proposition 5.7] that the natural homomorphism ${}^{\ddagger}F^{\times}(G_n) \to k^{\times}((D_{\mathfrak{C}})_n)$ [cf. Theorem 1.4, (6)] is **injective**. *Let us regard ${}^{\ddagger}F^{\times}(G_n)$ as a submodule of $k^{\times}((D_{\mathfrak{C}})_n)$ by means of this injection. Write $* \in k_{\times}((D_{\mathfrak{C}})_n)$ [cf. Theorem 1.4, (8)] for the unique element of the set $k_{\times}((D_{\mathfrak{C}})_n) \setminus k^{\times}((D_{\mathfrak{C}})_n)$. Then we define a map*

$$\boxplus_{{}^{\dagger}k((D_{\mathfrak{C}})_n)} \colon \ k_{\times}((D_{\mathfrak{C}})_n) \times k_{\times}((D_{\mathfrak{C}})_n) \longrightarrow k_{\times}((D_{\mathfrak{C}})_n)$$

as follows:

(a) *It holds that $\boxplus_{{}^{\dagger}k((D_{\mathfrak{C}})_n)}(*, a) = \boxplus_{{}^{\dagger}k((D_{\mathfrak{C}})_n)}(a, *) = a$ for every $a \in k_{\times}((D_{\mathfrak{C}})_n)$.*

(b) *The image of $(k^{\times}((D_{\mathfrak{C}})_n) \times k^{\times}((D_{\mathfrak{C}})_n))^{=0}$ [cf. Proposition 5.5, (1)] via $\boxplus_{{}^{\dagger}k((D_{\mathfrak{C}})_n)}$ is $\{*\}$.*

(c) *Let $(a, b) \in (k^{\times}((D_{\mathfrak{C}})_n) \times k^{\times}((D_{\mathfrak{C}})_n))^{\neq 0}$ [cf. Proposition 5.5, (1)]. Now it follows immediately from Lemma 5.4 that there exists a sequence $(a_i, b_i)_{i \geq 1}$ consisting of elements of $({}^{\ddagger}F^{\times}(G_n) \times {}^{\ddagger}F^{\times}(G_n)) \cap (k^{\times}((D_{\mathfrak{C}})_n) \times k^{\times}((D_{\mathfrak{C}})_n))^{\neq 0}$ such that $\lim_{i \to \infty}(a_i, b_i) = (a, b)$ [with respect to the topology of the topological module $k^{\times}((D_{\mathfrak{C}})_n) \times k^{\times}((D_{\mathfrak{C}})_n)$]. Then write $\boxplus_{{}^{\dagger}k((D_{\mathfrak{C}})_n)}(a, b) \stackrel{\mathrm{def}}{=} \lim_{i \to \infty} \boxplus_{{}^{\dagger}F^{\mathrm{slv}}(G_{\mathfrak{C}})}(a_i, b_i)$ [cf. Proposition 5.7, (2)]. Note that it follows from Proposition 3.11, (i), and Proposition 5.7 that this limit "$\boxplus_{{}^{\dagger}k((D_{\mathfrak{C}})_n)}(a, b)$"* **exists** *and does* **not depend** *on the choice of the sequence $(a_i, b_i)_{i \geq 1}$.*

One verifies immediately from Proposition 3.11, (i), and Proposition 5.7 that the map $\boxplus_{{}^{\dagger}k((D_{\mathfrak{C}})_n)}$, together with the monoid structure of $k_{\times}((D_{\mathfrak{C}})_n)$, determines a **structure of field** *on $k_{\times}((D_{\mathfrak{C}})_n)$.*

(2) *In the notation of* (1), *since it holds that*

$$\overline{k}_\times(D_{\mathfrak{C}}) \ = \ \varinjlim_n k_\times((D_{\mathfrak{C}})_n)$$

[*cf. Theorem* 1.4, (9)], *it follows immediately from the construction of* (1) *that the various maps* $\boxplus_{{}^\dagger k((D_{\mathfrak{C}})_n)}$, *where* n *ranges over the nonnegative integers, determine a map*

$$\boxplus_{\overline{k}(D_{\mathfrak{C}})}\colon \ \overline{k}_\times(D_{\mathfrak{C}}) \times \overline{k}_\times(D_{\mathfrak{C}}) \ \longrightarrow \ \overline{k}_\times(D_{\mathfrak{C}})$$

such that the map $\boxplus_{\overline{k}(D_{\mathfrak{C}})}$, *together with the monoid structure of* $\overline{k}_\times(D_{\mathfrak{C}})$, *determines a* **structure of field** *on* $\overline{k}_\times(D_{\mathfrak{C}})$. *We shall write*

$$\overline{k}(D_{\mathfrak{C}})$$

for the resulting field. Note that it follows from our construction that the map $\boxplus_{\overline{k}(D_{\mathfrak{C}})}$, *hence also the field structure of* $\overline{k}(D_{\mathfrak{C}})$, *does* **not depend** *on the choice of the sequence*

$$\cdots \ \subseteq \ G_{n+1} \ \subseteq \ G_n \ \subseteq \ \cdots \ \subseteq \ G_1 \ \subseteq \ G_0 \ \stackrel{\mathrm{def}}{=} \ G_{\mathfrak{C}}.$$

(3) *Let* $D \in \widetilde{\mathcal{V}}(G)$. *Thus, it holds that* $C_{G_{\mathfrak{C}}}(D) \in \widetilde{\mathcal{V}}(G_{\mathfrak{C}})$ [*cf. Proposition* 3.5, (iv)]. *Moreover, it follows immediately from the various definitions involved that* $\overline{k}_\times(D) = \overline{k}_\times(C_{G_{\mathfrak{C}}}(D))$. *We shall write*

$$\boxplus_{\overline{k}(D)}\colon \ \overline{k}_\times(D) \times \overline{k}_\times(D) \ \longrightarrow \ \overline{k}_\times(D)$$

for the map determined by $\boxplus_{\overline{k}(C_{G_{\mathfrak{C}}}(D))}$, *which thus determines a* **structure of field** *on* $\overline{k}_\times(D)$. *We shall write*

$$\overline{k}(D)$$

for the resulting field.

(4) *Let* $D \in v \in \mathcal{V}(G)$ [*cf. Proposition* 3.5, (1)]. *Then it follows from Theorem* 1.4, (iv), *that* $k_\times(D) = \overline{k}_\times(D)^D$. *We shall write*

$$\boxplus_{k(D)}\colon \ k_\times(D) \times k_\times(D) \ \longrightarrow \ k_\times(D)$$

for the map determined by $\boxplus_{\overline{k}(D)}$ *and*

$$\boxplus_{k(v)}\colon \ k_\times(v) \times k_\times(v) \ \longrightarrow \ k_\times(v)$$

[*cf. Proposition* 3.7, (2)] *for the map determined by* $\boxplus_{k(D)}$ *and the isomorphism of* $k_\times(v)$ *with* $k_\times(D)$ *of* (b) *of Proposition* 3.7, (1). *Then one verifies immediately that the map* $\boxplus_{k(D)}$ (*respectively,* $\boxplus_{k(v)}$), *together with the monoid structure of* $k_\times(D)$ (*respectively,* $k_\times(v)$), *determines a* **structure of field** *on* $k_\times(D)$ (*respectively,* $k_\times(v)$). *We shall write*

$$k(D) \quad (\textit{respectively, } \ k(v))$$

for the resulting field.

(5) *Observe that the field structures on the* $k_\times(v)$*'s of* (4)*, where* v *ranges over the elements of* $\mathcal{V}(G)$*, determine a* **structure of ring** *on*

$$\prod_{v\in\mathcal{V}(G)} k_\times(v).$$

Let

$$(F,\ \widetilde{F},\ \alpha\colon Q_F \overset{\sim}{\to} G)$$

be an **AGSC-envelope** *for* G *and* $D \in v \in \mathcal{V}(G)$*. Write* $\widetilde{v} \in \mathcal{V}_{\widetilde{F}}$ *for the element of* $\mathcal{V}_{\widetilde{F}}$ *corresponding, via* α*, to* $D \in \widetilde{\mathcal{V}}(G)$ [*cf. Proposition* 3.5, (i)]*. Then the commutative diagram of monoids*

$$\begin{array}{ccc} (F_v)_\times & \longrightarrow & (\overline{F}_{\widetilde{v}})_\times \\ \wr\downarrow & & \wr\downarrow \\ k_\times(v) & \longrightarrow & \overline{k}_\times(D) \end{array}$$

— where the horizontal arrows are the natural inclusions, the left-hand vertical arrow is the isomorphism of monoids of Theorem 1.4, (iii)*, and the right-hand vertical arrow is the isomorphism of monoids of Theorem* 1.4, (iv) *— determines a* **commutative diagram of fields**

$$\begin{array}{ccc} F_v & \longrightarrow & \overline{F}_{\widetilde{v}} \\ \wr\downarrow & & \wr\downarrow \\ k(v) & \longrightarrow & \overline{k}(D). \end{array}$$

PROOF. This follows from the various definitions involved. ☐

Lemma 5.9. *Suppose that* F *is* **of PmF-type** [*cf. Definition* 2.1]*. Let* $\widetilde{v} \in \mathcal{V}_{\widetilde{F}}$*. Write* $v \in \mathcal{V}_F$ *for the nonarchimedean prime of* F *determined by* $\widetilde{v}$*,*

$$\widetilde{F}[\widetilde{v}] \ \subseteq\ \overline{F}_{\widetilde{v}}$$

for the image of the inclusion $\widetilde{F} \hookrightarrow \overline{F}_{\widetilde{v}}$*, and*

$$\mathrm{Aut}^{\mathrm{fld}}(\widetilde{F}[\widetilde{v}])$$

for the group of field automorphisms of $\widetilde{F}[\widetilde{v}]$*.* [*Thus, the natural inclusion* $\widetilde{F}[\widetilde{v}] \hookrightarrow \overline{F}_{\widetilde{v}}$ *induces an injection* $\mathrm{Gal}(\overline{F}_{\widetilde{v}}/F_v) \hookrightarrow \mathrm{Aut}^{\mathrm{fld}}(\widetilde{F}[\widetilde{v}])$.] *Note that the various subfields of* $\widetilde{F}[\widetilde{v}]$ *which are NF's determine a structure of* **profinite group** *on* $\mathrm{Aut}^{\mathrm{fld}}(\widetilde{F}[\widetilde{v}])$ *with respect to which* $\mathrm{Aut}^{\mathrm{fld}}(\widetilde{F}[\widetilde{v}])$ *is isomorphic to* Q_F *as an abstract profinite group. Then the following hold:*

(i) *Let M be a subfield of $\overline{F}_{\widetilde{v}}$. Suppose that M is* **algebraic** *over F ($\subseteq \overline{F}_{\widetilde{v}}$),* **absolutely Galois**, *and* **solvably closed**, *and that the group of field automorphisms of M — equipped with the* **profinite topology** *determined by the various subfields of M which are NF's — is* **isomorphic** *to Q_F as an abstract profinite group. Then $M = \widetilde{F}[\widetilde{v}]$.*

(ii) *There exists a* **uniquely determined** *isomorphism*

$$\iota_{\widetilde{v}} \colon \mathrm{Aut}^{\mathrm{fld}}(\widetilde{F}[\widetilde{v}]) \stackrel{\sim}{\longrightarrow} Q_F$$

of profinite groups that **restricts** *to the natural identification of the subgroup $\mathrm{Gal}(\overline{F}_{\widetilde{v}}/F_v) \subseteq \mathrm{Aut}^{\mathrm{fld}}(\widetilde{F}[\widetilde{v}])$ with the decomposition subgroup of Q_F associated to $\widetilde{v}$.*

(iii) *Let $\widetilde{w} \in \mathcal{V}_{\widetilde{F}}$. Then the isomorphism*

$$\widetilde{F}[\widetilde{v}] \stackrel{\sim}{\longrightarrow} \widetilde{F}[\widetilde{w}]$$

obtained by forming the composite of the inverse of the natural isomorphism $\widetilde{F} \stackrel{\sim}{\to} \widetilde{F}[\widetilde{v}]$ [i.e., obtained by the definition of $\widetilde{F}[\widetilde{v}]$] and the natural isomorphism $\widetilde{F} \stackrel{\sim}{\to} \widetilde{F}[\widetilde{w}]$ [i.e., obtained by the definition of $\widetilde{F}[\widetilde{w}]$] may be **characterized** *as the unique isomorphism $\iota_{\widetilde{v},\widetilde{w}} \colon \widetilde{F}[\widetilde{v}] \stackrel{\sim}{\to} \widetilde{F}[\widetilde{w}]$ of fields that satisfies the following condition: The composite*

$$Q_F \xrightarrow[\sim]{\iota_{\widetilde{v}}^{-1}} \mathrm{Aut}^{\mathrm{fld}}(\widetilde{F}[\widetilde{v}]) \stackrel{\sim}{\longrightarrow} \mathrm{Aut}^{\mathrm{fld}}(\widetilde{F}[\widetilde{w}]) \xrightarrow[\sim]{\iota_{\widetilde{w}}} Q_F$$

[cf. (ii)] — where the second arrow is the isomorphism obtained by conjugation by $\iota_{\widetilde{v},\widetilde{w}}$ — coincides with the **identity automorphism** *of Q_F.*

PROOF. First, we verify assertion (i). Since M is *absolutely Galois* and *solvably closed*, and the group of field automorphisms of M is *isomorphic* to Q_F, it follows from Theorem 3.3 that M is *isomorphic* to $\widetilde{F}$. Thus, since $\widetilde{F}$ is *absolutely Galois*, we conclude that $M = \widetilde{F}[\widetilde{v}]$, as desired. This completes the proof of assertion (i).

Next, we verify assertion (ii). Since [one verifies easily that] the isomorphism $\mathrm{Aut}^{\mathrm{fld}}(\widetilde{F}[\widetilde{v}]) \stackrel{\sim}{\to} Q_F$ obtained by conjugation by the natural isomorphism $\widetilde{F} \stackrel{\sim}{\to} \widetilde{F}[\widetilde{v}]$ satisfies the condition imposed on "$\iota_{\widetilde{v}}$", to verify assertion (ii), it suffices to verify the uniqueness of such isomorphisms. To this end, let ι_1, $\iota_2 \colon \mathrm{Aut}^{\mathrm{fld}}(\widetilde{F}[\widetilde{v}]) \stackrel{\sim}{\to} Q_F$ be isomorphisms that satisfy the condition in the statement of assertion (ii). Then since [we have assumed that] F is *of PmF-type*, it follows from Theorem 3.3 that $\iota_2 \circ \iota_1^{-1}$ is an *inner automorphism* of Q_F. Thus, since the decomposition subgroup of Q_F associated to $\widetilde{v}$ is *commensurably terminal* in Q_F [cf. [6], Proposition 2.3, (v)] and *center-free* [cf. [6], Proposition 2.3, (iii)], we conclude from the condition in the statement of assertion (ii) that $\iota_2 \circ \iota_1^{-1}$ is the *identity automorphism* of Q_F, as desired. This completes the proof of assertion (ii).

Finally, we verify assertion (iii). Since [one verifies easily that] the composite of the inverse of the natural isomorphism $\widetilde{F} \overset{\sim}{\to} \widetilde{F}[\widetilde{v}]$ and the natural isomorphism $\widetilde{F} \overset{\sim}{\to} \widetilde{F}[\widetilde{w}]$ satisfies the condition imposed on "$\iota_{\widetilde{v},\widetilde{w}}$", to verify assertion (iii), it suffices to verify the uniqueness of such isomorphisms. On the other hand, the desired uniqueness follows immediately from the fact that Q_F is *center-free* [cf. [6], Corollary 2.2]. This completes the proof of assertion (iii), hence also of Lemma 5.9. □

Proposition 5.10. *Let G be a profinite group* **of AGSC-type**. *We construct various objects associated to G as follows:*

(1) *Let $D_{\mathfrak{C}} \in \widetilde{\mathcal{V}}(G_{\mathfrak{C}})$ [cf. Proposition* 3.5, (1)*; Proposition* 5.2, (3)*; Proposition* 5.2, (ii)]. *Then it follows from Lemma* 5.9, (i), *that there exists a* **uniquely determined** *subfield of $\overline{k}(D_{\mathfrak{C}})$ [cf. Proposition* 5.8, (3)]

$$\widetilde{F}(D_{\mathfrak{C}}) \;\subseteq\; \overline{k}(D_{\mathfrak{C}})$$

such that $\widetilde{F}(D_{\mathfrak{C}})$ is **algebraic** *over the prime field contained in $\overline{k}(D_{\mathfrak{C}})$,* **absolutely Galois**, *and* **solvably closed**, *and, moreover, the group of field automorphisms of $\widetilde{F}(D_{\mathfrak{C}})$ — equipped with the* **profinite topology** *determined by the various subfields of $\widetilde{F}(D_{\mathfrak{C}})$ which are NF's — is* **isomorphic** *to $G_{\mathfrak{C}}$ as an abstract profinite group. We shall write*

$$\mathrm{Aut}^{\mathrm{fld}}(\widetilde{F}(D_{\mathfrak{C}}))$$

for the [profinite] group of field automorphisms of $\widetilde{F}(D_{\mathfrak{C}})$. Thus, the natural inclusion $\widetilde{F}(D_{\mathfrak{C}}) \hookrightarrow \overline{k}(D_{\mathfrak{C}})$ and the action of $D_{\mathfrak{C}}$ on $\overline{k}(D_{\mathfrak{C}})$ determine an injection

$$D_{\mathfrak{C}} \;\hookrightarrow\; \mathrm{Aut}^{\mathrm{fld}}(\widetilde{F}(D_{\mathfrak{C}})).$$

(2) *Let $D_{\mathfrak{C}} \in \widetilde{\mathcal{V}}(G_{\mathfrak{C}})$. Then it follows from Lemma* 5.9, (ii), *that there exists a* **uniquely determined** *isomorphism of profinite groups*

$$\iota_{D_{\mathfrak{C}}}:\; \mathrm{Aut}^{\mathrm{fld}}(\widetilde{F}(D_{\mathfrak{C}})) \;\overset{\sim}{\longrightarrow}\; G_{\mathfrak{C}}$$

such that the composite of the injection $D_{\mathfrak{C}} \hookrightarrow \mathrm{Aut}^{\mathrm{fld}}(\widetilde{F}(D_{\mathfrak{C}}))$ of the final display of (1) *and this isomorphism $\iota_{D_{\mathfrak{C}}}$* **coincides** *with the natural inclusion $D_{\mathfrak{C}} \hookrightarrow G_{\mathfrak{C}}$.*

(3) *Let $D_{\mathfrak{C}}$, $E_{\mathfrak{C}} \in \widetilde{\mathcal{V}}(G_{\mathfrak{C}})$. Then it follows from Lemma* 5.9, (iii), *that there exists a* **uniquely determined** *isomorphism of fields*

$$\iota_{D_{\mathfrak{C}},E_{\mathfrak{C}}}:\; \widetilde{F}(D_{\mathfrak{C}}) \;\overset{\sim}{\longrightarrow}\; \widetilde{F}(E_{\mathfrak{C}})$$

such that the composite

$$G_{\mathfrak{C}} \;\overset{\iota_{D_{\mathfrak{C}}}^{-1}}{\underset{\sim}{\longrightarrow}}\; \mathrm{Aut}^{\mathrm{fld}}(\widetilde{F}(D_{\mathfrak{C}})) \;\overset{\sim}{\longrightarrow}\; \mathrm{Aut}^{\mathrm{fld}}(\widetilde{F}(E_{\mathfrak{C}})) \;\overset{\iota_{E_{\mathfrak{C}}}}{\underset{\sim}{\longrightarrow}}\; G_{\mathfrak{C}}$$

[*cf.* (2)] — *where the second arrow is the isomorphism obtained by conjugation by* $\iota_{D_{\mathfrak{C}},E_{\mathfrak{C}}}$ — **coincides** *with the identity automorphism of* $G_{\mathfrak{C}}$.

(4) *It follows from Lemma* 5.9, (iii), *that, by considering the "diagonal" via the various isomorphisms "*$\iota_{D_{\mathfrak{C}},E_{\mathfrak{C}}}$*" of* (3), *we obtain a subring of the product* $\prod_{D_{\mathfrak{C}}\in\widetilde{\mathcal{V}}(G_{\mathfrak{C}})} \widetilde{F}(D_{\mathfrak{C}})$. *We shall write*

$$\widetilde{F}(G) \subseteq \prod_{D_{\mathfrak{C}}\in\widetilde{\mathcal{V}}(G_{\mathfrak{C}})} \widetilde{F}(D_{\mathfrak{C}}) \subseteq \prod_{D_{\mathfrak{C}}\in\widetilde{\mathcal{V}}(G_{\mathfrak{C}})} \overline{k}(D_{\mathfrak{C}}) = \prod_{D\in\widetilde{\mathcal{V}}(G)} \overline{k}(D)$$

[*cf. Proposition* 3.5, (iv)] *for this subring.*

Then the following hold:

(i) *The ring* $\widetilde{F}(G)$ *of* (4) *is a* **field** *which is* **absolutely Galois** *and* **solvably closed**.

(ii) *The natural action of* G *on* $\prod_{D\in\widetilde{\mathcal{V}}(G)} \overline{k}(D)$ **preserves** *this subring* $\widetilde{F}(G)$.

(iii) *The subfield* $\widetilde{F}(G)^G$ *of* $\widetilde{F}(G)$ *consisting of* G*-invariants* [*cf.* (ii)] *is an* **NF**. *Moreover, the action of* G *on* $\widetilde{F}(G)$ *determines an* **isomorphism of profinite groups**

$$G \xrightarrow{\sim} \mathrm{Gal}(\widetilde{F}(G)/\widetilde{F}(G)^G).$$

PROOF. This follows from Lemma 5.9, together with the various definitions involved. □

Theorem 5.11. *Let* G *be a profinite group* **of AGSC-type** [*cf. Definition* 3.2]. *Thus, it follows from Proposition* 5.10 *that we have an* **absolutely Galois** *and* **solvably closed** *field* $\widetilde{F}(G)$ *equipped with an action of* G *such that the subfield* $\widetilde{F}(G)^G$ *of* $\widetilde{F}(G)$ *consisting of* G*-invariants is an* **NF**, *and, moreover, the action of* G *on* $\widetilde{F}(G)$ *determines an* **isomorphism of profinite groups**

$$G \xrightarrow{\sim} \mathrm{Gal}(\widetilde{F}(G)/\widetilde{F}(G)^G).$$

We shall write

$$F(G) \stackrel{\mathrm{def}}{=} \widetilde{F}(G)^G$$

for the **NF** *obtained by forming the subfield of* $\widetilde{F}(G)$ *consisting of* G*-invariants,*

$$(G \curvearrowright \widetilde{\mathcal{O}}^{\triangleright}(G))$$

for the **AGSC-Galois pair** [*cf. Definition* 4.1] *determined by the absolutely Galois and solvably closed field* $\widetilde{F}(G)$ *equipped with an action of* G, *and*

$$\mathcal{O}^{\triangleright}(G) \stackrel{\mathrm{def}}{=} \widetilde{\mathcal{O}}^{\triangleright}(G)^G$$

for the submonoid of $\widetilde{\mathcal{O}}^{\triangleright}(G)$ *of* G*-invariants. Then the following hold:*

(i) *Let* $D \in \widetilde{\mathcal{V}}(G)$ [*cf. Proposition* 3.5, (1)]. *Then the natural inclusion* $D \hookrightarrow G$ *determines a* **commutative diagram of fields**

$$\begin{array}{ccc} F(G) & \longrightarrow & \widetilde{F}(G) \\ \downarrow & & \downarrow \\ k(D) & \longrightarrow & \overline{k}(D) \end{array}$$

[*cf. Proposition* 5.8, (3), (4); *Proposition* 5.10, (4)] — *where the horizontal arrows are the natural inclusions, and the right-hand vertical arrow is* ***D*****-equivariant**.

(ii) *Let*

$$(F,\ \widetilde{F},\ \alpha\colon Q_F \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\widetilde{F}/F) \stackrel{\sim}{\to} G)$$

be an **AGSC-envelope** *for* G [*cf. Definition* 3.2]. *Then the arrows in the second display of Lemma* 3.10, (vi), *and the inclusion in the fourth display of Proposition* 3.11, *together with the isomorphism* α, *determine a* **commutative diagram of fields**

$$\begin{array}{ccc} F & \longrightarrow & \widetilde{F} \\ \wr\downarrow & & \wr\downarrow \\ F(Q_F) & \longrightarrow & \widetilde{F}(Q_F) \\ \wr\downarrow & & \wr\downarrow \\ F(G) & \longrightarrow & \widetilde{F}(G) \end{array}$$

— *where the horizontal arrows are the natural inclusions, the vertical arrows are* **isomorphisms**, *and the right-hand vertical arrows are* **compatible** *with the respective actions of* Q_F *and* G *relative to* α.

PROOF. These assertions follow from the various definitions involved. □

Remark 5.11.1. In the notation of Theorem 5.11, one verifies immediately that we have a natural isomorphism

$$(\widetilde{\mathcal{O}}^{\triangleright}(G)^{\mathrm{gp}})^{\circledast} \stackrel{\sim}{\longrightarrow} \widetilde{F}(G)_{\times}$$

and natural inclusions

$$\widetilde{F}(G)^{\times} \subseteq \varinjlim_{H} \mathcal{H}^{\times}(H) \subseteq \varinjlim_{H} H^1(H, \Lambda(H))$$

— where the injective limits are taken over the open subgroups $H \subseteq G$ of G.

Remark 5.11.2. Note that, in the establishment of our global reconstruction result, the Neukirch-Uchida theorem [i.e., Theorem 3.3] plays a crucial role [cf. the proofs of Lemma 5.1, (iii), and Lemma 5.9, (i), (ii)]. In particular, the proof of this global reconstruction result does *not yield* an alternative proof of the Neukirch-Uchida theorem.

Remark 5.11.3. We thus conclude from the global reconstruction result obtained in the present paper that every profinite group *of NF-type* admits a *ring-theoretic basepoint* [i.e., a *"ring-theoretic interpretation"* or a *"ring-theoretic label"*] *group-theoretically* constructed from the given profinite group.

Remark 5.11.4. Let $G_\circ$, $G_\bullet$ be profinite groups *of AGSC-type*; $\alpha\colon G_\circ \to G_\bullet$ an open homomorphism of profinite groups.

(i) Suppose that α is *injective*. Then one verifies immediately that α determines a *commutative diagram of fields*

$$\begin{array}{ccc} F(G_\bullet) & \longrightarrow & \widetilde{F}(G_\bullet) \\ \downarrow & & \downarrow \wr \\ F(G_\circ) & \longrightarrow & \widetilde{F}(G_\circ) \end{array}$$

— where the horizontal arrows are the natural inclusions, and the right-hand vertical arrow is an *isomorphism* that is *compatible* with the respective actions of $G_\bullet$, $G_\circ$ relative to α.

(ii) Suppose that α is *surjective*, and that $\mathrm{Ker}(\alpha)$ is *perfect*, i.e., $\mathrm{Ker}(\alpha) = [\mathrm{Ker}(\alpha), \mathrm{Ker}(\alpha)]$. Then one verifies immediately that the subfield $\widetilde{F}(G_\circ)^{\mathrm{Ker}(\alpha)}$ of $\widetilde{F}(G_\circ)$ consisting of $\mathrm{Ker}(\alpha)$-invariants is *solvably closed*. In particular, by applying Proposition 5.2, (1), to the various open subgroups of $G_\circ/\mathrm{Ker}(\alpha)$ ($\xrightarrow{\sim} G_\bullet$), we conclude that $G_\circ/\mathrm{Ker}(\alpha)$ is *of AGSC-type*, and that $\widetilde{F}(G_\circ)^{\mathrm{Ker}(\alpha)}$ is *absolutely Galois* and *solvably closed*. Thus, it follows immediately from the construction of "$\widetilde{F}(-)$" [cf. Proposition 5.10, (1), (2), (3), (4)] that the surjection α determines a *commutative diagram of fields*

$$\begin{array}{ccc} F(G_\bullet) & \longrightarrow & \widetilde{F}(G_\bullet) \\ \wr\downarrow & & \wr\downarrow \\ F(G_\circ/\mathrm{Ker}(\alpha)) & \longrightarrow & \widetilde{F}(G_\circ/\mathrm{Ker}(\alpha)) \\ \wr\downarrow & & \downarrow \\ F(G_\circ) & \longrightarrow & \widetilde{F}(G_\circ) \end{array}$$

— where the upper vertical arrows are the isomorphisms induced by the isomorphism $G_\circ/\mathrm{Ker}(\alpha) \xrightarrow{\sim} G_\bullet$ determined by α; the horizontal arrows are the natural inclusions;

the left-hand vertical arrows are *isomorphisms*; the right-hand upper vertical arrow is an *isomorphism* that is *compatible* with the respective actions of $G_\bullet$, $G_\circ$ relative to α.

(iii) Suppose that $\mathrm{Ker}(\alpha)$ is *perfect*. Then it follows from (i), (ii) that α determines a *commutative diagram of fields*

$$\begin{array}{ccc} F(G_\bullet) & \longrightarrow & \widetilde{F}(G_\bullet) \\ \downarrow & & \downarrow \\ F(G_\circ) & \longrightarrow & \widetilde{F}(G_\circ) \end{array}$$

— where the horizontal arrows are the natural inclusions, and the right-hand vertical arrow is *compatible* with the respective actions of $G_\bullet$, $G_\circ$ relative to α. In particular, one may assert that the *"group-theoretic" algorithm*

$$\text{“}G \quad \mapsto \quad (G \curvearrowright \widetilde{F}(G))\text{”}$$

established in the present paper is *functorial* with respect to *open homomorphisms of profinite groups of AGSC-type whose kernels are perfect.*

Theorem 5.12. *Let* $(G \curvearrowright M)$ *be an* **AGSC-Galois pair** [*cf. Definition* 4.1]. *Recall that we have an* **injections**

$$M^G \hookrightarrow M^{\mathrm{gp}} \hookrightarrow \varinjlim_H H^1(H, \Lambda(M))$$

— where the injective limit is taken over the open subgroups $H \subseteq G$ *of* G [*cf. Theorem* 4.4, (1), (2)]. *Moreover, let us also recall that we have inclusions*

$$\mathcal{O}^\triangleright(G) \subseteq \widetilde{F}(G)^\times \subseteq \varinjlim_H H^1(H, \Lambda(H))$$

— where the injective limit is taken over the open subgroups $H \subseteq G$ *of* G [*cf. Proposition* 3.7, (4); *Theorem* 5.11; *Remark* 5.11.1]. *Then there exists a* **uniquely determined** G**-equivariant isomorphism**

$$\Lambda(M) \xrightarrow{\sim} \Lambda(G)$$

such that the induced isomorphism [*cf. also Theorem* 3.8, (i)]

$$\varinjlim_H H^1(H, \Lambda(M)) \xrightarrow{\sim} \varinjlim_H H^1(H, \Lambda(H))$$

maps M^G **bijectively** *onto* $\mathcal{O}^\triangleright(G)$. *Moreover, this induced isomorphism*

$$\varinjlim_H H^1(H, \Lambda(M)) \xrightarrow{\sim} \varinjlim_H H^1(H, \Lambda(H))$$

also determines an **isomorphism** $M^{\mathrm{gp}} \xrightarrow{\sim} \widetilde{F}(G)^{\times}$ *that extends to a* ***G*-equivariant isomorphism of fields**

$$\widetilde{F}(G \curvearrowright M) \xrightarrow{\sim} \widetilde{F}(G)$$

[*cf. Theorem* 4.4, (5)*; Remark* 4.4.1]*. We shall refer to this uniquely determined isomorphism* $\Lambda(M) \xrightarrow{\sim} \Lambda(G)$ *as the* **cyclotomic synchronization isomorphism** *for* $(G \curvearrowright M)$*.*

PROOF. The existence of such an isomorphism $\Lambda(M) \xrightarrow{\sim} \Lambda(G)$ follows immediately from the definition of the notion of an AGSC-Galois pair [cf. also Theorem 4.4; Theorem 5.11, (ii)]. The uniqueness of such an isomorphism follows immediately from the following elementary observation: Let $a \in \widehat{\mathbb{Z}}^{\times}$. Then it holds that the automorphism of $\widehat{\mathbb{Z}}$ given by multiplication by a induces an automorphism of the submonoid $\mathbb{N} \subseteq (\mathbb{Z} \subseteq)$ $\widehat{\mathbb{Z}}$ if and only if $a = 1$. The final assertion follows immediately from the [existence and] *uniqueness* of such an isomorphism. This completes the proof of Theorem 5.12. □

Finally, we prove a certain *compatibility* between the *functorial "group-theoretic" algorithm* obtained in the present paper and the *functorial "group-theoretic" algorithm* obtained in [9], Theorem 1.9.

Theorem 5.13. *Let* Π *be a profinite group which is isomorphic to the étale fundamental group of a hyperbolic orbicurve* X [*cf. the discussion entitled "Curves" in* [7], §0] *over an NF. Write* $\Pi \twoheadrightarrow Q$ *for the* **arithmetic quotient** *of* Π*, i.e., the quotient of* Π *by the* [*uniquely determined — cf.* [7]*, Theorem* 2.6, (vi)] *maximal topologically finitely generated normal closed subgroup of* Π*. Thus, the quotient* Q *is a profinite group* **of NF-type** [*cf.* [7]*, Theorem* 2.6, (vi)]*. Suppose that* X *is* **of strictly Belyi type** [*cf.* [8]*, Definition* 3.5]*. Write*

$$\Pi \curvearrowright \overline{F}(\Pi)$$

for the algebraically closed field equipped with an action of Π *obtained by applying the* **functorial "group-theoretic" algorithm** *given in* [9]*, Theorem* 1.9 *to* Π [*i.e., the field "*$\overline{k}^{\times}_{\mathrm{NF}} \cup \{0\}$*" of* [9]*, Theorem* 1.9, (e)]*. Thus, by the construction of* $\overline{F}(\Pi)$*, we have an inclusion*

$$\overline{F}(\Pi)^{\times} \hookrightarrow \varinjlim_{V} H^1(\Pi_V, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi_V))$$

— where we refer to the statement of [9]*, Theorem* 1.9*, for an explanation of the notation "*$\varinjlim_V H^1(\Pi_V, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi_V))$*". Then the natural homomorphism*

$$\varinjlim_{H} H^1(H, \Lambda(H)) \longrightarrow \varinjlim_{V} H^1(\Pi_V, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi_V))$$

[*cf. Proposition* 3.7, (4)] — *where the first injective limit is taken over the open subgroups* $H \subseteq Q$ *of* Q — *induced by the various natural surjections from the "*Π_V*'s" to the "*H*'s"* [*where we observe that every sufficiently small "*H*" arises as the arithmetic quotient of some "*Π_V*"*], *together with the isomorphisms of the* $\Lambda(H)$*'s with the* $\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi_V)$*'s discussed in Lemma* 5.14 *below, determines* [*cf. Remark* 5.11.1] *a* **Π-equivariant isomorphism of fields**

$$\widetilde{F}(Q) \xrightarrow{\sim} \overline{F}(\Pi).$$

PROOF. Theorem 5.13 follows immediately from the fact that, in the situation where the profinite groups involved are not just "abstract profinite groups", but rather arise from familiar objects of scheme theory, the homomorphism

$$\varinjlim_{H} H^1(H, \Lambda(H)) \longrightarrow \varinjlim_{V} H^1(\Pi_V, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi_V))$$

under consideration *coincides with the conventional homomorphism* between the injective limits of cohomology groups involved that arise from *conventional scheme theory.* ☐

Lemma 5.14. *Let* Π *be a profinite group which is isomorphic to the étale fundamental group of a hyperbolic orbicurve over an NF. Write* $\Pi \twoheadrightarrow Q$ *for the* **arithmetic quotient** *of* Π [*cf. the statement of Theorem* 5.13]*. Let* $D \in \widetilde{\mathcal{V}}(Q)$ [*cf. Proposition* 3.5, (1)]*. Then the composite*

$$\Lambda(Q) \xrightarrow{\sim} \Lambda(D) \xrightarrow{\sim} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi \times_Q D) = \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi)$$

— *where we refer to* [9]*, Theorem* 1.9, (b) [*cf. also* [9]*, Remark* 1.10.1, (ii)]*, for an explanation of the notation "*$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(-)$*"; the first arrow is the* **local-global cyclotomic synchronization isomorphism** *of Theorem* 3.8, (ii) [*cf. also Theorem* 1.4, (9)*; Proposition* 3.7, (4)]*; the second arrow is the isomorphism of* [9]*, Corollary* 1.10, (ii), (c) [*note that one verifies easily that the D-module* $\Lambda(D)$ **coincides** *with the "*$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G)$*" defined in* [9]*, Corollary* 1.10, (i), (a)*, in the case where we take the "*G*" of loc. cit. to be* D]*; the "*$=$*" is the natural identification that arises from the definitions of* $\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi \times_Q D)$ *and* $\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi)$ — *is* **Π-equivariant** *and* **independent** *of the choice of* $D \in \widetilde{\mathcal{V}}(Q)$*.*

PROOF. These assertions follow immediately from the fact that, in the situation where the profinite groups involved are not just "abstract profinite groups", but rather arise from familiar objects of scheme theory, the composite under consideration *coincides with the conventional identification* between the cyclotomes involved that arise from *conventional scheme theory.* This completes the proof of Lemma 5.14. ☐

§ 6. Global Mono-anabelian Log-Frobenius Compatibility

In the present §6, we give an interpretation of the global reconstruction result obtained in the present paper in terms of a certain *compatibility* with the *NF-log-Frobenius functor* [cf. Theorem 6.10 below].

Definition 6.1. Let D be a profinite group *of MLF-type* [cf. Definition 1.1; Proposition 1.2, (i)]. Then we shall refer to a collection of data

$$(G,\ D \hookrightarrow G)$$

consisting of a profinite group G of NF-type [cf. Definition 3.2] and an injection $D \hookrightarrow G$ of profinite groups as an *NF-holomorphic structure* on D.

Definition 6.2. Let D be a profinite group *of MLF-type* and $\mathfrak{hol} \stackrel{\mathrm{def}}{=} (G,\ D \hookrightarrow G)$ an *NF-holomorphic structure* on D. Then it follows immediately from [11], Theorem 12.1.9; [6], Proposition 2.3, (v), that the injection $D \hookrightarrow G$ in $\mathfrak{hol}$ determines an *open injection* $D \hookrightarrow C_G(\mathrm{Im}(D))$ — where we write $\mathrm{Im}(D)$ for the image of the injection $D \hookrightarrow G$, and we observe that $C_G(\mathrm{Im}(D)) \in \widetilde{\mathcal{V}}(G)$ [cf. Proposition 3.5, (1)]. Thus, we have an isomorphism of monoids

$$\overline{k}_\times(C_G(\mathrm{Im}(D))) \ \stackrel{\sim}{\longrightarrow}\ \overline{k}_\times(D)$$

[cf. Theorem 1.4, (9)], which is *compatible* with the natural actions of $C_G(\mathrm{Im}(D))$ and D relative to the open injection $D \hookrightarrow C_G(\mathrm{Im}(D))$. In particular, the field structure on $\overline{k}_\times(C_G(\mathrm{Im}(D)))$ constructed in Proposition 5.8, (3), determines a *field structure* on $\overline{k}_\times(D)$. We shall write

$$\overline{k}(D, \mathfrak{hol})$$

for the resulting field [equipped with a natural action by D].

Remark 6.2.1. One verifies immediately from the various definitions involved that we have a natural identification

$$\overline{k}(D, \mathfrak{hol})_\times \ =\ \overline{k}_\times(D).$$

Definition 6.3. Let $(D \curvearrowright M)$ be an *MLF-Galois* $\mathbb{TM}$*-pair of mono-analytic type* [cf. [9], Definition 3.1, (ii)]. Thus, D is a profinite group *of MLF-type.* We shall refer to an NF-holomorphic structure on D as an *NF-holomorphic structure* on $(D \curvearrowright M)$.

Definition 6.4. Let $(D \curvearrowright M)$ be an MLF-Galois $\mathbb{TM}$-pair of mono-analytic type.

(i) We shall write

$$\boldsymbol{\mu}(M) \stackrel{\text{def}}{=} (M^\times)_{\text{tor}}$$

and

$$\Lambda(M) \stackrel{\text{def}}{=} \varprojlim_n \boldsymbol{\mu}(M)[n]$$

— where the projective limit is taken over $n \in \mathbb{Z}_{\geq 1}$. Note that D acts on $\boldsymbol{\mu}(M)$ and $\Lambda(M)$. We shall refer to the D-module $\Lambda(M)$ as the *cyclotome* associated to $(D \curvearrowright M)$. Note that one verifies immediately that the cyclotome $\Lambda(M)$ has a natural structure of *profinite* [cf. the above definition of $\Lambda(M)$], hence also topological, G-module; moreover, we have a natural identification $\boldsymbol{\mu}(M)[n] = \Lambda(M)/n\Lambda(M)$. [Let us observe that one verifies easily that the D-modules $\boldsymbol{\mu}(M)$, $\Lambda(M)$ *coincide* with the D-modules "$\boldsymbol{\mu}_{\mathbb{Q}/\mathbb{Z}}(M)$", "$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(M)$" defined in [9], Definition 3.1, (v), respectively.]

(ii) It follows from Lemma 1.3, (x), that the exact sequences of D-modules

$$1 \longrightarrow \Lambda(M)/n\Lambda(M) \longrightarrow M^{\text{gp}} \stackrel{n}{\longrightarrow} M^{\text{gp}} \longrightarrow 1$$

— where n ranges over the positive integers — determine an *injection*

$$(M^{\text{gp}})^D \hookrightarrow H^1(D, \Lambda(M)).$$

(iii) Note that one verifies easily that the D-module $\Lambda(D)$ [cf. Theorem 1.4, (9)] *coincides* with the D-module "$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G)$" defined in [9], Corollary 1.10, (i), (a), in the case where we take the "G" of *loc. cit.* to be D. Thus, by [9], Remark 3.2.1, we have a *functorial algorithm* for constructing, from $(D \curvearrowright M)$, a D-equivariant isomorphism

$$\Lambda(M) \stackrel{\sim}{\longrightarrow} \Lambda(D)$$

such that the induced isomorphism

$$H^1(D, \Lambda(M)) \stackrel{\sim}{\longrightarrow} H^1(D, \Lambda(D))$$

determines — relative to the injection $\text{Kmm}(D)$ of Theorem 1.4, (10), and the injection of (ii) — an *isomorphism* of modules

$$(M^{\text{gp}})^D \stackrel{\sim}{\longrightarrow} k^\times(D).$$

(iv) By applying the discussion of (iii) to the various open subgroups of D, we obtain an *isomorphism* of modules

$$M^{\text{gp}} \stackrel{\sim}{\longrightarrow} \overline{k}^\times(D)$$

[cf. Theorem 1.4, (9)], hence also an *isomorphism* of monoids

$$(M^{\mathrm{gp}})^{\circledast} \stackrel{\sim}{\longrightarrow} \overline{k}_{\times}(D)$$

[cf. Theorem 1.4, (9)]. It follows immediately from the various definitions involved that these isomorphisms of monoids are *D-equivariant.*

(v) Let $\mathfrak{hol}$ be an NF-holomorphic structure on $(D \curvearrowright M)$. Then the D-equivariant isomorphism of monoids of (iv), together with the field structure of $\overline{k}(D, \mathfrak{hol})$ obtained in Definition 6.2 [cf. also Remark 6.2.1], determines a *field structure* on $(M^{\mathrm{gp}})^{\circledast}$. We shall write

$$\overline{k}(D \curvearrowright M, \mathfrak{hol})$$

for the resulting field [equipped with a natural action by D].

Remark 6.4.1. One verifies immediately from the various definitions involved that we have a natural identification

$$\overline{k}(D \curvearrowright M, \mathfrak{hol})_{\times} \ = \ (M^{\mathrm{gp}})^{\circledast}.$$

Definition 6.5. Let $(D \curvearrowright M)$ be an MLF-Galois $\mathbb{TM}$-pair of mono-analytic type and $\mathfrak{hol}$ an NF-holomorphic structure on $(D \curvearrowright M)$.

(i) Write

$$(M^{\mathrm{gp}})^{\circledast}_{+}$$

for the module [whose underlying set is $(M^{\mathrm{gp}})^{\circledast}$] obtained by forming the underlying *additive* module of the field $\overline{k}(D \curvearrowright M, \mathfrak{hol})$ of Definition 6.4, (v) [cf. Remark 6.4.1]. Then the $p(D)$-adic [cf. Theorem 1.4, (1)] logarithm on $\overline{k}(D \curvearrowright M, \mathfrak{hol})$ determines a *D-equivariant isomorphism* of modules

$$(M^{\times})^{\mathrm{pf}} \stackrel{\sim}{\longrightarrow} (M^{\mathrm{gp}})^{\circledast}_{+}.$$

Thus, the field structure on $(M^{\mathrm{gp}})^{\circledast}_{+}$ [i.e., the field structure of $\overline{k}(D \curvearrowright M, \mathfrak{hol})$] determines a *field structure* on $(M^{\times})^{\mathrm{pf}}$. We shall write

$$\mathfrak{log}(\overline{k})(D \curvearrowright M, \mathfrak{hol})$$

for the resulting field [equipped with a natural action by D].

(ii) We shall write

$$\mathcal{O}_{\mathfrak{log}(\overline{k})(D \curvearrowright M, \mathfrak{hol})} \subseteq \mathfrak{log}(\overline{k})(D \curvearrowright M, \mathfrak{hol})$$

for the ring of integers of $\mathfrak{log}(\overline{k})(D \curvearrowright M, \mathfrak{hol})$ [cf. Remark 6.5.2 below] and

$$\mathfrak{log}(D \curvearrowright M, \mathfrak{hol}) \stackrel{\mathrm{def}}{=} (\mathcal{O}_{\mathfrak{log}(\overline{k})(D \curvearrowright M, \mathfrak{hol})})^{\triangleright}.$$

(iii) One verifies immediately from the various definitions involved [cf. Remark 6.5.2 below] that the action of D on $\mathfrak{log}(\overline{k})(D \curvearrowright M, \mathfrak{hol})$ determines a natural action of D on $\mathfrak{log}(D \curvearrowright M, \mathfrak{hol})$; moreover, the collection of data

$$(D \curvearrowright \mathfrak{log}(D \curvearrowright M, \mathfrak{hol}))$$

consisting of the profinite group D and the topological [cf. Remark 6.5.2 below] D-monoid $\mathfrak{log}(D \curvearrowright M, \mathfrak{hol})$ forms an *MLF-Galois* $\mathbb{TM}$*-pair of mono-analytic type.*

Remark 6.5.1. One verifies immediately that if we write

$$\mathfrak{log}(\overline{k})(D \curvearrowright M, \mathfrak{hol})_+$$

for the underlying additive module of the field $\mathfrak{log}(\overline{k})(D \curvearrowright M, \mathfrak{hol})$, then we have a natural identification

$$\mathfrak{log}(\overline{k})(D \curvearrowright M, \mathfrak{hol})_+ \ = \ (M^\times)^{\mathrm{pf}}.$$

Remark 6.5.2. One verifies immediately that the field structure on $\mathfrak{log}(\overline{k})(D \curvearrowright M, \mathfrak{hol})$, together with the natural action by D, determines [cf. the discussion entitled "Fields" in §0, applied to the various subfields of invariants of $\mathfrak{log}(\overline{k})(D \curvearrowright M, \mathfrak{hol})$ by the open subgroups of D] a *topology* on $\mathfrak{log}(\overline{k})(D \curvearrowright M, \mathfrak{hol})$, i.e., the "$p(D)$*-adic topology*" of $\mathfrak{log}(\overline{k})(D \curvearrowright M, \mathfrak{hol})$. Moreover, this topology on $\mathfrak{log}(\overline{k})(D \curvearrowright M, \mathfrak{hol})$ determines a *topology* on $\mathfrak{log}(D \curvearrowright M, \mathfrak{hol})^\times$. We shall regard $\mathfrak{log}(D \curvearrowright M, \mathfrak{hol})$ as a *topological monoid* by means of the topology determined by the topology on $\mathfrak{log}(D \curvearrowright M, \mathfrak{hol})^\times$.

Definition 6.6.

(i) We shall say that a collection of data

$$((G \curvearrowright M),\ \{(D \curvearrowright M_D)\}_{D \in \widetilde{\mathcal{V}}(G)},\ \{\rho_D \colon M \hookrightarrow M_D\}_{D \in \widetilde{\mathcal{V}}(G)})$$

consisting of an NF-Galois pair $(G \curvearrowright M)$ [cf. Definition 4.1], an MLF-Galois $\mathbb{TM}$-pair $(D \curvearrowright M_D)$ of mono-analytic type for each $D \in \widetilde{\mathcal{V}}(G)$ [cf. Proposition 3.5, (1)], and a D-equivariant injection $\rho_D \colon M \hookrightarrow M_D$ of monoids for each $D \in \widetilde{\mathcal{V}}(G)$ is an *NF-Galois theater* if, for each $D \in \widetilde{\mathcal{V}}(G)$, the diagram

$$\begin{array}{ccc} M^{\mathrm{gp}} & \xrightarrow{\sim} & \widetilde{F}(G)^\times \\ \downarrow & & \downarrow \\ M_D^{\mathrm{gp}} & \xrightarrow{\sim} & \overline{k}^\times(D) \end{array}$$

— where the left-hand vertical arrow is the homomorphism determined by ρ_D; the right-hand vertical arrow is the homomorphism induced by the right-hand vertical arrow of the diagram of Theorem 5.11, (i), together with the natural identification of $\overline{k}^{\times}(D)$ with the monoid "$\overline{k}(D)^{\times}$" of Proposition 5.8, (3); the upper horizontal arrow is the [non-displayed] isomorphism of the final assertion of Theorem 5.12; the lower horizontal arrow is the isomorphism of Definition 6.4, (iv) — commutes.

(ii) Let

$$\mathcal{T}_1 \overset{\text{def}}{=} ((G_1 \curvearrowright M_1),\ \{(D_1 \curvearrowright M_{D_1})\}_{D_1 \in \widetilde{\mathcal{V}}(G_1)},\ \{\rho_{D_1} \colon M_1 \hookrightarrow M_{D_1}\}_{D_1 \in \widetilde{\mathcal{V}}(G_1)}),$$

$$\mathcal{T}_2 \overset{\text{def}}{=} ((G_2 \curvearrowright M_2),\ \{(D_2 \curvearrowright M_{D_2})\}_{D_2 \in \widetilde{\mathcal{V}}(G_2)},\ \{\rho_{D_2} \colon M_2 \hookrightarrow M_{D_2}\}_{D_2 \in \widetilde{\mathcal{V}}(G_2)})$$

be *NF-Galois theaters*. Then we shall say that a collection of data

$$(\alpha \colon G_1 \hookrightarrow G_2,\ \tau \colon \widetilde{\mathcal{V}}(G_1) \xrightarrow{\sim} \widetilde{\mathcal{V}}(G_2),\ \beta \colon M_1 \xrightarrow{\sim} M_2,\ \{\beta_{D_1} \colon M_{D_1} \xrightarrow{\sim} M_{\tau(D_1)}\}_{D_1 \in \widetilde{\mathcal{V}}(G_1)})$$

consisting of an open injection $\alpha \colon G_1 \hookrightarrow G_2$ of profinite groups, a bijection $\tau \colon \widetilde{\mathcal{V}}(G_1) \xrightarrow{\sim} \widetilde{\mathcal{V}}(G_2)$, an isomorphism $\beta \colon M_1 \xrightarrow{\sim} M_2$ of monoids, and, for each $D_1 \in \widetilde{\mathcal{V}}(G_1)$, an isomorphism $\beta_{D_1} \colon M_{D_1} \xrightarrow{\sim} M_{\tau(D_1)}$ of topological monoids is a *morphism of NF-Galois theaters*

$$\mathcal{T}_1 \longrightarrow \mathcal{T}_2$$

if the following three conditions are satisfied:

(a) The isomorphism $\beta \colon M_1 \xrightarrow{\sim} M_2$ of monoids is *compatible* with the actions of G_1, G_2 relative to the open injection α.

(b) For each $D_1 \in \widetilde{\mathcal{V}}(G_1)$, it holds that $\tau(D_1) = C_{G_2}(\alpha(D_1))$, which thus implies [cf. [11], Theorem 12.1.9] that α restricts to an open injection $D_1 \hookrightarrow \tau(D_1)$.

(c) For each $D_1 \in \widetilde{\mathcal{V}}(G_1)$, the isomorphism $\beta_{D_1} \colon M_{D_1} \xrightarrow{\sim} M_{\tau(D_1)}$ of topological monoids is *compatible* with the actions of D_1, $\tau(D_1)$ relative to the open injection $D_1 \hookrightarrow \tau(D_1)$ of (b).

(iii) We shall write

$$\mathfrak{Th}^{\mathrm{NF}}$$

for the category of *NF-Galois theaters* and *morphisms of NF-Galois theaters* [cf. (i), (ii)].

(iv) We shall write

$$\mathbb{NF}$$

for the category whose objects are *profinite groups of NF-type* [cf. Definition 3.2], and whose morphisms are *open injections of profinite groups*. Thus, the assignment

$$((G \curvearrowright M),\ \{(D \curvearrowright M_D)\}_{D \in \widetilde{\mathcal{V}}(G)},\ \{\rho_D \colon M \hookrightarrow M_D\}_{D \in \widetilde{\mathcal{V}}(G)}) \ \mapsto\ G$$

determines a *functor*

$$\mathfrak{Th}^{\mathrm{NF}} \longrightarrow \mathbb{NF}.$$

(v) We shall write

$$\mathfrak{An}[\mathbb{NF}]$$

for the category defined as follows: An object of $\mathfrak{An}[\mathbb{NF}]$ is a collection of data of the form

$$\mathfrak{An}(G) \stackrel{\mathrm{def}}{=} ((G \curvearrowright \widetilde{\mathcal{O}}^{\triangleright}(G)), \{(D \curvearrowright \overline{\mathcal{O}}^{\triangleright}(D))\}_{D \in \widetilde{\mathcal{V}}(G)}, \{\widetilde{\mathcal{O}}^{\triangleright}(G) \hookrightarrow \widetilde{\mathcal{O}}^{\triangleright}(D)\}_{D \in \widetilde{\mathcal{V}}(G)})$$

— where $\widetilde{\mathcal{O}}^{\triangleright}(G) \hookrightarrow \overline{\mathcal{O}}^{\triangleright}(D)$ [cf. Theorem 1.4, (9); Theorem 5.11] is the inclusion determined by the right-hand vertical arrow of the diagram of Theorem 5.11, (i), together with the natural inclusion $\overline{\mathcal{O}}^{\triangleright}(D) \hookrightarrow \overline{k}(D)$ [cf. Proposition 5.8, (3)] — for some object G of $\mathbb{NF}$. The morphisms of $\mathfrak{An}[\mathbb{NF}]$ are the morphisms induced by morphisms of $\mathbb{NF}$. Thus, the assignment

$$G \mapsto \mathfrak{An}(G)$$

determines a *functor*

$$\mathbb{NF} \longrightarrow \mathfrak{An}[\mathbb{NF}].$$

Remark 6.6.1. In the notation of Definition 6.6, (i), if we write $\mathfrak{hol}_D$ for the *NF-holomorphic structure* on D [cf. Definition 6.1] determined by the natural inclusion $D \hookrightarrow G$, then it follows immediately from Theorem 5.11, (i); Theorem 5.12, together with the definition of the field $\overline{k}(D \curvearrowright M_D, \mathfrak{hol}_D)$ given in Definition 6.4, (v), that the morphism of modules induced by ρ_D

$$M^{\mathrm{gp}} \longrightarrow M_D^{\mathrm{gp}}$$

extends to an *inclusion of fields*

$$\widetilde{F}(G \curvearrowright M) \hookrightarrow \overline{k}(D \curvearrowright M_D, \mathfrak{hol}_D)$$

[cf. Theorem 4.4, (5); Remark 4.4.1; Remark 6.4.1].

Remark 6.6.2. In the notation of Definition 6.6, (ii), it follows immediately from [11], Theorem 12.1.9, that the bijection τ is *completely determined* by the open injection α as follows: The element $\tau(D_1)$ is the *unique* element of $\widetilde{\mathcal{V}}(G_2)$ that contains $\alpha(D_1)$.

Proposition 6.7. *The three functors*

$$\mathbb{NF} \longrightarrow \mathfrak{An}[\mathbb{NF}] \longrightarrow \mathfrak{Th}^{\mathrm{NF}} \longrightarrow \mathbb{NF}$$

— *where the first arrow is the functor of Definition* 6.6, (v), *the second arrow is the functor obtained by forgetting the way in which the object* $\mathfrak{An}(G)$ *arose from* G, *and the third arrow is the functor of Definition* 6.6, (iv) — *are* **equivalences of categories**. *Moreover, the composite* $\mathbb{NF} \to \mathbb{NF}$ *of these three functors is naturally isomorphic to the* **identity** *functor.*

PROOF. This follows immediately from Theorem 5.12; Remark 6.6.2; [9], Proposition 3.2, (iv), together with the various definitions involved. □

Definition 6.8. Let

$$\mathcal{T} \overset{\text{def}}{=} ((G \curvearrowright M),\ \{(D \curvearrowright M_D)\}_{D \in \widetilde{\mathcal{V}}(G)},\ \{\rho_D \colon M \hookrightarrow M_D\}_{D \in \widetilde{\mathcal{V}}(G)})$$

be an NF-Galois theater and $D_0 \in \widetilde{\mathcal{V}}(G)$ [cf. Proposition 3.5, (1)]. Write

$$(G \curvearrowright {}^{\dagger}M,\ \{(D \curvearrowright {}^{\dagger}M_D)\}_{D \in \widetilde{\mathcal{V}}(G)},\ \{{}^{\dagger}\rho_D \colon {}^{\dagger}M \hookrightarrow {}^{\dagger}M_D\}_{D \in \widetilde{\mathcal{V}}(G)})$$

for the NF-Galois theater obtained by forming the image of the profinite group G of NF-type by the composite of the first two functors of the display of Proposition 6.7 and $\mathfrak{hol}_{D_0}$ for the NF-holomorphic structure on $(D_0 \curvearrowright M_{D_0})$ determined by the natural inclusion $D_0 \hookrightarrow G$. Thus, by Definition 6.5, (iii), we have an MLF-Galois $\mathbb{TM}$-pair of mono-analytic type

$$(D_0 \curvearrowright \mathfrak{log}(D_0 \curvearrowright M_{D_0}, \mathfrak{hol}_{D_0})).$$

Moreover, one verifies easily that the second Kummer map of the display of [9], Proposition 3.2, (ii) [cf. also the displayed isomorphism of [9], Remark 3.2.1] — applied to the MLF-Galois $\mathbb{TM}$-pairs of mono-analytic type

$$(D_0 \curvearrowright \mathfrak{log}(D_0 \curvearrowright M_{D_0}, \mathfrak{hol}_{D_0})) \quad \text{and} \quad (D_0 \curvearrowright {}^{\dagger}M_{D_0})$$

— determines a D_0*-equivariant isomorphism*

$${}^{\dagger}\iota_{D_0} \colon\ \mathfrak{log}(D_0 \curvearrowright M_{D_0}, \mathfrak{hol}_{D_0}) \ \overset{\sim}{\longrightarrow}\ {}^{\dagger}M_{D_0}.$$

Now one verifies immediately from the various definitions involved that the collection of data

$$\mathfrak{log}(\mathcal{T}) \overset{\text{def}}{=} (G \curvearrowright {}^{\dagger}M,\ \{(D \curvearrowright \mathfrak{log}(D \curvearrowright M_D, \mathfrak{hol}_D))\}_{D \in \widetilde{\mathcal{V}}(G)},\ \{{}^{\dagger}\iota_D^{-1} \circ {}^{\dagger}\rho_D\}_{D \in \widetilde{\mathcal{V}}(G)})$$

forms an *NF-Galois theater*. Thus, we obtain a functor

$$\mathfrak{log} \colon\ \mathfrak{Th}^{\mathrm{NF}} \longrightarrow \mathfrak{Th}^{\mathrm{NF}}.$$

We shall refer to this functor $\mathfrak{log}$ as the *NF-log-Frobenius functor*.

Remark 6.8.1. One verifies immediately that the *NF-log-Frobenius functor* of Definition 6.8 is *naturally isomorphic to the identity functor*, hence, in particular, an *equivalence of categories*.

Definition 6.9. Let G be a profinite group *of NF-type* [cf. Definition 3.2], i.e., an object of the category $\mathbb{NF}$, and p a prime number.

(i) We shall write

$$\mathbb{NF}[G] \subseteq \mathbb{NF}$$

for the *full subcategory* of $\mathbb{NF}$ consisting of profinite groups which are isomorphic to G. [Thus, it follows from Theorem 3.3 that every morphism in this full subcategory is an *isomorphism*.] This full subcategory determines, in an evident fashion [cf. also the equivalences of categories of Proposition 6.7], *full subcategories*

$$\mathfrak{Th}^{\mathrm{NF}}[G] \subseteq \mathfrak{Th}^{\mathrm{NF}}, \quad \mathfrak{An}[\mathbb{NF}[G]] \subseteq \mathfrak{An}[\mathbb{NF}].$$

Moreover, one verifies immediately that the *NF-log-Frobenius functor* $\mathfrak{log}\colon \mathfrak{Th}^{\mathrm{NF}} \to \mathfrak{Th}^{\mathrm{NF}}$ determines a *functor*

$$\mathfrak{Th}^{\mathrm{NF}}[G] \longrightarrow \mathfrak{Th}^{\mathrm{NF}}[G].$$

By abuse of notation, we shall denote this functor by $\mathfrak{log}$.

(ii) We shall write

$$\mathcal{N}_p \quad (\text{respectively, } \mathcal{N}_p^{\boxplus})$$

for the category defined as follows: An object of the category $\mathcal{N}_p$ (respectively, $\mathcal{N}_p^{\boxplus}$) is a collection of data

$$((H \curvearrowright M),\ \{(D \curvearrowright M_D)\}_{D\in\widetilde{\mathcal{V}}(H)},\ \{\rho_D\colon M \hookrightarrow M_D\}_{D\in\widetilde{\mathcal{V}}(H)},\ \{(D \curvearrowright N_D)\}_{D\in\widetilde{\mathcal{V}}(H),\, p(D)=p})$$

— where

$$((H \curvearrowright M),\ \{(D \curvearrowright M_D)\}_{D\in\widetilde{\mathcal{V}}(H)},\ \{\rho_D\colon M \hookrightarrow M_D\}_{D\in\widetilde{\mathcal{V}}(H)})$$

is an object of $\mathfrak{Th}^{\mathrm{NF}}[G]$ [i.e., a certain NF-Galois theater], and, for each $D \in \widetilde{\mathcal{V}}(H)$ [cf. Proposition 3.5, (1)] such that $p(D) = p$ [cf. Theorem 1.4, (1)],

$$(D \curvearrowright N_D)$$

is an MLF-Galois $\mathbb{TS}$-pair (respectively, $\mathbb{TS}\boxplus$-pair) [necessarily of mono-analytic type] [cf. [9], Definition 3.1, (ii)]. A morphism in the category $\mathcal{N}_p$ (respectively, $\mathcal{N}_p^{\boxplus}$) is a pair consisting of a morphism of NF-Galois theaters and a compatible [in the evident sense] morphism of MLF-Galois $\mathbb{TS}$-pairs (respectively, $\mathbb{TS}\boxplus$-pairs) [cf. [9], Definition 3.1, (ii)]. Thus, we have *natural functors*

$$\mathcal{N}_p^{\boxplus} \longrightarrow \mathcal{N}_p \longrightarrow \mathbb{NF}[G].$$

(iii) Let ν be a vertex of the oriented graph "$\vec{\Gamma}^{\mathfrak{log}}_{\mathrm{non}}$" of [9], Definition 5.4, (iii). Then, by a similar procedure to the procedure applied in [9], Definition 5.4, (iv), to define the functor "$\lambda^{\boxplus}_{v,\nu}$", one may define a *functor*

$$\lambda^{\boxplus}_{p,\nu} \colon \mathfrak{Th}^{\mathrm{NF}}[G] \longrightarrow \mathcal{N}_p^{\boxplus}.$$

(iv) Let ϵ be an edge of the oriented graph "$\vec{\Gamma}^{\ltimes}_{\mathrm{non}}$" of [9], Definition 5.4, (iii), running from a vertex ν_1 to a vertex ν_2. Then, by a similar procedure to the procedure applied in [9], Definition 5.4, (vii), to define the natural transformation "$\iota^{\boxplus}_{v,\epsilon}$", one may define a *natural transformation*

$$\iota^{\boxplus}_{p,\,\epsilon} \colon \lambda^{\boxplus}_{p,\,\nu_1} \circ \Lambda_{\nu_1} \longrightarrow \lambda^{\boxplus}_{p,\,\nu_2}$$

— where, for each *pre-log* (respectively, *post-log*) vertex ν [cf. [9], Definition 5.4, (iii)] of the oriented graph "$\vec{\Gamma}^{\mathfrak{log}}_{\mathrm{non}}$" of [9], Definition 5.4, (iii), we take Λ_ν to be the *identity functor* on $\mathfrak{Th}^{\mathrm{NF}}[G]$ (respectively, *NF-log-Frobenius functor* $\mathfrak{log} \colon \mathfrak{Th}^{\mathrm{NF}}[G] \to \mathfrak{Th}^{\mathrm{NF}}[G]$ — cf. (i)).

Finally, we prove the following *global mono-anabelian log-Frobenius compatibility*:

Theorem 6.10. *Let G be a profinite group* **of NF-type** [*cf. Definition* 3.2]*.*

Consider the diagram of categories $\mathcal{D}$ [*cf.* [9], *Definition* 3.5, (i)]

$$
\begin{array}{ccccccccc}
\dots \xrightarrow{\mathfrak{log}} & \mathcal{X} & & \xrightarrow{\mathfrak{log}} & \mathcal{X} & \xrightarrow{\mathfrak{log}} & & \mathcal{X} & \xrightarrow{\mathfrak{log}} \dots \\
\dots & & \mathrm{id}_{i-1} \searrow & & \downarrow \mathrm{id}_i & & \swarrow \mathrm{id}_{i+1} & & \dots \\
 & & & & \mathcal{X} & & & & \\
\dots & \lambda^{\boxplus}_{p^\dagger} \swarrow \dots \swarrow & & & \downarrow \dots \downarrow \lambda^{\boxplus}_{p} & & \searrow \dots \searrow \lambda^{\boxplus}_{p^\ddagger} & & \dots \\
\dots & \mathcal{N}^{\boxplus}_{p^\dagger} & & & \mathcal{N}^{\boxplus}_{p} & & & \mathcal{N}^{\boxplus}_{p^\ddagger} & \dots \\
\dots & \downarrow & & & \downarrow & & & \downarrow & \dots \\
\dots & \mathcal{N}_{p^\dagger} & & & \mathcal{N}_{p} & & & \mathcal{N}_{p^\ddagger} & \dots \\
\dots & & \searrow & & \downarrow & & \swarrow & & \dots \\
 & & & & \mathcal{E} & & & & \\
 & & & & \downarrow \kappa & & & & \\
 & & & & \mathfrak{An} & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \mathcal{E} & & & &
\end{array}
$$

— *where we write*

$$\mathcal{X} \stackrel{\mathrm{def}}{=} \mathfrak{Th}^{\mathrm{NF}}[G], \quad \mathcal{E} \stackrel{\mathrm{def}}{=} \mathbb{NF}[G], \quad \mathfrak{An} \stackrel{\mathrm{def}}{=} \mathfrak{An}[\mathbb{NF}[G]]$$

[*cf. Definition* 6.9, (i)]*; we write "*$\mathcal{N}^{\boxplus}_p$*", "*$\mathcal{N}_p$*" for the categories defined in Definition* 6.9, (ii)*; we think of the vertices of the first row of* $\mathcal{D}$ *as being indexed by the elements of* $\mathbb{Z}$*; we write* $\mathbb{Z}^{(\infty)} \stackrel{\mathrm{def}}{=} \mathbb{Z} \cup \{\infty\}$ *for the ordered set obtained by appending to* $\mathbb{Z}$ *a formal symbol "*∞*" — which we think of as corresponding to the unique vertex of the second row of* $\mathcal{D}$ *— such that* $i < \infty$ *for all* $i \in \mathbb{Z}$*; we write* id_i *for the identity functor at the vertex* $i \in \mathbb{Z}$*; for an element* $n \in \{1, \dots, 7\}$*, we write* $\mathcal{D}_{\leq n}$ *for the subdiagram of categories* [*cf.* [9], *Definition* 3.5, (i)] *of* $\mathcal{D}$ *determined by the first* n [*of the seven*] *rows of* $\mathcal{D}$*; the vertices of the third and fourth rows of* $\mathcal{D}$ *are indexed by the prime numbers*

p, $p^\dagger$, $p^\ddagger$ ...; the arrows from the second row to the category $\mathcal{N}_p^{\boxplus}$ in the third row are given by the collection of functors $\lambda_p^{\boxplus} \stackrel{\mathrm{def}}{=} \{\lambda_{p,\nu}^{\boxplus}\}_\nu$ of Definition 6.9, (iii)*, where ν ranges over the* **pre-log** *vertices of the oriented graph "$\vec{\Gamma}^{\mathfrak{log}}_{\mathrm{non}}$" of* [9]*, Definition* 5.4, (iii) [*or, alternatively, over* **all** *the vertices of the oriented graph "$\vec{\Gamma}^{\mathfrak{log}}_{\mathrm{non}}$" of* [9]*, Definition* 5.4, (iii)*, subject to the proviso that we identify the functors associated to the* **space-link** *and* **post-log** *vertices*]*; the arrows from the third to fourth and from the fourth to fifth rows are the natural functors $\mathcal{N}_p^{\boxplus} \to \mathcal{N}_p \to \mathcal{E}$ of Definition* 6.9, (ii)*; the arrows from the fifth to sixth and from the sixth to seventh rows are the* **natural equivalences of categories** *$\mathcal{E} \to \mathfrak{An} \to \mathcal{E}$ — the first of which we shall denote by κ — of Proposition* 6.7 *restricted to "$[G]$"; we shall apply the notation "$[-]$" to the names of arrows appearing in $\mathcal{D}$ to denote the* **path** [*cf. the discussion entitled "Combinatorics" in* [9], §0] *of length* 1 *associated to the arrow. Also, let us write*

$$\phi\colon\ \mathfrak{An} \longrightarrow \mathcal{X}$$

*for the equivalence of categories given by the "***forgetful functor***" of Proposition* 6.7 *restricted to "$[G]$";*

$$\pi\colon\ \mathcal{X} \longrightarrow \mathcal{E} \stackrel{\kappa}{\longrightarrow} \mathfrak{An}$$

for the quasi-inverse for ϕ given by the composite of the natural projection functor $\mathcal{X} \to \mathcal{E}$ with κ;

$$\eta\colon\ \phi \circ \pi \stackrel{\sim}{\longrightarrow} \mathrm{id}_{\mathcal{X}}$$

for the isomorphism that exhibits ϕ, π as quasi-inverses to one another. Then the following hold:

(i) *For $n \in \{5,6,7\}$, $\mathcal{D}_{\leq n}$ admits a natural structure of* **core** [*cf.* [9]*, Definition* 3.5, (iii)] *on $\mathcal{D}_{\leq n-1}$. That is to say, loosely speaking, $\mathcal{E}$, $\mathfrak{An}$ "form cores" of the functors in $\mathcal{D}$.*

(ii) *The "***forgetful functor***" ϕ gives rise to a* **telecore structure** $\mathfrak{T}$ [*cf.* [9]*, Definition* 3.5, (iv)] *on $\mathcal{D}_{\leq 5}$ — whose underlying diagram of categories we denote by $\mathcal{D}_{\mathfrak{T}}$ — by appending to $\mathcal{D}_{\leq 6}$* **telecore edges** [*cf.* [9]*, Definition* 3.5, (iv), (a)]

$$\begin{array}{ccccccc} & & & \mathfrak{An} & & & \\ \dots & & \phi_{i-1} \swarrow & \downarrow \phi_i & \searrow \phi_{i+1} & & \dots \\ \dots \stackrel{\mathfrak{log}}{\longrightarrow} & \mathcal{X} & \stackrel{\mathfrak{log}}{\longrightarrow} & \mathcal{X} & \stackrel{\mathfrak{log}}{\longrightarrow} & \mathcal{X} & \stackrel{\mathfrak{log}}{\longrightarrow} \dots \end{array}$$

$$\mathfrak{An} \stackrel{\phi_\infty}{\longrightarrow} \mathcal{X}$$

from the **core** $\mathfrak{An}$ [*cf.* (i)] *to the various copies of* $\mathcal{X}$ *in* $\mathcal{D}_{\leq 2}$ *given by copies of* ϕ *— which we denote by* ϕ_i *— for* $i \in \mathbb{Z}^{(\infty)}$*. That is to say, loosely speaking,* ϕ *determines a telecore structure on* $\mathcal{D}_{\leq 5}$*. Finally, for each* $i \in \mathbb{Z}^{(\infty)}$*, let us write* $[\beta_i^0]$ *for the path on* $\vec{\Gamma}_{\mathcal{D}_{\mathfrak{T}}}$ *of length* 0 *at* i *and* $[\beta_i^1]$ *for* **some** [*cf. the* **coricity** *of* (i)] *path on* $\vec{\Gamma}_{\mathcal{D}_{\mathfrak{T}}}$ *of length* $\in \{5, 6\}$ [*i.e., depending on whether or not* $i = \infty$] *that starts from* i*, descends via some path of length* $\in \{4, 5\}$ *to the core vertex "*$\mathfrak{An}$*"* [*cf.* (i)], *and returns to* i *via the telecore edge* ϕ_i*. Then the collection of natural transformations*

$$\{\eta_{\infty,\, i},\ \eta_{\infty,\, i}^{-1},\ \eta_j,\ \eta_j^{-1}\}_{i \in \mathbb{Z},\, j \in \mathbb{Z}^{(\infty)}}$$

— where we write

$$\eta_{\infty,\, i}\colon\ \phi_{\infty} \ \stackrel{\sim}{\longrightarrow}\ \mathrm{id}_i \circ \phi_i$$

for the identity natural transformation and

$$\eta_j\colon\ (\mathcal{D}_{\mathfrak{T}})_{[\beta_j^1]} \ \stackrel{\sim}{\longrightarrow}\ (\mathcal{D}_{\mathfrak{T}})_{[\beta_j^0]}$$

[*cf.* [9], *Definition* 3.5, (i)] *for the isomorphism arising from* η *— generate a* **contact structure** $\mathcal{H}$ [*cf.* [9], *Definition* 3.5, (iv)] *on the telecore* $\mathfrak{T}$*.*

(iii) *The natural transformations*

$$\iota^{\boxplus}_{p,\,\epsilon}\colon\ \lambda^{\boxplus}_{p,\,\nu_1} \circ \Lambda_{\nu_1}\ \longrightarrow \lambda^{\boxplus}_{p,\,\nu_2}$$

[*cf. Definition* 6.9, (iv)] *— where* p *is a prime number;* ϵ *is an edge of the oriented graph "*$\vec{\Gamma}^{\ltimes}_{\mathrm{non}}$*" of* [9], *Definition* 5.4, (iii), *running from a vertex* ν_1 *to a vertex* ν_2*; if* ν_1 *is a* **pre-log** *vertex, then we interpret the domain and codomain of* $\iota^{\boxplus}_{p,\,\epsilon}$ *as the arrows associated to the paths of length* 1 *from the second to third rows of* $\mathcal{D}$ *determined by* p *and* ν_1, ν_2*; if* ν_1 *is a* **post-log** *vertex, then we interpret the domain of* $\iota^{\boxplus}_{p,\,\epsilon}$ *as the arrow associated to the path of length* 3 *from the first to the third rows of* $\mathcal{D}$ *determined by* p*,* ν_1*, and the condition that the initial length* 2 *portion of the path be a path of the form* $[\mathrm{id}_i] \circ [\mathfrak{log}]$ [*for* $i \in \mathbb{Z}$], *and we interpret the codomain of* $\iota^{\boxplus}_{p,\,\epsilon}$ *as the arrow associated to the path of length* 2 *from the first to the third rows of* $\mathcal{D}$ *determined by* p*,* ν_2*, and the condition that the initial length* 1 *portion of the path be a path of the form* $[\mathrm{id}_{i-1}]$ [*for the* **same** $i \in \mathbb{Z}$] *— belong to a* **family of homotopies** [*cf.* [9], *Definition* 3.5, (ii)] *on* $\mathcal{D}_{\leq 3}$ *that determines on the portion of* $\mathcal{D}_{\leq 3}$ *indexed by* p *a structure of* **observable** $\mathfrak{S}_{\mathfrak{log}}$ [*cf.* [9], *Definition* 3.5, (iii)] *on* $\mathcal{D}_{\leq 2}$*. Moreover, the family of homotopies that constitute* $\mathfrak{S}_{\mathfrak{log}}$ *is* **compatible** [*cf.* [9], *Definition* 3.5, (ii)] *with the families of homotopies that constitute the* **core** *and* **telecore** *structures of* (i), (ii)*.*

(iv) *The diagram of categories* $\mathcal{D}_{\leq 2}$ *does* **not admit** *a structure of* **core** *on* $\mathcal{D}_{\leq 1}$ *which* [*i.e., whose constituent family of homotopies*] *is* **compatible** *with* [*the constituent* *family of homotopies of*] *the* **observable** $\mathfrak{S}_{\mathfrak{log}}$ *of* (iii)*. Moreover, the* **telecore structure** $\mathfrak{T}$ *of* (ii)*, the* **contact structure** $\mathcal{H}$ *of* (ii)*, and the* **observable** $\mathfrak{S}_{\mathfrak{log}}$ *of* (iii) *are* **not simultaneously compatible***.*

(v) *The unique vertex* ∞ *of the second row of* $\mathcal{D}$ *is a* **nexus** [*cf. the discussion entitled "Combinatorics" in* [9]*,* §0] *of* $\vec{\Gamma}_{\mathcal{D}}$*. Moreover,* $\mathcal{D}$ *is* **totally** ∞**-rigid** [*cf. Proposition* 6.7*;* [6]*, Corollary* 2.2*;* [9]*, Definition* 3.5*,* (vi)]*, and the* **natural action** *of* $\mathbb{Z}$ *on the infinite linear oriented graph* $\vec{\Gamma}_{\mathcal{D}_{\leq 1}}$ **extends** *to an action of* $\mathbb{Z}$ *on* $\mathcal{D}$ *by* **nexus-classes of self-equivalences** *of* $\mathcal{D}$ [*cf.* [9]*, Definition* 3.5*,* (vi)]*. Finally, the self-equivalences in these nexus-classes are* **compatible** *with the* **families of homotopies** *that constitute the* **cores** *and* **observable** *of* (i)*,* (iii)*; these self-equivalences also extend naturally* [*cf. the technique of extension applied in* [9]*, Definition* 3.5*,* (vi)] *to the diagram of categories* [*cf.* [9]*, Definition* 3.5*,* (iv)*,* (a)] *that constitutes the* **telecore** *of* (ii)*, in a fashion that is* **compatible** *with both the* **family of homotopies** *that constitutes this telecore structure* [*cf.* [9]*, Definition* 3.5*,* (iv)*,* (b)] *and the contact structure* $\mathcal{H}$ *of* (ii)*.*

PROOF. This follows immediately from a similar argument to the argument applied in the proof of [9], Corollary 5.5. □

Remark 6.10.1. The "*general formal content*" of the remarks following [9], Corollaries 3.6, 3.7, applies to the situation discussed in Theorem 6.10, as well. We leave the routine details of translating these remarks into the language of the situation of Theorem 6.10 to the interested reader.

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References

[1] Y. Hoshi, Conditional results on the birational section conjecture over small number fields, *Automorphic Forms and Galois Representations*, vol. **2**, 187–230, London Math. Soc. Lecture Note Ser., **415**, Cambridge Univ. Press, Cambridge, 2014.

[2] Y. Hoshi, A note on the geometricity of open homomorphisms between the absolute Galois groups of p-adic local fields, *Kodai Math. J.* **36** (2013), no. **2**, 284–298.

[3] Y. Hoshi, On the field-theoreticity of homomorphisms between the multiplicative groups of number fields, *Publ. Res. Inst. Math. Sci.* **50** (2014), no. **2**, 269–285.

[4] M. Jarden and J. Ritter, On the characterization of local fields by their absolute Galois groups, *J. Number Theory* **11** (1979), no. **1**, 1–13.
[5] S. Mochizuki, A version of the Grothendieck conjecture for p-adic local fields, *Internat. J. Math.* **8** (1997), no. **4**, 499–506.
[6] S. Mochizuki, Global solvably closed anabelian geometry, *Math. J. Okayama Univ.* **48** (2006), 57–71.
[7] S. Mochizuki, Topics in absolute anabelian geometry I: Generalities, *J. Math. Sci. Univ. Tokyo* **19** (2012), no. **2**, 139–242.
[8] S. Mochizuki, Topics in absolute anabelian geometry II: Decomposition groups and endomorphisms, *J. Math. Sci. Univ. Tokyo* **20** (2013), no. **2**, 171–269.
[9] S. Mochizuki, Topics in absolute anabelian geometry III: Global reconstruction algorithms, *J. Math. Sci. Univ. Tokyo* **22** (2015), no. **4**, 939–1156.
[10] J. Neukirch, *Algebraic number theory.* Translated from the 1992 German original and with a note by Norbert Schappacher. With a foreword by G. Harder, *Grundlehren der Mathematischen Wissenschaften*, **322**. Springer-Verlag, Berlin, 1999.
[11] J. Neukirch, A. Schmidt, and K. Wingberg, *Cohomology of number fields.* Second edition, *Grundlehren der Mathematischen Wissenschaften*, **323**. Springer-Verlag, Berlin, 2008.
[12] N. Nikolov and D. Segal, Finite index subgroups in profinite groups, *C. R. Math. Acad. Sci. Paris* **337** (2003), no. **5**, 303–308.
[13] K. Uchida, Isomorphisms of Galois groups of solvably closed Galois extensions, *Tôhoku Math. J.* (**2**) **31** (1979), no. **3**, 359–362.