Causality of General Input-Output Systems and Extended Small-Gain Theorem for Their Feedback Connection

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Abstract

For the small-gain theorem derived by Zames in 1966, the later studies after a few decades elaborated on its derivation through defining system causality, which was not assumed by Zames. In connection with the treatment of causality, however, these studies made some unnecessary assumptions on the subsystems in feedback connection and failed to handle general systems described by an input-output relation rather than mapping (which we call input-intolerant/output-unsolitary systems). On the other hand, although the treatment by Zames can handle such subsystems, it instead turns out to lead to larger values for the induced norms of subsystems compared with the later treatment. This paper is concerned with developing an extended form of the small-gain theorem through the same induced norms as in the later studies while dealing with general input-output causal subsystems. Since causality of subsystems plays a key role in such development, our research direction strongly motivates us to study how causality should be defined for general input-output systems. Thus, much of the arguments in this paper is devoted to such a study, which provides us with profound and thorough understandings on causality of different restricted classes of general input-output systems. Mutual relationships among adequate causality definitions for different classes are also clarified, which should be important in its own right. After deriving an extended form of the small-gain theorem, an example

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illustrates the importance of dealing with such general subsystems, as well as usefulness of the extension.

Keywords: causality, small-gain theorem, input-intolerant systems, output-unsolitary systems, truncation-invariant sets

1. Introduction

The small-gain theorem, which was derived by Zames [1, 2] in 1966 for the first time, is known as one of the most important instruments for control theory. Roughly speaking, the discrete-time version of it states that if the product of the l_p induced norms of two subsystems is less than 1, then the closed-loop system consisting of the subsystems is l_p stable. Those later studies such as [3, 4, 5, 6, 7] dealing with the small gain theorem also elaborate on its derivation through defining system causality, although it was not assumed in [1, 2].

Relevant to this difference in the original study by Zames and those later studies is another difference in defining the induced norm of each subsystem. More precisely, Zames defined the induced norm through the treatment of the truncated input and output of each subsystem, even though the entire time horizon was eventually taken into account in the definition. This definition, however, turns out to possibly lead to a larger value for the induced norm than the treatment in the later studies, which are free from truncation at the stage of the induced norm definition. In connection with the treatment of causality, however, those later studies instead made some unnecessary assumptions on the subsystems in feedback connection in deriving the small-gain theorem.

This paper is thus motivated by our interest in reverting to the same assumption on the (causal) subsystems as that made by Zames, while exploiting improved evaluation of their induced norms in deriving an extended form of the small-gain theorem. More precisely, we aim at considering the following types of subsystems described by the input-output relation [3, 4, 5, 6, 7] (rather than the input-output mapping) on the product space of the input space and the output space:

(a) such (sub)systems that possibly fall into a situation in which two or more signals can be determined as the "corresponding output" under some input;

(b) such (sub)systems that possibly have some input leading to a situation in which no signal can be determined as the "corresponding output."

We say that the systems of type (a) are output-unsolitary, while those systems that are not output-unsolitary are said to be output-solitary[1]. For example, let us consider a static system in which a solution to a nonlinear equation associated with the current input together with some parameters is regarded as its possible current output. Then, such a system could naturally be output-unsolitary. Quite importantly, we will also give another example of output-unsolitary systems resulting from a feedback connection of output-solitary subsystems. Hence, if we consider a general case of complex systems where such a feedback connection of subsystems actually has a further interaction in a feedback form with a similar kind of feedback connection of other subsystems, then it is natural for us to study a closed-loop system consisting of output-unsolitary subsystems. Furthermore, it may be worth noting that if we take such a standpoint that picking out a state-space system does not determine its initial state and leaves it completely free, ambiguous or unknown, then the state-space system viewed as a general input-output system will naturally be output-unsolitary.

On the other hand, the implication of the systems of type (b) is that some input should possibly be ruled out in dealing with such systems so that we can always talk about the output. In this sense, such systems are said to be input-intolerant, while those systems that are not input-intolerant are said to be input-tolerant. As easily seen, the aforementioned example of nonlinear static systems could also be an example of input-intolerant systems. Furthermore, such systems in which some nonlinear functions such as the reciprocal and logarithm functions are applied to each entry of the current input are simple examples of input-intolerant systems. In connection with these terms, the class of subsystems dealt with in the aforementioned later studies is precisely the one consisting of those systems that are both input-tolerant and output-solitary (i.e., with an input-output mapping).

Accommodating input-intolerant/output-unsolitary input-output systems

[1] For convenience, however, we sometimes take a standpoint that output-solitary systems form a special subclass of output-unsolitary systems (and thus are output-unsolitary, too). We believe that this would not cause any serious ambiguity in the arguments. Similarly for the term of input-tolerant systems with respect to the term of input-intolerant systems introduced later.

is quite important in exploiting the full power of the small-gain type of arguments in the stability analysis of complicated systems. As further developments of stability analysis based on the small-gain theorem, techniques with multipliers have been studied for many decades [8, 9, 10, 11, 12, 13]. In particular, the study on integral quadratic constraints has been carried out intensively in recent years [14, 15, 16, 17]. Deriving the extended small-gain theorem for general input-output causal systems is expected to contribute in further enriching multipliers theory.

Since causality of subsystems plays a key role in the development of the small-gain theorem in the treatment of the aforementioned later studies on the small-gain theorem [3, 4, 5, 6, 7], our research direction strongly motivates us to study how causality should be defined for such general input-output systems. Thus, much of the arguments in this paper is devoted to such a study, as we will first see in Section 2. There, we begin with the well-known causality definition of input-tolerant output-solitary systems [3, 4, 5, 6, 7], from which a series of amendments of definitions will be motivated until we eventually arrive at an adequate definition applicable to general input-output systems and some further modified definitions are inspired. After clarifying some trivial mutual relationships among those definitions, Section 3 uses the relationships to continue further arguments on causality to provide profound and thorough understandings on adequate definitions of causality for different restricted classes of general input-output systems. On the basis of the definitions that eventually turn out to be applicable to the most general and wide class of input-output systems, Section 4 then establishes the extended small-gain theorem for closed-loop systems consisting of input-intolerant and/or output-unsolitary subsystems. In particular, we show that for a general input-intolerant output-unsolitary system, the definition by Zames can lead to an arbitrarily larger value for its induced norm compared with the treatment in this paper through the definition in the later studies [3, 4, 5, 6, 7]. This clearly demonstrates the advances in our arguments through the study on the causality of such general input-output systems. Lastly, we give an example illustrating usefulness of the extended small-gain theorem and, at the same time, importance of accommodating input-intolerant and/or output-unsolitary systems in stability analysis of complicated systems.

Throughout the paper, $\mathbf{N}$ denotes the set of positive integers while $\mathbf{N}_0$ denotes that of nonnegative positive integers.

2. Causality definitions and their trivial relationships

For the discrete-time signal x and $K \in \mathbf{N}_0$, we denote by $x_{[K]}$ the truncated signal of x defined as

$$x_{[K]k} = \begin{cases} x_k & (0 \le k \le K) \\ 0 & (k > K), \end{cases} \tag{1}$$

where $(.)_k$ denotes the value of the signal $(.)$ at time k. The studies such as [3, 4, 5, 6, 7] elaborate on the derivation of the small gain theorem by defining system causality through this truncation as follows, under the implicit standpoint that attention is paid only on input-tolerant and output-solitary systems (i.e., systems with an input-output mapping): the system G is causal if, for each pair of the input w and the corresponding output z and for every $K \in \mathbf{N}_0$, the output z' corresponding to the input $w_{[K]}$ satisfies $z'_{[K]} = z_{[K]}$.

In deriving an extended form of the small-gain theorem for more general input-output systems with input-intolerant as well as output-unsolitary subsystems, it is hence quite important to first consider generalizing the above causality definition to accommodate those types of systems. This section studies possible directions for such generalization. More precisely, we start with a rather naive idea for extending the definition and next state its inadequacy as a general definition. We then proceed to introducing more appropriate and modified definitions, leading eventually to five "definitions," where only some of them are actually adequate as the definition for the most general (i.e., input-intolerant and output-unsolitary) input-output systems. Importantly, however, some of the definitions are still adequate if one is confined to some classes consisting of, e.g., input-tolerant systems and/or output-solitary systems. The former part of this paper is interested in introducing such definitions for system causality and clarifying their mutual relationships. In particular, this section is devoted to motivating the serial introduction of such definitions. After summarizing their mutual relationships that are more or less trivial from the definition statements before closing this section, further discussions on deeper insight into their relationships will be carried out in the following section by restricting our attention to some particular classes of systems.

We begin with the notation and terms used in this paper. The extended l_p space [1, 2], the set of x such that $x_{[K]} \in l_p$ whenever $K \in \mathbf{N}_0$, is denoted by[2] l_{pe}. Consider the system $G = G : (l_{pe}, l_{qe})$, by which we mean that the input and output of G belong to l_{pe} and l_{qe}, respectively (where p, $q \in [1, \infty]$ are given independently). The input-output relation of G is denoted by R_G. In other words, the notation

$$(w, z) \in R_G \tag{2}$$

is used to mean that G admits the situation where the given $z \in l_{qe}$ may be the outcome when the given $w \in l_{pe}$ is applied to it. Then, we define

$$\mathcal{W}_{pe}(G) := \{w \in l_{pe} : \exists\, z \in l_{qe} \text{ s.t. } (w, z) \in R_G\} \tag{3}$$

and call it the admissible input set of G. To rule out meaningless situations, the condition $\mathcal{W}_{pe}(G) \neq \emptyset$ is assumed for every system G dealt with in this paper. Those studies elaborating on the small gain theorem such as [3, 4, 5, 6, 7] considered only the case where $\mathcal{W}_{pe}(G) = l_{pe}$, which in turn implies that only input-tolerant systems are dealt with and that the condition $w_{[K]} \in \mathcal{W}_{pe}(G)$ is satisfied whenever $w \in \mathcal{W}_{pe}(G)$ and $K \in \mathbf{N}_0$. When this condition is satisfied, we say that $\mathcal{W}_{pe}(G)$ is truncation-invariant.

As a naive idea for extending the aforementioned definition of causality for input-tolerant and output-solitary systems and accommodating input-intolerant and/or output-unsolitary systems, we begin with the following "definition," in which the truncated input $w_{[\cdot]}$ plays an important role.

Definition 1. *(weak $w_{[\cdot]}$-causality) The system $G : (l_{pe}, l_{qe})$ is weakly $w_{[\cdot]}$-causal if $w_{[K]} \in \mathcal{W}_{pe}(G)$ and the following condition are satisfied for each $w \in \mathcal{W}_{pe}(G)$ and every $K \in \mathbf{N}_0$.*

- *For each $z \in l_{qe}$ such that $(w, z) \in R_G$, there exists $z' \in l_{qe}$ such that $(w_{[K]}, z') \in R_G$ and $z'_{[K]} = z_{[K]}$.*

However, Definition 1 turns out to be inadequate for output-unsolitary systems (even though it will indeed turn out to be an adequate definition for some class of output-solitary systems). To see this, we provide the following example.

[2] As a matter of fact, the extended space l_{pe} does not depend on p in spite of its notation. However, it should be meaningful to keep the underlying p highlighted because recklessly dropping the symbol p could rather cause confusion; we must eventually return to the treatment of l_p whenever we are dealing with l_{pe}, and this is why we utilize the notation l_{pe} rather than "l_{e}" throughout this paper.

Example 1. Let δ denote the impulse signal $(1, 0, 0, \cdots)$. Consider the system $G : (l_{2e}, l_{2e})$ whose input-output relation is described by $(\delta, 2\delta) \in R_G$ and $(w, w) \in R_G$ for every $w \in l_{2e}$ (including $w = \delta$). Clearly, G is output-unsolitary and input-tolerant (since $\mathcal{W}_{2e}(G) = l_{2e}$). Hence, $\mathcal{W}_{2e}(G)$ is obviously truncation-invariant and thus $w_{[K]} \in \mathcal{W}_{2e}(G)$ regardless of $K \in \mathbf{N}_0$ for each $w \in \mathcal{W}_{2e}(G)$. Furthermore, it is not hard to confirm that G satisfies the remaining condition in Definition 1. Hence, the system G here must be judged to be causal in the sense of Definition 1. Nevertheless, we next argue that regarding this system to be causal would be inadequate, whose implication is that Definition 1 should not be employed for general input-output systems (even though it will turn out, for output-solitary systems, to be equivalent to an adequate definition in the following arguments). To see this, let us consider $(w, z) := (\delta, 2\delta) \in R_G$. Then, one can readily conclude that the specific value at $k = 0$ for this z (i.e., $z_0 = 2$) is indeed consistent with the underlying input-output relation of the system G only through the knowledge that this w satisfies $w_k = 0$ for $k > 0$, where the knowledge is impossible to obtain at time $k = 0$.

Since Example 1 shows that Definition 1 (adequate for some class of output-solitary systems) is inadequate for output-unsolitary systems, we introduce the following definition by amending Definition 1.

Definition 2. *($w_{[\cdot]}$-causality) The system G is $w_{[\cdot]}$-causal if $w_{[K]} \in \mathcal{W}_{pe}(G)$ and the following two conditions are satisfied for each $w \in \mathcal{W}_{pe}(G)$ and every $K \in \mathbf{N}_0$.*

(i) *For each $z \in l_{qe}$ such that $(w, z) \in R_G$, there exists $z' \in l_{qe}$ such that $(w_{[K]}, z') \in R_G$ and $z'_{[K]} = z_{[K]}$.*
(ii) *For each $z' \in l_{qe}$ such that $(w_{[K]}, z') \in R_G$, there exists $z'' \in l_{qe}$ such that $(w, z'') \in R_G$ and $z''_{[K]} = z'_{[K]}$.*

Note that Definition 2 (as well as Definition 1) obviously leads to requiring that $\mathcal{W}_{pe}(G)$ is truncation-invariant. In this sense, and as we also discuss shortly, these definitions are not yet fully satisfactory for our ultimate purpose. However, our arguments starting from these definitions are believed to be quite meaningful in the overall picture on our entire arguments. This is because removing the assumption on truncation-invariance turns out to be very closely related with our outperforming the small-gain theorem by

Zames through the causality treatment (see Remark 6 in Section 4). Definition 2 is stronger than Definition 1 because of the additional condition (ii) (the condition (i) in Definition 2 is the same as that in Definition 1). The additional condition plays an important role when there exists $z' \in l_{qe}$ satisfying $(w_{[K]}, z') \in R_G$ as in (i) but $z'_{[K]} \neq z_{[K]}$ unlike in (i). Even though Definition 1 pays no attention to possible existence of such z', the condition (ii) in Definition 2 requires that such z' may exist only under some relevance to another possible output z'' of G to the same untruncated input w. Hence, Definition 2 naturally becomes equivalent to Definition 1 when G is output-solitary.

However, Definition 2 would still turn out to be unsatisfactory for input-intolerant systems. To see this, we further provide the following example.

Example 2. Consider the system $G : (l_{2e}, l_{2e})$ whose input-output relation is given by

$$z_k = \begin{cases} 0 & (w_i = 0 \ (i = 0, \ldots, k)) \\ w_k^{-1} & (\text{otherwise}). \end{cases} \tag{4}$$

This G is input-intolerant since the admissible input set $\mathcal{W}_{2e}(G)$ consists of signals whose value is nonzero except for its leading part of arbitrary time length consisting of zeros. It is thus easy to see that this G does not satisfy the conditions of Definition 2, because for $w = (1, 1, \cdots) \in \mathcal{W}_{2e}(G)$ and $K = 0$, we have $w_{[0]} = (1, 0, \ldots) \notin \mathcal{W}_{2e}(G)$ (more essentially because $\mathcal{W}_{2e}(G)$ is not truncation-invariant). Nevertheless, it is questionable whether we should actually regard this G to be noncausal because its output at time k can simply be determined by the value of the input at the same instant without referring to the future values of the input (as long as the input is assumed to belong to the admissible set, which would be a natural assumption because considering a nonadmissible input would usually be nonsense.

The above example suggests that an additional modification is needed for defining causality. Hence, for $w \in \mathcal{W}_{pe}(G)$ and $K \in \mathbf{N}_0$, let us define

$$\mathcal{W}_{pe}(w, K) := \{w' \in \mathcal{W}_{pe}(G) : w'_{[K]} = w_{[K]}\} \subset \mathcal{W}_{pe}(G). \tag{5}$$

Using this set consisting of signals that are identical with w up to time K, we introduce the following definition by further amending Definition 2.

Definition 3. *($\mathcal{W}_{pe}(w,\cdot)$-causality) The system G is $\mathcal{W}_{pe}(w,\cdot)$-causal if the following two conditions are satisfied for each $w \in \mathcal{W}_{pe}(G)$, every $K \in \mathbf{N}_0$ and every $w' \in \mathcal{W}_{pe}(w,K)$.*

(i) *For each $z \in l_{qe}$ such that $(w,z) \in R_G$, there exists $z' \in l_{qe}$ such that $(w',z') \in R_G$ and $z'_{[K]} = z_{[K]}$.*

(ii) *For each $z' \in l_{qe}$ such that $(w',z') \in R_G$, there exists $z'' \in l_{qe}$ such that $(w,z'') \in R_G$ and $z''_{[K]} = z'_{[K]}$.*

The importance of introducing the above definition lies in the point that the problematic issue of referring to the truncation $w_{[K]}$ in Definition 2 has been avoided with only a modest and natural amendment (most of the structure of the definition is preserved compared with the preceding definition). Thus, unlike Definition 2, $\mathcal{W}_{pe}(G)$ need not be truncation-invariant under the requirement of Definition 3. In this sense, Definition 2 may seem "stronger" than Definition 3. On the other hand, even though only a single input $w_{[K]}$ (if K is fixed) is involved in the conditions in Definition 2, those in Definition 3 involve all $w' \in \mathcal{W}_{pe}(w,K)$. It is not always the case that $w_{[K]} \in \mathcal{W}_{pe}(w,K)$ but it would often be the case that $\mathcal{W}_{pe}(w,K)$ is an infinite set. In this sense, Definition 3 may seem "stronger" than Definition 2. The mutual relationship of these definitions will be made clear in the following section.

One of the assertions of this paper is that Definition 3 should be an adequate definition of causality for general output-unsolitary systems as well as input-intolerant systems. This is partly supported by the following further observations about Examples 1 and 2.

Example 1 (continued). Recall that the system G is output-unsolitary and is a system that should not be regarded as being causal. In accordance with this situation and as desired, it turns out to fail to satisfy the amended conditions in Definition 3. To see this, consider $w = \delta$ and $z = 2\delta$ satisfying $(w,z) \in R_G$. Then, for $K = 0$ and $w' = (1,1,0,\cdots) \in \mathcal{W}_{2e}(w,K)$, we have $z' = (1,1,0,\cdots)$. This obviously implies that $z'_{[0]} \neq z_{[0]}$ and thus G fails to satisfy the condition (i) in Definition 3 (as well as that in Definition 2).

Example 2 (continued). Recall that the system G is input-intolerant and is a system that would better be regarded as being causal. To see that G successfully satisfies the amended conditions in Definition 3, we consider the following two cases for each $K \in \mathbf{N}_0$.

1) Take $w = 0$ and $w' \in \mathcal{W}_{2e}(w, K)$ $(= \mathcal{W}_{2e}(0, K))$. Then, $(w, z) \in R_G$ leads to $z = 0$. On the other hand, it is easy to see that $(w', z') \in R_G$ $(z' \in l_{2e})$ implies that $z'_{[K]} = w'_{[K]} = 0 = z_{[K]}$.
2) Take $w \in \mathcal{W}_{2e}(G) \setminus \{0\}$ such that $w_k = 0$ $(k < i)$ for $i \in \mathbf{N}_0$ and $w_k \neq 0$ $(k \geq i)$ and also take $w' \in \mathcal{W}_{2e}(w, K)$. For this w, take w^- such that $w_k^- = 0$ $(k < i)$ and $w_k^- = w_k^{-1}$ $(k \geq i)$. Then, $z \in l_{2e}$ such that $(w, z) \in R_G$ equals to w^-. Similarly, w'^- is also well-defined for the signal $w' \in \mathcal{W}_{2e}(w, K)$ and the corresponding signal z' such that $(w', z') \in R_G$ equals to w'^-. For the above z, we have $z'_{[K]} = w'^-_{[K]} = w^-_{[K]} = z_{[K]}$ regardless of whether $i \geq K$ or $i \leq K$.

The above observations immediately imply that G satisfies the condition (i) in Definition 3 (as well as that in Definition 2). It also satisfies the condition (ii) since it is output-solitary (recall the arguments below Definition 3), and thus G is successfully regarded as being causal in the sense of Definition 3.

To reach more profound and entire understandings on how causality should and may be defined, let us further consider the following definition obtained by removing the condition (ii) from Definition 3.

Definition 4. *(weak $\mathcal{W}_{pe}(w, \cdot)$-causality) The system G is weakly $\mathcal{W}_{pe}(w, \cdot)$-causal if the following condition is satisfied for each $w \in \mathcal{W}_{pe}(G)$, every $K \in \mathbf{N}_0$ and every $w' \in \mathcal{W}_{pe}(w, K)$.*

- *For each $z \in l_{qe}$ such that $(w, z) \in R_G$, there exists $z' \in l_{qe}$ such that $(w', z') \in R_G$ and $z'_{[K]} = z_{[K]}$.*

One might argue that Definition 4 is readily acceptable as a quite natural definition of causality for general input-output systems and all the preceding arguments are just redundant. Indeed, it turns out to correspond to the definition of nonanticipation provided in [18], if we interpret our input-output system in the context of the behavioral framework. Nevertheless, what this paper precisely aims at deeply investigating is clarifying the mutual relationships among possible definitions and their adequacy/inadequacy for general and/or some restricted classes of input-output systems. Indeed, Definition 4 is actually revealed to be equivalent to Definition 3. Roughly speaking, this can be explained as follows: since w and w' in Definition 4 are taken arbitrarily (within some associated restrictions), they could somehow be mutually interchanged, and doing so in the condition therein equivalently leads to the same condition as (ii) in Definition 3. This leads to equivalence between Definitions 3 and 4, and the following section indeed shows this rigorously as Theorem 3. This equivalence of Definition 3 to Definition 4 will further support the assertion of this paper that the former is also one possible definition of causality for general input-output systems.

For the sake of further arguments, we now consider comparing Definitions 1 and 4. Unlike Definition 1, Definition 4 does not lead to requiring that $\mathcal{W}_{pe}(G)$ is truncation-invariant. In this sense, Definition 1 may seem "stronger" than Definition 4. On the other hand, while a single input $w_{[K]}$ (if w and K are fixed) is involved in the conditions in Definition 1, the conditions in Definition 4 involve all $w' \in \mathcal{W}_{pe}(w, K)$. In this sense, Definition 4 may seem "stronger" than Definition 1. Even though their mutual relationship is not clear at this stage, it is meaningful to introduce the following definition obtained by (equivalently) reimposing the condition that $\mathcal{W}_{pe}(G)$ is truncation-invariant on top of the requirement in Definition 4. Introducing this definition will be helpful in reaching entire understandings on the mutual relationships among all the definitions in this section and their adequacy/inadequacy for general and/or some restricted classes of input-output systems.

Definition 5. *(truncation-invariant $\mathcal{W}_{pe}(w, \cdot)$-causality) The system G is truncation-invariant $\mathcal{W}_{pe}(w, \cdot)$-causal if $w_{[K]} \in \mathcal{W}_{pe}(G)$ and the following condition are satisfied for each $w \in \mathcal{W}_{pe}(G)$, every $K \in \mathbf{N}_0$ and every $w' \in \mathcal{W}_{pe}(w, K)$.*

- *For each $z \in l_{qe}$ such that $(w, z) \in R_G$, there exists $z' \in l_{qe}$ such that $(w', z') \in R_G$ and $z'_{[K]} = z_{[K]}$.*

Note that when the assumptions of Definition 5 are satisfied, one of $w' \in \mathcal{W}_{pe}(w, K)$ is obviously $w_{[K]}$. Hence, if the conditions in Definition 5 are satisfied, then those of Definition 1 are also satisfied.

Before closing this section, we give the following theorem by enumerating trivial relationships (following readily from their statements as well as our earlier arguments) among the five "definitions," which are also illustrated in Fig. 1 with solid arrows.

Theorem 1. *The following relationships hold.*

1) *If G is $w_{[\cdot]}$-causal (Definition 2), then it is weakly $w_{[\cdot]}$-causal (Definition 1).*

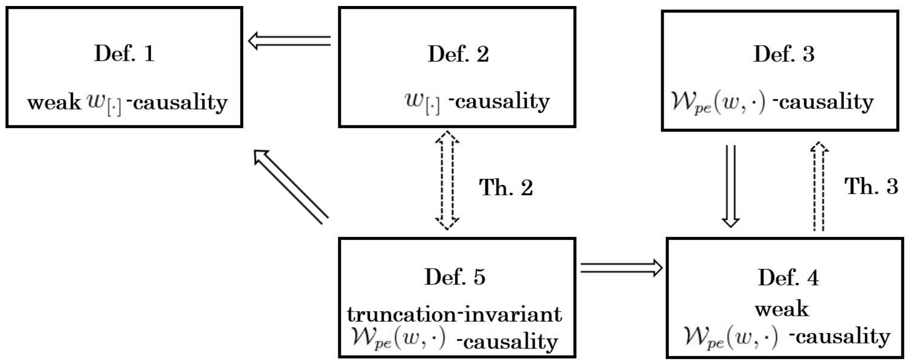

Figure 1: Relationships among the five definitions of causality.

2) *If G is $\mathcal{W}_{pe}(w,\cdot)$-causal (Definition 3), then it is weakly $\mathcal{W}_{pe}(w,\cdot)$-causal (Definition 4).*
3) *If G is truncation-invariant $\mathcal{W}_{pe}(w,\cdot)$-causal (Definition 5), then it is weakly $\mathcal{W}_{pe}(w,\cdot)$-causal (Definition 4).*
4) *If G is truncation-invariant $\mathcal{W}_{pe}(w,\cdot)$-causal (Definition 5), then it is weakly $w_{[\cdot]}$-causal (Definition 1).*

3. Further equivalence relations among causality definitions

The preceding section introduced "five definitions" for causality together with some trivial mutual relationships of them. This section is devoted to reaching more profound and entire understandings on their further nontrivial relationships and adequacy/inadequacy for general and/or some restricted classes of input-output systems. In particular, the definitions are integrated into three groups (by integrating mutually equivalent ones as an equivalence group after the former part of the arguments in this section), and it is shown that these groups can further be integrated into two groups or even one group once we introduce some restrictions on the class of systems for which the definition of causality is sought for.

3.1. General further equivalence relations

We first show that there are actually two pairs of equivalence relations in the "five definitions." In particular, Definition 2 is equivalent to Definition 5 while Definition 3 is to Definition 4, which is illustrated in Fig. 1 with dashed arrows.

We begin by showing the following first equivalence relation.

Theorem 2. *The following two conditions are equivalent.*

1. *G is $w_{[\cdot]}$-causal (Definition 2).*
2. *G is truncation-invariant $\mathcal{W}_{pe}(w,\cdot)$-causal (Definition 5).*

Proof. 1 ⇒ 2: Suppose G is $w_{[\cdot]}$-causal. For each pair of $w \in \mathcal{W}_{pe}(G)$ and $z \in l_{qe}$ such that $(w,z) \in R_G$ and for every $K \in \mathbf{N}_0$, there exists $z' \in l_{qe}$ such that $(w_{[K]}, z') \in R_G$ and $z'_{[K]} = z_{[K]}$ by the condition (i) in Definition 2. Whatever $w' \in \mathcal{W}_{pe}(w,K) \subset \mathcal{W}_{pe}(G)$ we may take, on the other hand, it satisfies $w'_{[K]} = w_{[K]}$ and thus

$$(w'_{[K]}, z') \in R_G \tag{6}$$

for the above z'. Furthermore, let us note that the fact $w' \in \mathcal{W}_{pe}(G)$ allows us to think of the w' here as one specifically chosen w in Definition 2 (if we start a fresh and independent look at this definition again), in which case its condition (ii) (with w replaced by w') particularly implies that

- For the above $z' \in l_{qe}$ satisfying (6), there exists $z'' \in l_{qe}$ such that $(w', z'') \in R_G$ and $z''_{[K]} = z'_{[K]}$ (and thus, $z''_{[K]} = z_{[K]}$ since $z'_{[K]} = z_{[K]}$ by the original assumption).

The final key in this part of the proof is to note that we can identify the above z'' with z', whose existence is required (for each w, K and w') in Definition 5.

2 ⇒ 1: Suppose G is truncation-invariant $\mathcal{W}_{pe}(w,\cdot)$-causal. For each pair of $w \in \mathcal{W}_{pe}(G)$ and $z \in l_{qe}$ such that $(w,z) \in R_G$ and for every $K \in \mathbf{N}_0$, there exists $z' \in l_{qe}$ such that

$$(w_{[K]}, z') \in R_G \quad \text{and} \quad z'_{[K]} = z_{[K]}, \tag{7}$$

because we may take $w' = w_{[K]}$ as a specific choice of $w' \in \mathcal{W}_{pe}(w,K)$ in the condition (i) in Definition 5. This precisely implies that the condition (i) in Definition 2 is satisfied. Next, consider the condition (i) that is obtained by taking the above $w_{[K]} =: w^\bullet \in \mathcal{W}_{pe}(G)$ and $w =: w^\star \in \mathcal{W}_{pe}(w^\bullet, K)$ as w and w' in Definition 5, respectively, which reads as follows (after z' is rewritten as z'' while z is written as z' to avoid conflicts of symbols).

- For each (and thus the above) $z' \in l_{qe}$ satisfying $(w^\bullet, z') \in R_G$ and $z'_{[K]} = z_{[K]}$ (i.e., (7)), there exists $z'' \in l_{qe}$ such that $(w^\star, z'') \in R_G$ and $z''_{[K]} = z'_{[K]}$.

Since this is true (when the conditions of Definition 5 is satisfied) regardless of z' satisfying (7) and since the above $w^\star$ is nothing but the original w for which (together with z such that $(w, z) \in R_G$) the condition (7) was considered, we see that the conditions of Definition 2 are satisfied. Q.E.D.

We next show the following second equivalence relation.

Theorem 3. *The following two conditions are equivalent.*

1. *G is $\mathcal{W}_{pe}(w, \cdot)$-causal (Definition 3).*
2. *G is weakly $\mathcal{W}_{pe}(w, \cdot)$-causal (Definition 4).*

Proof. It is sufficient to show 2 $\Rightarrow$ 1. The condition (i) in Definition 3 is clearly satisfied since it is same as that in Definition 4. For each pair of $w \in \mathcal{W}_{pe}(G)$ and $z \in l_{qe}$ such that $(w, z) \in R_G$, and for every $K \in \mathbf{N}_0$ and every $w' \in \mathcal{W}_{pe}(w, K)$, there exists $z' \in l_{qe}$ such that

$$(w', z') \in R_G \quad \text{and} \quad z'_{[K]} = z_{[K]}. \tag{8}$$

Next, consider the condition that is obtained by taking the above $w' =: w^\bullet \in \mathcal{W}_{pe}(G)$ and $w =: w^\star \in \mathcal{W}_{pe}(w^\bullet, K)$ as w and w' in Definition 4, which reads as follows (again after z' and z are rewritten as z'' and z', respectively).

- For each (and thus the above) $z' \in l_{qe}$ such that $(w^\bullet, z') \in R_G$, there exists $z'' \in l_{qe}$ such that $(w^\star, z'') \in R_G$ and $z''_{[K]} = z'_{[K]}$.

Since this is true (when the conditions of Definition 4 are satisfied) regardless of z' satisfying (8) and since the above $w^\star$ is nothing but the original w for which (together with z such that $(w, z) \in R_G$) the condition (8) was considered, we see that the condition (ii) of Definition 3 is satisfied. Hence, G satisfies both conditions in Definition 3. Q.E.D.

As a result of Theorems 2 and 3 together with Fig. 1 with the dashed arrows taken into account, the five definitions of causality in the preceding section are integrated into three groups A, B and C as shown in Fig. 2 with solid arrows.

3.2. Further equivalence relations of causality definitions under restricted classes of systems

Fig. 2 (with the dashed arrows yet to be established in the rest of this section and thus ignored for the moment) suggests that the "five definitions" introduced in the preceding section are not mutually equivalent as far as general input-intolerant and output-unsolitary systems are concerned. Indeed, Example 2 shows that the dashed arrow * in the left direction fails for an input-intolerant system with the admissible input set being not truncation-invariant while Example 1 shows that the dashed arrow ** in the right direction also fails for an output-unsolitary system. Nevertheless, there could be chances that either or both of the dashed arrows could hold if the attention is restricted on some particular classes of systems. The rest of this section is devoted to establishing that it is indeed the case and thus consequently showing that there could exist further equivalence relations among the three groups (called A, B and C) of causality definitions in Fig. 2 if we restrict ourselves to some classes of systems.

In particular, we consider the two cases below in the following:

(I) The case when $\mathcal{W}_{pe}(G)$ is truncation-invariant (which obviously contains the special case when G is input tolerant);

(II) The case when G is output-solitary.

We first consider case (I), for which we readily have the following theorem, leading to the dashed arrow * in Fig. 2 in the left direction and inducing a new grouping of A and (B,C).

Theorem 4. *If $\mathcal{W}_{pe}(G)$ is truncation-invariant, then the following two conditions are equivalent.*

1. *G is weakly $\mathcal{W}_{pe}(w,\cdot)$-causal (Definition 4).*

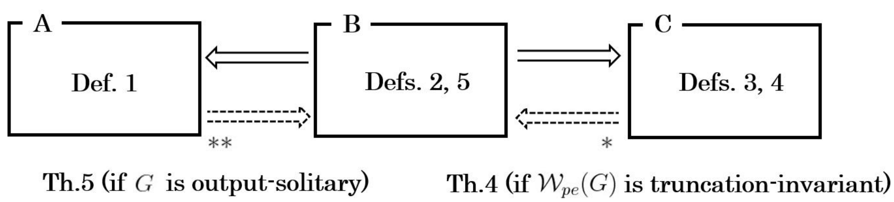

Figure 2: Relationships among three groups of causality definitions.

2. *G is truncation-invariant* $\mathcal{W}_{pe}(w, \cdot)$*-causal (Definition 5).*

Proof. It suffice to show that 1. implies 2. This, however, is obvious because Definition 5 is obtained by adding the condition that $\mathcal{W}_{pe}(G)$ is truncation-invariant to the requirement in Definition 4. Q.E.D.

By Theorems 2–4, we have the following equivalence relation, where each type of causality in it is qualified for systems whose admissible input set is truncation-invariant.

Corollary 1. *If* $\mathcal{W}_{pe}(G)$ *is truncation-invariant, then the following four conditions are equivalent.*

1. *G is* $w_{[\cdot]}$*-causal (Definition 2).*
2. *G is* $\mathcal{W}_{pe}(w, \cdot)$*-causal (Definition 3).*
3. *G is weakly* $\mathcal{W}_{pe}(w, \cdot)$*-causal (Definition 4).*
4. *G is truncation-invariant* $\mathcal{W}_{pe}(w, \cdot)$*-causal (Definition 5).*

We remark, as stated earlier, that Definition 1 cannot belong to this equivalence relation as disproved with Example 1.

We next consider case (II), for which we readily have the following theorem, leading to the dashed arrow ** in Fig. 2 in the right direction and inducing a new grouping of (A,B) and C.

Theorem 5. *If G is output-solitary, then the following two conditions are equivalent.*

1. *G is weakly* $w_{[\cdot]}$*-causal (Definition 1).*
2. *G is* $w_{[\cdot]}$*-causal (Definition 2).*

Proof. It suffices to show that 1. implies 2. but this is obvious as noted in the paragraph below Definition 2. Q.E.D.

By Theorems 2 and 5, we have the following equivalence relation, where each type of causality in it is qualified for output-solitary systems.

Corollary 2. *If G is output-solitary, then the following three conditions are equivalent.*

1. *G is weakly $w_{[\cdot]}$-causal (Definition 1).*
2. *G is $w_{[\cdot]}$-causal (Definition 2).*
3. *G is truncation-invariant $\mathcal{W}_{pe}(w,\cdot)$-causal (Definition 5).*

It readily follows that if G is output-solitary and its admissible input set is truncation-invariant, then all the "five definitions" are equivalent by Corollaries 1 and 2, even though G could still be input-intolerant under such assumptions. In [3], Vidyasagar restricted his attention on systems that are both output-solitary and input-tolerant. He then gave two definitions of causality corresponding to Definition 1 (which happens to belong to group A) and Definition 4 (to group C) of this paper and showed their equivalence, an assertion consistent with the general relationship revealed as Fig. 2. Compared with this earlier study, the advances in the present paper are obvious in establishing when some equivalence relations are possibly lost if the system under concern fails to satisfy the implicit assumption by Vidyasagar. Indeed, the dashed arrows in Fig. 2, which are supported by Examples 1 and 2, do imply when such a kind of loss could actually occur.

Remark 1. Since group C consists of the definitions that are valid even for the most general (i.e., largest) class of systems allowing input-intolerant output-unsolitary systems, it may be natural to expect that group C leads to the strongest requirement. In this respect, however, "B $\Rightarrow$ C" in Fig. 2 actually implies that group B leads to a stronger requirement than group C. The reason for coming to such a possibly confusing situation can be understood once we recall that group B consists of such definitions that include the requirement that $\mathcal{W}_{pe}(G)$ is truncation-invariant. Note that this requirement, however, is actually unnecessary and overdemanding as a requirement for causality. This makes group B, if it is viewed as definitions for input-intolerant output-unsolitary systems, unnecessarily stronger than the adequate requirement by group C. Indeed, recall that Example 2 gave an example of a causal system (in the sense of group C) for which $\mathcal{W}_{pe}(G)$ is not truncation-invariant.

4. Small-gain theorem for general input-output systems

This section is devoted to extending the small-gain theorem for closed-loop systems consisting of two subsystems that are possibly input-intolerant and/or output-unsolitary. In the small-gain theorem, the definition of the induced norm or gain (as well as causality) of subsystems is quite important from the viewpoint of reducing conservativeness in stability analysis. This paper uses the definition employed in the later studies [3, 4, 5, 6, 7], which does not resort to truncation treatment unlike the original study by Zames and leads to a smaller value as confirmed later with an example. Note that these later studies only dealt with closed-loop systems consisting of subsystems that are both input-tolerant and output-solitary. Hence, the arguments in this section corresponds to extension accommodating more general input-intolerant output-unsolitary input-output subsystems, which is believed to be significant in exploiting the full power of the small-gain type of arguments. In such extension, causality of subsystems plays a key role as in those later studies, and Definitions 3 and 4 in group C applicable to input-intolerant output-unsolitary systems become quite relevant in the arguments. Usefulness of the extended small-gain theorem will be demonstrated with an example.

4.1. Problem formulation

For p_1, $p_2 \in [1, \infty]$, let $p'_1 := p_2$ and $p'_2 := p_1$. Consider the closed-loop system Σ consisting of $G_1 = G_1(l_{p_1 e}, l_{p'_1 e})$ and $G_2 = G_2(l_{p_2 e}, l_{p'_2 e})$ in Fig. 3 (which is in positive feedback). We regard $u := [u^{1T}, u^{2T}]^T$ as the input of Σ while $w := [w^{1T}, w^{2T}]^T$ and $z := [z^{1T}, z^{2T}]^T$ as the output. Furthermore, we say $(u, [w^T, z^T]^T) \in R_\Sigma$ to mean that $(w^i, z^i) \in R_{G_i}$ $(i = 1, 2)$, $w^1 = u^1 + z^2$ and $w^2 = u^2 + z^1$. As a subspace of $\mathcal{W}_{pe}(G)$, we define

$$\mathcal{W}_p(G) := \mathcal{W}_{pe}(G) \cap l_p \tag{9}$$

and call it the admissible bounded input set of G.

We assume the following for the system Σ.

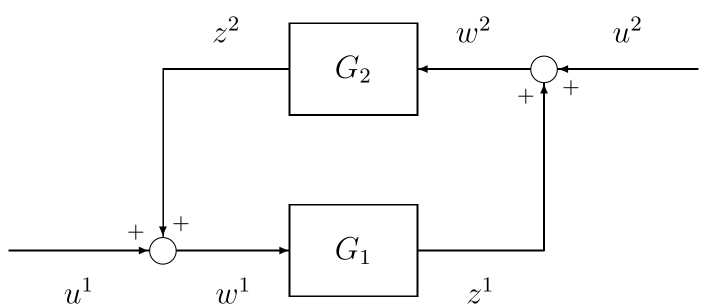

Figure 3: The closed-loop system Σ.

Assumption 1. The closed-loop system Σ satisfies the following conditions.

1. For $i = 1, 2$, the admissible input set $\mathcal{W}_{p_i e}(G_i)$ is closed under addition.
2. Whenever $w^1 \in \mathcal{W}_{p_1 e}(G_1)$, every $z^1 \in l_{p_1' e} = l_{p_2 e}$ such that $(w^1, z^1) \in R_{G_1}$ belongs to $\mathcal{W}_{p_2 e}(G_2)$ $(\subset l_{p_2 e})$.
3. Whenever $w^2 \in \mathcal{W}_{p_2 e}(G_2)$, every $z^2 \in l_{p_2' e} = l_{p_1 e}$ such that $(w^2, z^2) \in R_{G_2}$ belongs to $\mathcal{W}_{p_1 e}(G_1)$ $(\subset l_{p_1 e})$.
4. For each pair of $u^1 \in \mathcal{W}_{p_1 e}(G_1)$ and $u^2 \in \mathcal{W}_{p_2 e}(G_2)$, there exist w and z such that $(u, [w^T, z^T]^T) \in R_\Sigma$, w^1, $z^2 \in \mathcal{W}_{p_1 e}(G_1)$ and w^2, $z^1 \in \mathcal{W}_{p_2 e}(G_2)$.

Fig. 4 illustrates the closed-loop system Σ with the assumptions on the signal spaces explicitly taken into account. Note that $[w^T, z^T]^T$ is not assumed to be unique in the last condition in the above assumption, meaning that Σ is allowed to be output-unsolitary. The external signals u^i $(i = 1, 2)$ will be further restricted to those belonging to $\mathcal{W}_{p_i}(G_i)$ $(i = 1, 2)$, respectively, in the arguments of stability and the small-gain theorem for Σ. Roughly speaking, we say that Σ is stable if $w^i \in l_{p_i}$ $(i = 1, 2)$ and $z^i \in l_{p_i'}$ $(i = 1, 2)$ as long as $(u, [w^T, z^T]^T) \in R_\Sigma$ for such $u = [u^{1T}, u^{2T}]^T$. A precise definition for stability follows shortly.

Remark 2. Assumption 1 corresponds to the well-posedness conditions for Σ in the context involving input-intolerant and output-unsolitary subsystems. In this connection, Willems [8, p. 105] describes that stability of feedback systems could very well be defined without requiring well-posedness in the sense that such a fundamental property should be verified anyway as a prerequisite before tackling the stability issue so that physically meaningless considerations could be avoided. This is because the lack of well-posedness implies that the subsystems do not in fact adequately describe the physical phenomena, and once the descriptions of the subsystems are modified, then it would very well lead to altering such fundamental properties of the closed-loop system as stability properties. If we were also to take such a position, Assumption 1 would become unnecessary in the latter arguments.

The problem setting under $p_i = p \in [1, \infty]$ reduces to that in [1, 2]. In addition, the problem setting under $p_i = p \in [1, \infty]$ and $\mathcal{W}_{p_i}(G_i) = l_p$ $(i = 1, 2)$ together with the assumption that G_i $(i = 1, 2)$ are input-tolerant and output-solitary reduces to that in [3, 4, 5, 6, 7]; under this situation, the conditions in Assumption 1 are satisfied except for the last, corresponding to the standard well-posedness assumption. In contrast, this paper obviously considers more general cases such as the treatment of input-intolerant subsystems, by which we can consider, e.g., such closed-loop systems consisting of subsystems whose input and output take only positive values.

We are now in a position to define stability of Σ, for which we begin by defining stability of G_i $(i = 1, 2)$, where $\|\cdot\|_p$ denotes the l_p norm.

Definition 6. *For $i = 1,\ 2$, the system G_i is $l_{p_i'}/\mathcal{W}_{p_i}(G_i)$ stable if there exist $\gamma_i \geq 0$ and $\beta_i \geq 0$ satisfying the following conditions.*

- *Whenever $w \in \mathcal{W}_{p_i}(G_i)$, every $z \in l_{p_i'e}$ such that $(w, z) \in R_{G_i}$ belongs to $l_{p_i'}$, and*
- *satisfies*

$$\|z\|_{p_i'} \leq \gamma_i \|w\|_{p_i} + \beta_i \tag{10}$$

For $l_{p_i'}/\mathcal{W}_{p_i}(G_i)$ stable G_i, its gain is defined as

$$\gamma_{p_i'/p_i}(G_i) := \inf\{\gamma_i \geq 0 : \exists \beta_i \geq 0 \text{ s.t. } (10)\}. \tag{11}$$

We then define the input-output stability of Σ as follows.

Definition 7. *The closed-loop system Σ is $(\mathcal{W}_{p_1}(G_1), \mathcal{W}_{p_2}(G_2))$ stable if there exist $\gamma \geq 0$ and $\beta \geq 0$ satisfying the following conditions.*

- *Whenever $u = [u^{1T}, u^{2T}]^T$ satisfies $u^i \in \mathcal{W}_{p_i}(G_i)$ $(i = 1, 2)$, every quadruple of $w^i \in l_{p_ie}$ $(i = 1, 2)$ and $z^i \in l_{p_i'e}$ $(i = 1, 2)$ such that $(u, [w^T, z^T]^T) \in R_\Sigma$ satisfies $w^i \in \mathcal{W}_{p_i}(G_i)$ $(i = 1, 2)$, $z^1 \in \mathcal{W}_{p_2}(G_2)$ and $z^2 \in \mathcal{W}_{p_1}(G_1)$, and*

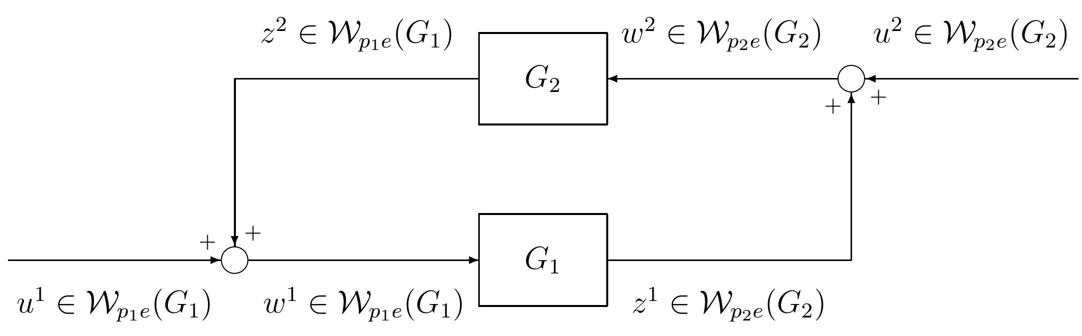

Figure 4: The closed-loop system Σ with the signal space explicitly referred to.

- *satisfies*

$$\max\{\|[w^{1T}, z^{2T}]^T\|_{p_1}, \|[w^{2T}, z^{1T}]^T\|_{p_2}\} \leq \gamma(\|u^1\|_{p_1} + \|u^2\|_{p_2}) + \beta \tag{12}$$

4.2. Small-gain theorem

We are in a position to extend the small-gain theorem so that it can also handle subsystems that are input-intolerant and/or output-unsolitary while exploiting the induced norm definition in [3, 4, 5, 6, 7]. As in the derivation of the small-gain theorem for such systems, causality of subsystems that are in feedback connection should somehow be exploited. In this respect, it is quite important to note that the definitions of causality adequate for the most general input-intolerant output-unsolitary systems (i.e., Definitions 3 and 4) are, unlike the standard definition for input-tolerant output-solitary systems, not based on mere truncation of input signals. Instead, they require us to consider the input set $\mathcal{W}_{pe}(w, K)$ introduced in (5). To circumvent the associated difficulty in the derivation of the small-gain theorem for the input-intolerant output-unsolitary case, the following additional assumption is introduced for the subsystems G_i $(i = 1, 2)$.

Assumption 2. *For $i = 1, 2$, the subsystem G_i satisfies the following conditions.*

1. $\mathcal{W}_{p_i}(G_i) \neq \emptyset$.
2. *There exists $\delta_i \geq 0$ such that whenever $w \in \mathcal{W}_{p_ie}(G_i)$ and $K \in \mathbf{N}_0$, one can find $w' \in \mathcal{W}_{p_ie}(w, K) \cap \mathcal{W}_{p_i}(G_i)$ satisfying*

$$\|w'\|_{p_i} \leq \|w'_{[K]}\|_{p_i} + \delta_i \ (= \|w_{[K]}\|_{p_i} + \delta_i). \tag{13}$$

Roughly speaking, what the above assumption says is as follows: even though an admissible input of G_i may become inadmissible if it is exactly truncated at some K, one can always consider some "approximate truncation" (as long as the trailing part of the input is concerned) within the admissible (bounded) input set. Another rough rephrasing would be that there always exists an admissible w' close to the exact truncation in the sense that w' belongs to $\mathcal{W}_{p_i}(G_i) \subset l_{p_i}$ (as in the exact truncation) and its l_{p_i} norm increases only by δ_i at most compared with the norm of the (possibly inadmissible) exact truncation of the admissible input. This assumption is believed to be weak enough because δ_i is not required to be small and it would be possible, in most systems, to cease applying the input at least gradually, if not immediately.

Remark 3. The study [16], which develops the integral quadratic constraints framework for stability analysis based on the small-gain theorem, also discusses "approximate truncation" by introducing a condition similar to (13):

$$\|w'\|_{p_i} \le (1+\delta_i)\|w'_{[K]}\|_{p_i}. \tag{14}$$

However, we note that the above study is actually under the assumption that $\mathcal{W}_{p_i e}(G_i) = l_{p_i e}$. This implies that $\mathcal{W}_{p_i e}(G_i)$ is truncation invariant and thus we could take $\delta_i = 0$ and thus *exact truncation* if we were to consider such $\mathcal{W}_{p_i e}(G_i)$ in our present context. What this suggests is that the circumstance of introducing the idea of "approximate truncation" in [16] is considerably different from that in this paper.

Remark 4. Regarding Assumption 2, we may replace (13) with

$$\|w'\|_{p_i} \le \mu_i\|w'_{[K]}\|_{p_i} + \delta_i \tag{15}$$

where $\mu_i \ge 1$ is also required to exist. This μ_i relaxes the assumption and thus enables us to deal with a wider class of subsystems G_1 and G_2. However, the correspondingly modified version of Theorem 7 (to be given below as one of the main results in this paper) will be obtained only at the sacrifice of a more demanding small-gain condition due to μ_i, and thus the details are omitted.

The above assumption readily leads us to the following lemma as in [3], successfully relating the norms of the truncated versions of w (not necessarily in $\mathcal{W}_{p_i}(G_i)$) and z such that $(w, z) \in R_{G_i}$.

Lemma 1. *For $i = 1,\ 2$, let $p_i, p'_i \in [1, \infty]$ and suppose $G_i : (l_{p_i e}, l_{p'_i e})$ is $l_{p'_i}/\mathcal{W}_{p_i}(G_i)$ stable and weakly $\mathcal{W}_{p_i e}(w, \cdot)$-causal. Whenever $w \in \mathcal{W}_{p_i e}(G_i)$ and $z \in l_{p'_i e}$ satisfy $(w, z) \in R_{G_i}$, we have*

$$\|z_{[K]}\|_{p'_i} \le \gamma_i\|w_{[K]}\|_{p_i} + \zeta_i \quad (\forall K \in \mathbf{N}_0), \tag{16}$$

where ζ_i is given by $\gamma_i\delta_i + \beta_i$ for $\delta_i \ge 0$ satisfying (13) together with $\gamma_i \ge 0$ and $\beta_i \ge 0$ satisfying (10).

Proof. Fix $i \in \{1, 2\}$. Take $w \in \mathcal{W}_{p_i e}(G_i)$ and $z \in l_{p'_i e}$ such that $(w, z) \in$

R_{G_i}, and also take $K \in \mathbf{N}_0$. By Assumption 2, we can take $\delta_i \geq 0$ such that there exists $w' \in \mathcal{W}_{p_i e}(w, K) \cap \mathcal{W}_{p_i}(G_i)$ satisfying

$$\|w'\|_{p_i} \leq \|w'_{[K]}\|_{p_i} + \delta_i \ (= \|w_{[K]}\|_{p_i} + \delta_i). \tag{17}$$

Since G_i is weakly $\mathcal{W}_{p_i e}(w, \cdot)$-causal, there exists $z' \in l_{p'_i e}$ such that $(w', z') \in R_G$ and $z'_{[K]} = z_{[K]}$. In addition, since G_i is $l_{p'_i}/\mathcal{W}_{p_i}(G_i)$ stable, this z' in fact satisfies $z' \in l_{p'_i}$ and

$$\|z'\|_{p'_i} \leq \gamma_i \|w'\|_{p_i} + \beta_i. \tag{18}$$

By (17), (18) and $\|z'\|_{p'_i} \geq \|z'_{[K]}\|_{p'_i} = \|z_{[K]}\|_{p'_i}$, we have

$$\|z_{[K]}\|_{p'_i} \leq \gamma_i \|w_{[K]}\|_{p_i} + \zeta_i \tag{19}$$

where $\zeta_i = \gamma_i \delta_i + \beta_i$. Since this is true for every $K \in \mathbf{N}_0$, this completes the proof. Q.E.D.

We are in a position to give a very important result on the relation between the gain of the subsystem G_i defined in (11) and that defined by Zames in [1, 2] (assuming $p_i = p'_i$); roughly speaking, the truncated versions of w and z such that $(w, z) \in R_G$ were considered there, and the inequality (16) without the bias term ζ_i was considered. The gain defined by Zames may be rephrased and generalized as follows[3]:

$$\gamma^{\star}_{p'_i/p_i}(G_i) := \inf\{\gamma_i \geq 0 : (16) \text{ holds under } \zeta_i = 0\}. \tag{20}$$

For the sake of deeper understanding, let us further consider the case where the bias term β_i in Definition 6 is prohibited to be positive and thus the definition of the gain is modified accordingly in (11) with $\beta_i = 0$, which we denote by $\overline{\gamma}_{p'_i/p_i}(G_i)$ ($\geq \gamma_{p'_i/p_i}(G_i)$). The following theorem plays a key role in our assertion that the extension of the small-gain theorem provided shortly indeed contains meaningful advances over the arguments by Zames in reducing conservativeness in stability analysis (as long as feedback systems consisting of causal subsystems are concerned).

[3] We admit the case with $p_i \neq p'_i$, and also allow such a case that $\mathcal{W}_{p_i e}(G_i) = \{0\}$, which is rather abnormal but leads to a situation that the gain becomes undefinable in the treatment of Zames.

Theorem 6. *If* $\gamma^\star_{p_i'/p_i}(G_i) < \infty$*, then* G_i *is* $l_{p_i'}/\mathcal{W}_{p_i}(G_i)$ *stable (with* $\beta_i = 0$ *in (10)) and*

$$\gamma_{p_i'/p_i}(G_i) \le \overline{\gamma}_{p_i'/p_i}(G_i) \le \gamma^\star_{p_i'/p_i}(G_i) \tag{21}$$

If G_i *is weakly* $\mathcal{W}_{p_ie}(w,\cdot)$*-causal and* $\mathcal{W}_{p_ie}(G_i)$ *is truncation-invariant in addition to the above assumption, then*

$$\gamma_{p_i'/p_i}(G_i) \le \overline{\gamma}_{p_i'/p_i}(G_i) = \gamma^\star_{p_i'/p_i}(G_i). \tag{22}$$

Proof. The fact that $\gamma^\star_{p_i'/p_i}(G_i) < \infty$ implies $l_{p_i'}/\mathcal{W}_{p_i}(G_i)$ stability of G_i (with $\beta_i = 0$ in (10)) and the second inequality in (21) are shown in [1] while the first inequality is obvious. Hence, it suffices to show that $\overline{\gamma}_{p_i'/p_i}(G_i) = \gamma^\star_{p_i'/p_i}(G_i)$ if G_i is weakly $\mathcal{W}_{p_ie}(w,\cdot)$-causal and $\mathcal{W}_{p_ie}(G_i)$ is truncation-invariant. To show this, we first note that we can take $\delta_i = 0$ in (13) since $\mathcal{W}_{p_ie}(G_i)$ is truncation-invariant. We also note from the definition of $\overline{\gamma}_{p_i'/p_i}(G_i)$ that for every $\epsilon > 0$, the inequality (10) with $\beta_i = 0$ holds for $\gamma_i = \overline{\gamma}_{p_i'/p_i}(G_i) + \epsilon$. Hence, by Lemma 1, we are led to the inequality (16) with $\zeta_i = 0$ under the same $\gamma_i = \overline{\gamma}_{p_i'/p_i}(G_i) + \epsilon$. Since this is true for whatever small $\epsilon > 0$, it follows from the definition of $\gamma^\star_{p_i'/p_i}(G_i)$ that $\gamma^\star_{p_i'/p_i}(G_i) \le \overline{\gamma}_{p_i'/p_i}(G_i)$. This completes the proof. Q.E.D.

Example 3. Let us denote by $n(w)$ the number of k such that $w_k \ne 0$. Consider $G = G(l_{1e}, l_{1e})$ such that (i) the admissible input set $\mathcal{W}_{1e}(G)$ is the set of w such that either $n(w) = 0$ or $n(w) = 2$, and $w_k = 1$ whenever $w_k \ne 0$; (ii) $(0, z) \in R_G$ if and only if $z = 0$; (iii) for nonzero $w \in \mathcal{W}_{1e}(G)$ such that $w_{\bar{k}} = 1$ and $w_k = 0$ $(0 \le k < \bar{k})$, the only z satisfying $(w, z) \in R_G$ is given by $z_{\bar{k}} = 1$ and $z_k = 0$ $(k \ne \bar{k})$. It is easy to see that G is weakly $\mathcal{W}_{1e}(w,\cdot)$-causal and $l_1/\mathcal{W}_1(G)$ stable. For this system, we can readily see that $\overline{\gamma}_{1/1}(G) = 1/2$ while $\gamma^\star_{1/1}(G) = 1$, and thus the equality in (22) fails. This is consistent with the assertion of Theorem 6 because we readily see that $\mathcal{W}_{1e}(G)$ is not truncation-invariant. It is also easy to see that $\gamma_{1/1}(G) = 0$.

Remark 5. It is easy to see that the example can be modified, with an arbitrary integer N (≥ 2), to an example such that $n(w) = N$ for every nonzero $w \in \mathcal{W}_{1e}(G)$. From this observation, it is obvious that the magnification factor of $\gamma^\star_{1/1}(G)$ relative to $\overline{\gamma}_{1/1}(G)$ (which is equal to N in such an example) can be arbitrarily large. Note that the admissible input set $\mathcal{W}_{1e}(G)$ still fails to be truncation-invariant in such a modified example. This suggests that as long as our interest is restricted to causal subsystems, our study would outperform that of the original study by Zames at least when either of $\mathcal{W}_{p_ie}(G_i)$ $(i = 1, 2)$ is not truncation-invariant (as suggested by Theorem 6). Indeed, the magnification factor of $\gamma^\star_{1/1}(G)$ relative to $\gamma_{1/1}(G)$ (rather than $\overline{\gamma}_{1/1}(G)$) is infinity in the modified (as well as the above) example because $\gamma_{1/1}(G) = 0$. As a side remark, we can further construct a similar example with $\mathcal{W}_{1e}(G)$ such that $2 \le n(w) < \infty$ for every nonzero $w \in \mathcal{W}_{1e}(G)$ and $w_k \ge 1$ whenever $w_k \ne 0$. In this case, $\mathcal{W}_{1e}(G)$ becomes closed under addition while $\gamma^\star_{1/1}(G) > \overline{\gamma}_{1/1}(G)$ remains true.

We are in a position to state and prove the following extended small-gain theorem that can handle subsystems that are input-intolerant and/or output-unsolitary while using the improved gain evaluation $\gamma_{p'_i/p_i}(G_i)$.

Theorem 7. *Suppose the closed-loop system Σ in Fig. 4 satisfies Assumption 1, $G_i : (l_{p_ie}, l_{p'_ie})$ is $l_{p'_i}/\mathcal{W}_{p_i}(G_i)$ stable and weakly $\mathcal{W}_{p_ie}(w, \cdot)$-causal, and satisfies Assumption 2 for each $i = 1$, 2. If $\gamma_{p'_1/p_1}(G_1)\gamma_{p'_2/p_2}(G_2) < 1$, then Σ is $(\mathcal{W}_{p_1}(G_1), \mathcal{W}_{p_2}(G_2))$ stable.*

Proof. Let $u^i \in \mathcal{W}_{p_i}(G_i)$ $(i = 1, 2)$, and consider whatever quadruple of $w^i \in l_{p_ie}$ $(i = 1, 2)$ and $z^i \in l_{p_{i'}e}$ $(i = 1, 2)$ such that $(u, [w^T, z^T]^T) \in R_\Sigma$ (whose existence is ensured by Assumption 1).

Before proceeding, we first take a sufficiently small ϵ such that $\gamma_1\gamma_2 < 1$, where $\gamma_i := \gamma_{p'_i/p_i}(G_i) + \epsilon$ $(i = 1, 2)$. This is helpful because it can alleviate a subtle issue relevant to the gain of G_i defined in (11) through the use of infimum; the above γ_i together with an appropriate $\beta_i \ge 0$ does satisfy (10). Hence, by Lemma 1, the following inequalities hold for the above quadruple.

$$\|z^1_{[K]}\|_{p_2} \le \gamma_1\|w^1_{[K]}\|_{p_1} + \zeta_1,\ \|z^2_{[K]}\|_{p_1} \le \gamma_2\|w^2_{[K]}\|_{p_2} + \zeta_2 \quad (\forall K \in \mathbf{N}_0) \tag{23}$$

It is also obvious that

$$\begin{aligned} &\|w^1_{[K]}\|_{p_1} \le \|u^1_{[K]}\|_{p_1} + \|z^2_{[K]}\|_{p_1},\\ &\|w^2_{[K]}\|_{p_2} \le \|u^2_{[K]}\|_{p_2} + \|z^1_{[K]}\|_{p_2}. \end{aligned} \tag{24}$$

By (23), (24) and $1-\gamma_1\gamma_2>0$, we obtain

$$\begin{bmatrix}\|w^1_{[K]}\|_{p_1}\\ \|w^2_{[K]}\|_{p_2}\end{bmatrix} \le \frac{1}{1-\gamma_1\gamma_2}\begin{bmatrix}1 & \gamma_2\\ \gamma_1 & 1\end{bmatrix}\begin{bmatrix}\|u^1_{[K]}\|_{p_1}\\ \|u^2_{[K]}\|_{p_2}\end{bmatrix} + \frac{1}{1-\gamma_1\gamma_2}\begin{bmatrix}\gamma_2\zeta_1+\zeta_2\\ \gamma_1\zeta_2+\zeta_1\end{bmatrix} \ \ (\forall K \in \mathbf{N}_0) \tag{25}$$

where the inequality is elementwise. Since $u^1 \in \mathcal{W}_{p_1}(G_1)$ and $u^2 \in \mathcal{W}_{p_2}(G_2)$, we have

$$\begin{bmatrix}\|w^1_{[K]}\|_{p_1}\\ \|w^2_{[K]}\|_{p_2}\end{bmatrix} \le \frac{1}{1-\gamma_1\gamma_2}\begin{bmatrix}1 & \gamma_2\\ \gamma_1 & 1\end{bmatrix}\begin{bmatrix}\|u^1\|_{p_1}\\ \|u^2\|_{p_2}\end{bmatrix} + \frac{1}{1-\gamma_1\gamma_2}\begin{bmatrix}\gamma_2\zeta_1+\zeta_2\\ \gamma_1\zeta_2+\zeta_1\end{bmatrix} \ \ (\forall K \in \mathbf{N}_0). \tag{26}$$

In addition, by (23) and (26), we have

$$\begin{bmatrix}\|z^1_{[K]}\|_{p_2}\\ \|z^2_{[K]}\|_{p_1}\end{bmatrix} \le \frac{1}{1-\gamma_1\gamma_2}\begin{bmatrix}\gamma_1 & \gamma_1\gamma_2\\ \gamma_1\gamma_2 & \gamma_2\end{bmatrix}\begin{bmatrix}\|u^1\|_{p_1}\\ \|u^2\|_{p_2}\end{bmatrix} + \frac{1}{1-\gamma_1\gamma_2}\begin{bmatrix}\zeta_1+\gamma_1\zeta_2\\ \zeta_2+\gamma_2\zeta_1\end{bmatrix} \ \ (\forall K \in \mathbf{N}_0). \tag{27}$$

Hence, (26) and (27) further lead us to

$$\left\|\begin{bmatrix}w^1\\ z^2\end{bmatrix}_{[K]}\right\|_{p_1} \le \frac{1+\gamma_1\gamma_2}{1-\gamma_1\gamma_2}\|u^1\|_{p_1} + \frac{2\gamma_2}{1-\gamma_1\gamma_2}\|u^2\|_{p_2} + \frac{2(\gamma_2\zeta_1+\zeta_2)}{1-\gamma_1\gamma_2}, \tag{28}$$

$$\left\|\begin{bmatrix}w^2\\ z^1\end{bmatrix}_{[K]}\right\|_{p_2} \le \frac{2\gamma_1}{1-\gamma_1\gamma_2}\|u^1\|_{p_1} + \frac{1+\gamma_1\gamma_2}{1-\gamma_1\gamma_2}\|u^2\|_{p_2} + \frac{2(\gamma_1\zeta_2+\zeta_1)}{1-\gamma_1\gamma_2} \tag{29}$$

for all $K \in \mathbf{N}_0$. Therefore, $[w^{1T}, z^{2T}]^T \in l_{p_1}$ and $[w^{2T}, z^{1T}]^T \in l_{p_2}$, and the proof is completed by taking

$$\gamma = \frac{\max\{2\gamma_1, 1+\gamma_1\gamma_2, 2\gamma_2\}}{1-\gamma_1\gamma_2},\ \beta = \frac{2\max\{\gamma_2\zeta_1+\zeta_2,\ \gamma_1\zeta_2+\zeta_1\}}{1-\gamma_1\gamma_2}. \tag{30}$$

Q.E.D.

The above theorem is for the closed-loop system Σ consisting of input-intolerant and/or output-unsolitary causal subsystems in the sense of group C and also group B because of "B $\Rightarrow$ C" in Fig. 2. Similarly, since "A $\Rightarrow$ B" in Fig. 2 if G_i is output-solitary, this theorem applies also to Σ consisting of input-tolerant and output-solitary causal subsystems in the sense of group A. This assertion is actually nothing but the well-known small-gain theorem, but what is suggested as a converse of the above observation is that the well-known theorem would not hold as it is when it is applied to the case with input-tolerant and output-*unsolitary* causal subsystems in the sense of group A.

Remark 6. If G_i $(i = 1, 2)$ satisfy the requirements in the modified version of Definition 6 with $\beta_i = 0$, then the statement of the above theorem remains true even if $\gamma_{p'_i/p_i}(G_i)$ is replaced by $\overline{\gamma}_{p'_i/p_i}(G_i)$ $(i = 1, 2)$ (whose proof proceeds under $\beta_i = 0$ $(i = 1, 2)$). Since $\overline{\gamma}_{p'_i/p_i}(G_i) \geq \gamma_{p'_i/p_i}(G_i)$, however, such an alternative assertion is virtually meaningless unless some extra desirable consequence follows the use of $\overline{\gamma}_{p'_i/p_i}(G_i)$ as a meaningful price. This is indeed the case when $\mathcal{W}_{p_ie}(G_i)$ $(i = 1, 2)$ are truncation-invariant, in which case we can take $\delta_i = 0$ $(i = 1, 2)$ in Assumption 2. Then, both ζ_1 and ζ_2 in Lemma 1 become 0 and thus β given by (30) also becomes 0. This implies that the stability of Σ can be ensured in such a way that the bias term β is not necessary in (12). Nevertheless, Theorem 6 implies that the resulting arguments (with such restriction to $\beta_i = 0$) reduce exactly to those in the original study by Zames (if $p_i = p'_i$). To put it conversely, our extended form of the small-gain theorem is indeed an improvement over the original one by Zames when either of $\mathcal{W}_{p_ie}(G_i)$ $(i = 1, 2)$ is not truncation-invariant (or when the induced norm of G_i is allowed to be evaluated with a nonzero bias β_i).

Before closing the theoretical part of this paper, we stress that our arguments have been consistently based on the standing assumption that the subsystems G_1 and G_2 are causal, unlike the recent study in [19] (which deal only with input-tolerant output-solitary systems) as well as the original study by Zames. In this connection, it would be worth remarking that, in spite of the understanding by the authors of the aforementioned recent study, the original study by Zames does not assume causality (even implicitly) on the subsystems G_1 and G_2. More precisely, the fact that $\gamma^{\star}_{p'_i/p_i}(G_i) < \infty$ does not imply that G_i is (weakly $\mathcal{W}_{p_ie}(w, \cdot)$-)causal. This can be confirmed by the following example.

Example 4. Consider $G = G(l_{1e}, l_{1e})$ such that

(i) $(w,w)\in R_G$ for $w=(\overbrace{1,\ldots,1}^{k},0,0,\ldots)$ for all $k\in\mathbf{N}_0$;

(ii) $(w,w)\in R_G$ and $(w,2w)\in R_G$ for $w=(\overbrace{1,\ldots,1}^{k},-1,0,0,\ldots)$ for all $k\in\mathbf{N}_0$.

and there are no other (w,z) satisfying $(w,z)\in R_G$. We can readily see that $\gamma^{\star}_{1/1}(G)=2<\infty$ although it is easy to verify that G is not weakly $\mathcal{W}_{1e}(G)$-causal.

4.3. Example of stability analysis

We now demonstrate that Theorem 7 is useful with the following example.

Example 5. Consider the closed-loop system Σ in Fig. 3 consisting of the single-input single-output systems $G_i=G_i(l_{2e},l_{2e})$ $(i=1,2)$. Let l_{2e+} be the subset of l_{2e} consisting of positive signals (signals whose value at every $k\in\mathbf{N}_0$ is positive), which actually coincides with the set of positive signals itself, and let l_{2+} be the subset of l_2 consisting of positive signals, i.e., $l_{2+}:=l_{2e+}\cap l_2$. Let G_1 be the input-intolerant output-solitary system satisfying

$$G_1:\quad z^1_k=\frac{2}{3}w^1_k \tag{31}$$

for every $k\in\mathbf{N}_0$, with $\mathcal{W}_{2e}(G_1)$ given by l_{2e+}, which is not truncation-invariant. Let G_2 be the input-intolerant output-solitary system satisfying

$$G_2:\quad z^2_k=\begin{cases}\frac{1}{2}w^2_k & (0<w^2_k\le 1)\\ \frac{1}{2}(w^2_k)^2 & (1<w^2_k\le 2)\\ w^2_k & (2<w^2_k)\end{cases} \tag{32}$$

for every $k\in\mathbf{N}_0$, with $\mathcal{W}_{2e}(G_2)$ also given by l_{2e+}. Note that the closed-loop system consisting of these subsystems G_1 and G_2 cannot be dealt with in [1, 2] since $(0,0)\in R_{G_i}$ $(i=1,2)$ is assumed therein. We assume that $u^i\in\mathcal{W}_2(G_i)=l_{2+}$ $(i=1,2)$ and examine whether Σ is (l_{2+},l_{2+}) stable. More specifically, we aim at demonstrating that the small-gain theorem derived in this paper is helpful for such a study.

To this end, we first confirm that Σ indeed has a structure compatible with those systems that can be handled by the small-gain theorem. We begin by confirming that Assumption 1 is satisfied. The first and second conditions in this assumption are obviously satisfied while the third is also satisfied essentially by the fact that $l_2 \subset l_4$. To show that the fourth condition is also satisfied, we introduce the continuous function

$$g_2(y) := \begin{cases} \frac{1}{2} & (0 < y \le 1) \\ \frac{y}{2} & (1 < y \le 2) \\ 1 & (2 < y) \end{cases} \tag{33}$$

on the positive real axis, which is relevant to G_2 in the sense that (32) can be represented as $z_k^2 = g_2(w_k^2) w_k^2$. Then, Σ can be represented by

$$\begin{bmatrix} 1 & -g_2(w_k^2) \\ -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} w_k^1 \\ w_k^2 \end{bmatrix} = \begin{bmatrix} u_k^1 \\ u_k^2 \end{bmatrix} \quad (\forall k \in \mathbf{N}_0). \tag{34}$$

In the following, we first consider $k = 0$ in (34) without loss of generality. We must first show that it admits a solution $[w_0^1, w_0^2]^T$ whenever $u_0^1 > 0$ and $u_0^2 > 0$. To show this, we first note from (33) that the range of g_2 is the interval $[1/2, 1]$, and thus that of $1 - \frac{2}{3} g_2$ does not contain 0. Hence, we can equivalently deal with the solution to (34) through the inversion of the matrix on the left-hand side, which leads to the equivalent equation

$$w_0^1 = \frac{u_0^1 + g_2(w_0^2) u_0^2}{1 - \frac{2}{3} g_2(w_0^2)} =: f_1(w_0^2, u_0), \quad w_0^2 = \frac{\frac{2}{3} u_0^1 + u_0^2}{1 - \frac{2}{3} g_2(w_0^2)} =: f_2(w_0^2, u_0), \tag{35}$$

where $u_0 := [u_0^1, u_0^2]^T$. Let us fix $u_0^1 > 0$ and $u_0^2 > 0$ and show that the above equation always admits a solution. We first note that f_2 is continuous on the positive real axis, on which it is uniformly bounded by

$$f_2(\cdot, u_0) \le 2u_0^1 + 3u_0^2, \tag{36}$$

and has a limit as $w_0^2 \to +0$, which is positive. Hence, there exist at least one $w_0^2 > 0$ satisfying the second equation in (35), and thus at least one solution $[w_0^1, w_0^2]^T$ to (35) such that $w_0^1 > 0$ and $w_0^2 > 0$. If we repeat the same arguments to $[w_k^1, w_k^2]^T$ $(k \in \mathbf{N}_0)$ and determine the corresponding $z_k^1 > 0$ and $z_k^2 > 0$ $(k \in \mathbf{N}_0)$ by (31) and (32), respectively, then it is obvious that $w^i \in l_{2e+}$ $(i = 1, 2)$, $z^i \in l_{2e+}$ $(i = 1, 2)$ and $(u, [w^T, z^T]) \in R_\Sigma$. Since we can take $u^i \in l_{2e+}$ $(i = 1, 2)$ arbitrarily within the above arguments, we see that the fourth condition of Assumption 1 is satisfied.

Next, it is obvious that Assumption 2 is satisfied. Furthermore, it is also obvious that G_1 is l_{2+} stable with $\gamma_{2/2}(G_1) = 2/3$ while G_2 is also l_{2+} stable (again by $l_2 \subset l_4$) with $\gamma_{2/2}(G_2) = 1$ because the range of g_2 is $[1/2, 1]$. Finally, we can directly confirm that they are both $\mathcal{W}_{2e}(w, \cdot)$-causal, which is easy since they are output-solitary and static. It follows that Σ fulfills all the conditions in the extended small-gain theorem, including the decisive condition $\gamma_{2/2}(G_1)\gamma_{2/2}(G_2) < 1$. Hence, Σ is ensured to be (l_{2+}, l_{2+}) stable.

In order to show a potential application of Theorem 7, let us further consider the closed-loop system Σ in Example 5, where G_1 is given by a causal nonlinear dynamical (strictly) positive system with $\gamma_{2/2}(G_1) = 2/3$, instead of (31). Here, if we take Willems' standpoint in Remark 2 and regard that verifying well-posedness in the modified example is a matter independent from our stability study itself, then a result similar to Example 5 can be obtained clearly for such a system, and thus this example provides an outlook on a potential application of the extended small-gain theorem as long as well-posedness under a given G_1 could be verified.

In addition to this, it would be significant to note that Σ in Example 5 is actually output-unsolitary even though both G_1 and G_2 are output-solitary. To see this, let us reconsider the rearranged input-output relation (35) of Σ, and consider the case with $u_0^1 = 1/2$ and $u_0^2 = 1/3$; it has two quadruples of solution given by $(w_0^1, w_0^2, z_0^1, z_0^2) = (1, 1, 2/3, 1/2)$, $(5/2, 2, 5/3, 2)$. This observation sheds light on the importance of introducing output-unsolitary systems in the (stability) study of complicated systems. This is because it could often be the case in quite complicated systems that G_1 and/or G_2 in Σ could actually be another feedback connection like Σ itself consisting of other two subsystems. This implies that even if all the irreducible components in such complicated systems were assumed to be output-solitary, it would be quite natural to also introduce and investigate output-unsolitary systems, as the present paper has done consistently and thoroughly. As such, the small-gain theorem extended in this paper to the feedback connection of general input-intolerant and output-unsolitary input-output systems is believed to be of great importance in the stability analysis of quite complicated systems.

5. Conclusion

This paper dealt with general input-intolerant output-unsolitary input-output systems, which were dealt with in the study of Zames [1, 2] but not in the later studies [3, 4, 5, 6, 7] on the small-gain theorem. Our ultimate goal lied in deriving an extended version of the small-gain theorem over the original result in [1, 2], which is based on the definition of the gain of systems that generally leads to a value larger than that in the treatment in the later studies. Such a difference in the gain definition is deeply related to the fact that causality of subsystems in the feedback loop was utilized in the later studies while the original study did not assume such causality. This motivated us to first tackle the problem of adequately defining causality for general input-output systems, which itself is believed to be an important issue in its own right. We began with reviewing the well-known causality definition of input-tolerant output-solitary systems, from which a series of amendments of (possible) definitions were motivated until we arrive at adequate mutually equivalent definitions for general input-intolerant output-unsolitary systems. We also clarified the mutual relationship among all the (possible) definitions as well as the subclass of input-intolerant output-unsolitary systems for which each of the possible definitions can stand as an indeed adequate definition. We then proceeded to the ultimate issue on the derivation of an extended version of the small-gain theorem and its usefulness was demonstrated with an example.

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