NP-Hardness and Fixed-Parameter Tractability of the Minimum Spanner Problem

Yusuke Kobayashi *

June 14, 2018

Abstract

For a positive integer t, a *t-spanner* of a graph G is a spanning subgraph in which the distance between every pair of vertices is at most t times of their distance in G. In this paper, we consider the problem of finding a t-spanner with minimum number of edges in a given graph, which we call MINIMUM t-SPANNER PROBLEM. For $t \geq 2$, MINIMUM t-SPANNER PROBLEM is known to be NP-hard in general graphs. When the input graph is planar, it is shown by Brandes and Handke in 1997 that this problem is NP-hard for $t \geq 5$. Since then, the case of $t \in \{2, 3, 4\}$ has been open for more than two decades. The main contribution of this paper is to settle this open problem by showing the NP-hardness of MINIMUM t-SPANNER PROBLEM in planar graphs for $t \in \{2, 3, 4\}$. As a byproduct, we show the NP-hardness of the problem on degree-bounded graphs, which improves previously known degree-bounds. We also present a fixed-parameter algorithm for this problem in which the number of removed edges is regarded as a parameter.

1 Introduction

For a positive integer t, a *t-spanner* of a graph G is a spanning subgraph H such that the distance between every pair of vertices in H is at most t times of that in G. Spanners were introduced in [17, 18] in the context of synchronization in networks. Since then, spanners and related concepts have been studied with applications to several areas such as space efficient routing tables [11, 19], computation of approximate shortest paths [9, 10, 14], distance oracles [2, 21], and so on. Even today, finding a good spanner or its variants in dense graphs is regarded as an important topic in algorithm theory, see recent papers such as [1, 7, 8].

The topic of this paper is a classical but natural and important problem that finds a spanner of minimum size. For a fixed positive integer t, we consider the following problem.

MINIMUM t-SPANNER PROBLEM

Instance. A graph $G = (V, E)$.

*Kyoto University. E-mail: yusuke@kurims.kyoto-u.ac.jp. Supported by ACT-I, JST and by JSPS KAKENHI Grant Numbers JP16K16010 and JP16H03118.

Question. Find a t-spanner $H = (V, E_H)$ of G that minimizes $|E_H|$.

This problem is sometimes called SPARSEST t-SPANNER PROBLEM. If $t = 1$, then this problem is trivial since the optimal solution is obtained from G by just removing parallel edges. Thus, we consider the case of $t \geq 2$. Since MINIMUM t-SPANNER PROBLEM is known to be NP-hard for any $t \geq 2$ in general graphs [4, 17], the main focus of the study is the polynomial solvability for the case when the input graph is in a certain graph class. In [23], Venkatesan et al. studied MINIMUM t-SPANNER PROBLEM for several graph classes such as chordal graphs, convex bipartite graphs, and split graphs. For each graph class, they showed a condition of t for which the problem can be solved in polynomial time. When the input graph is a 4-connected planar triangulation, a PTAS is proposed for MINIMUM 2-SPANNER PROBLEM in [13]. For the weighted version of the problem in which each edge has a positive integer length, Cai and Corneil [5] showed the NP-hardness of MINIMUM t-SPANNER PROBLEM for $t > 1$.

In this paper, we first consider the case when the input graph is planar. For $t \geq 5$, the NP-hardness of MINIMUM t-SPANNER PROBLEM in planar graphs was shown by Brandes and Handke [3] in 1997. Since then, the time complexity of the case of $t \in \{2, 3, 4\}$ has been open for more than two decades. The main contribution of this paper is to settle this open problem by showing the NP-hardness of MINIMUM t-SPANNER PROBLEM in planar graphs for $t \in \{2, 3, 4\}$. Since the case of $t \geq 5$ is settled in [3], our main result is stated as follows.

Theorem 1.1. *For any $t \geq 2$,* MINIMUM t-SPANNER PROBLEM *is NP-hard even if G is restricted to be planar.*

Another interesting special case is the problem on degree-bounded graphs. Cai and Keil [6] showed that MINIMUM 2-SPANNER PROBLEM can be solved in linear time if the maximum degree of the input graph is at most 4, whereas this problem is NP-hard even if the maximum degree is at most 9. They also gave a remark in [6] that MINIMUM 3-SPANNER PROBLEM can be solved in polynomial time if the maximum degree of the input graph is at most 3, whereas this problem is NP-hard even if the maximum degree is at most 8. Since we use degree-bounded graphs in our proof of Theorem 1.1, our argument can improve the degree conditions as follows.

Theorem 1.2. MINIMUM 2-SPANNER PROBLEM *is NP-hard even in planar graphs whose maximum degree is at most* 8.

Theorem 1.3. *For $t \in \{3, 4\}$,* MINIMUM t-SPANNER PROBLEM *is NP-hard even in planar bipartite graphs whose maximum degree is at most* 6.

Determining the exact complexity of the MINIMUM 2-SPANNER PROBLEM on graphs of bounded degree k with $5 \leq k \leq 8$ is posed as an open question in [23]. Theorem 1.2 solves a part of this question, and the case of $5 \leq k \leq 7$ is still open. We summarize the current status of the degree-bounded case in Table 1.

In this paper, we also consider a parameterized version of MINIMUM t-SPANNER PROBLEM and give a fixed-parameter algorithm for it. Since a t-spanner of a connected graph contains $\Omega(|V|)$ edges, the number of edges of a minimum t-spanner is not an appropriate parameter. A natural

	$t = 2$	$t = 3$	$t = 4$
$k = 3$	P [6]	P [6]	open
$k = 4$	P [6]	open	open
$k = 5$	open	open	open
$k = 6$	open	NP-hard (*)	NP-hard (*)
$k = 7$	open	NP-hard (*)	NP-hard (*)
$k = 8$	NP-hard (*)	NP-hard [6]	NP-hard [6]
$k \geq 9$	NP-hard [6]	NP-hard [6]	NP-hard [6]

Table 1: Polynomial solvability of Minimum t-Spanner Problem for graphs of maximum degree at most k, where (*) indicates our results.

parameter is the number of edges that are removed to obtain a minimum t-spanner. That is, we consider the following parameterized problem for fixed t.

Parameterized Minimum t-Spanner Problem

Instance. A graph $G = (V, E)$.

Parameter. A positive integer k.

Question. Find an edge set $E' \subseteq E$ with $|E'| \geq k$ such that $H = (V, E \setminus E')$ is a t-spanner of G or conclude that such E' does not exist.

Our objective is to show that there exists a fixed-parameter algorithm for this problem, where an algorithm is called a *fixed-parameter algorithm* (or an *FPT algorithm*) if its running time is bounded by $f(k)(|V| + |E|)^{O(1)}$ for some function f. Formally, our result is stated as follows.

Theorem 1.4. *For a positive integer t, there exists a fixed-parameter algorithm for* Parameterized Minimum t-Spanner Problem *that runs in $O(k(k^2t(t+1))^{k+1} + |V||E|)$ time.*

To the best of our knowledge, this is the first result on spanners using the number of removed edges as a parameter. We believe that this parameterization is natural and useful also in other problems in which we want to find a maximum edge/vertex set that can be removed under some conditions.

The remainder of this paper is organized as follows. In Section 2, we give some preliminaries. In order to prove Theorem 1.1, we first show the NP-hardness of the problem of finding a minimum dominating set with additional constraints, which is described in Section 3. Then, by showing a reduction from Minimum t-Spanner Problem to this problem, we give a proof of Theorem 1.1 in Section 4.1. A crucial part of this reduction is Proposition 4.1, which shows a relationship between a dominating set in a graph G and a minimum t-spanner in the dual graph G^*. Note that a dominating set in G corresponds to a set of faces in G^*, whereas a minimum t-spanner is a set of edges in G^*. Therefore, they look like completely unrelated objects. Although the proof of

Proposition 4.1 is not so difficult, it is not an easy task to find out this non-intuitive relationship between these two objects. In Section 4.2, we observe properties of graphs used in Section 4.1 and give proofs of Theorems 1.2 and 1.3. Finally, in Section 5, we give a fixed-parameter algorithm for Parameterized Minimum t-Spanner Problem and prove Theorem 1.4.

2 Preliminaries

In this paper, we deal with only undirected graphs with unit length edges. For a graph $G = (V, E)$ and for $u, v \in V$, let $d_G(u, v)$ denote the distance of the shortest path between u and v in G. For a positive integer t, a subgraph $H = (V, E_H)$ of $G = (V, E)$ is said to be a *t-spanner* if $d_H(u, v) \leq t \cdot d_G(u, v)$ for any $u, v \in V$. Since we can remove all the parallel edges and self-loops when we consider t-spanners, we may assume that the input graph is simple. However, since we use graphs with parallel edges in our NP-hardness proof, for convenience in this paper, the term *graph* is used to represent an undirected graph that may contain parallel edges but no self-loops. For a subgraph H of G, the set of edges in H is denoted by $E(H)$. For a vertex set $X \subseteq V$, let $\delta_G(X)$ denote the set of all edges in G connecting X and $V \setminus X$. For a vertex $v \in V$, $\delta_G(\{v\})$ is simply denoted by $\delta_G(v)$, and $|\delta_G(v)|$ is called the *degree* of v. For a positive integer k, a graph is said to be *k-regular* if its every vertex has degree k.

In order to deal with t-spanner, we use an easy but important observation which is stated as follows. Although this idea was used in [6], and almost the same statement was shown in [16], we give a proof for completeness.

Lemma 2.1. *Let t be a positive integer. For a graph $G = (V, E)$, its subgraph $H = (V, E_H)$ is a t-spanner if and only if $d_H(u, v) \leq t$ for any $uv \in E \setminus E_H$.*

Proof. Necessity is obvious, because $d_G(u, v) = 1$ for any $uv \in E \setminus E_H$. To show sufficiency, suppose that $d_H(u, v) \leq t$ for any $uv \in E \setminus E_H$, that is, H contains a path P_{uv} of length at most t connecting u and v. For any pair of vertices $u, v \in V$, we take a shortest u-v path in G that traverses $u = v_0, v_1, \ldots, v_{\ell-1}, v_\ell = v$ in this order, where $\ell = d_G(u, v)$. Then, by concatenating $P_{v_0v_1}, P_{v_1v_2}, \ldots, P_{v_{\ell-1}v_\ell}$ in this order, we obtain a u-v walk in H whose length is at most $t\ell$. This shows that $d_H(u, v) \leq t\ell = t \cdot d_G(u, v)$. □

3 Dominating Set with Degree Constraint

In this section, we show the NP-hardness of the problem of finding a minimum dominating set with additional constraints, which will be used in our proof of Theorem 1.1. For a graph $G = (V, E)$, a vertex set $S \subseteq V$ is called a *dominating set* if every vertex in $V \setminus S$ is adjacent to at least one vertex in S. For a fixed positive integer k, we consider the following problem.

Dominating Set with Degree-k-Constraint

Instance. A graph $G = (V, E)$.

Question. Find a dominating set S that minimizes $|S|$ subject to S contains every vertex of degree at least k.

In this section, we consider the case of $k \in \{4, 5, 6\}$. We note that when $k \geq 4$ and the maximum degree of the input graph is at most three, this problem is equivalent to the problem of finding a minimum dominating set, which is known to be NP-hard [15]. Thus, Dominating Set with Degree-k-Constraint is an NP-hard problem for $k \geq 4$. The objective of this section is to show that this problem is NP-hard even if the input graph is nearly k-edge-connected and planar. Here, we say that a graph $G = (V, E)$ is *nearly k-edge-connected* if

- the minimum degree of G is at least $k - 1$, and
- $|\delta_G(X)| \leq k - 1$ implies $|X| \leq 1$ or $|V \setminus X| \leq 1$ for any $X \subseteq V$.

That is, we show the following proposition in this section.

Proposition 3.1. *For $k \in \{4, 5, 6\}$,* Dominating Set with Degree-k-Constraint *in nearly k-edge-connected planar graphs is NP-hard.*

We first give a proof for the case of $k = 4$ in Section 3.1. Then, by applying a similar argument, we prove the case of $k \in \{5, 6\}$ in Section 3.2. As we will see in Section 4.2, the case of $k = 6$ is not necessary to show the NP-hardness of Minimum 4-Spanner Problem, but we consider this case to simplify the description of Section 4.1.

3.1 Case of $k = 4$

To show the NP-hardness for the case of $k = 4$, we use the fact that the problem of finding a minimum vertex cover is NP-hard even if the input graph is restricted to be planar, 3-regular, and 3-connected [22]. Here, for a graph $G = (V, E)$, a vertex set $S \subseteq V$ is called a *vertex cover* if each edge in E has at least one endpoint in S. We reduce this problem to Dominating Set with Degree-4-Constraint in nearly 4-edge-connected planar graphs.

Suppose we are given a graph $G = (V, E)$ that is planar, 3-regular, and 3-connected. In particular, since G is 3-regular and 3-connected, it is a simple graph. We fix a planar embedding of G. Let $\mathcal{F}$ denote the set of all the faces of G, where each open region of $\mathbb{R}^2 \setminus G$ is called a *face* (see e.g. [12]). For notational convenience, in this paper, a closed region consisting of a face and its boundary is also called a face if no confusion may arise. We construct an instance of Dominating Set with Degree-4-Constraint from G as follows.

- For each face $F \in \mathcal{F}$, we create a new vertex v_F. Let $V_{\mathcal{F}} = \{v_F \mid F \in \mathcal{F}\}$.
- For each edge $e = uv \in E$, we execute the following (see Fig. 1).
  - We replace e with a path of length 11 connecting u and v. Its internal vertices are denoted by $w^e_1, \ldots, w^e_{10}$ in this order from u to v. We add an edge connecting w^e_3 and w^e_8.

– For $i = 1, 2, 4, 5, 6, 7, 9, 10$, we create a new vertex x_i^e and add an edge connecting w_i^e and x_i^e.

– Among the two faces in $\mathcal{F}$ that are adjacent to e, we arbitrarily choose one face, say F, and denote the other face by F'. For $i = 2, 9$, we add two parallel edges connecting x_i^e and v_F. For $i = 1, 4, 5, 6, 7, 10$, we add two parallel edges connecting x_i^e and $v_{F'}$.

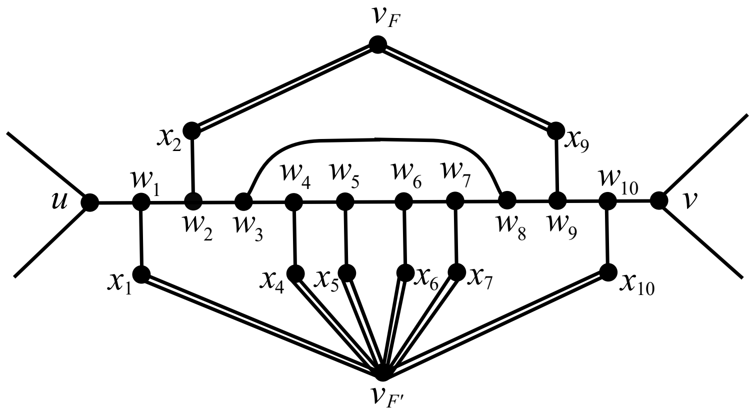

Figure 1: A gadget corresponding to an edge, where w_i^e and x_i^e are simply denoted by w_i and x_i, respectively.

The obtained graph is denoted by $G' = (V', E')$. For $e \in E$, let G'_e be the subgraph of G' induced by

$$\{w_i^e \mid i \in \{1, \dots, 10\}\} \cup \{x_i^e \mid i \in \{1, 2, 4, 5, 6, 7, 9, 10\}\} \cup \{v_F, v_{F'}\},$$

where F and F' are the faces in $\mathcal{F}$ that are adjacent to e.

We now prove the following two claims.

Claim 3.2. *The graph G' defined as above is nearly 4-edge-connected and planar.*

Proof. We can easily see that G' is planar and the minimum degree is three. In what follows, we show that $|\delta_{G'}(X)| \leq 3$ implies $|X| \leq 1$ or $|V' \setminus X| \leq 1$ for any $X \subseteq V'$. Let $X \subseteq V'$ be a vertex set with $|\delta_{G'}(X)| \leq 3$.

In G, suppose that two faces F_1 and F_2 share an edge $e = uv \in E$. Since the degree of v is three in G, there exists another face $F_3 \in \mathcal{F}$ such that F_3 and F_1 share an edge $e_1 \in \delta_G(v)$, and F_3 and F_2 share an edge $e_2 \in \delta_G(v)$. By the construction of G', G'_e contains two edge-disjoint paths P_1 and P_2 each connecting v_{F_1} and v_{F_2}. Similarly, G'_{e_1} contains two edge-disjoint paths Q_1 and Q_2 between v_{F_1} and v_{F_3}, and G'_{e_2} contains two edge-disjoint paths R_1 and R_2 between v_{F_2} and v_{F_3} (see Fig. 2). By concatenating $\{Q_1, Q_2\}$ and $\{R_1, R_2\}$, we can see that the union of G'_{e_1} and G'_{e_2} contains two edge-disjoint paths P_3 and P_4 between v_{F_1} and v_{F_2}. Thus, G' has four edge-disjoint paths P_1, P_2, P_3, and P_4 each connecting v_{F_1} and v_{F_2}. Since $|\delta_{G'}(X)| \leq 3$, v_{F_1} and v_{F_2} are both contained in X or both contained in $V' \setminus X$.

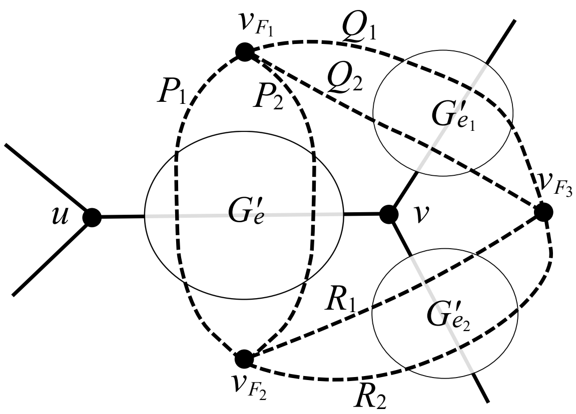

Figure 2: Four edge-disjoint paths between v_{F_1} and v_{F_2}.

By applying this argument for each pair of adjacent faces repeatedly, we have either $V_{\mathcal{F}} \subseteq X$ or $V_{\mathcal{F}} \subseteq V' \setminus X$. By changing the roles of X and $V' \setminus X$ if necessary, we may assume that $V_{\mathcal{F}} \subseteq V' \setminus X$, and our objective is to show that $|X| \leq 1$.

For any edge $zz' \in E'$ with $z, z' \in V' \setminus V_{\mathcal{F}}$, we can easily see by a case analysis that G' contains four edge-disjoint paths from $\{z, z'\}$ to $V_{\mathcal{F}}$,[1] which shows that X contains at most one of z and z', because $|\delta_{G'}(X)| \leq 3$. Thus, X is an independent set, and hence $|\delta_{G'}(X)| = \sum_{z \in X} |\delta_{G'}(z)|$. Since $|\delta_{G'}(z)| = 3$ for any $z \in V' \setminus V_{\mathcal{F}}$, we have $|X| \leq 1$. This completes the proof. □

Claim 3.3. *Let* OPT *be the size of a minimum vertex cover in G, and let* OPT′ *be the optimal value of* Dominating Set with Degree-4-Constraint *in G'. Then,*

$$\mathrm{OPT}' = \mathrm{OPT} + |V_{\mathcal{F}}| + 3|E|.$$

Proof. In order to show $\mathrm{OPT}' \leq \mathrm{OPT} + |V_{\mathcal{F}}| + 3|E|$, let $S \subseteq V$ be a minimum vertex cover in G, that is, $|S| = \mathrm{OPT}$. By using S, we will construct a feasible solution of Dominating Set with Degree-4-Constraint in G'. We first add $V_{\mathcal{F}}$ to S. Then, for each edge $e = uv \in E$, we add w^e_3, w^e_6, and w^e_{10} to S if $u \in S$, and we add w^e_1, w^e_5, and w^e_8 to S otherwise. Here, we recall that e is replaced with a path of length 11 whose internal vertices are denoted by $w^e_1, \dots, w^e_{10}$ in this order from u to v. The obtained vertex set $S' \subseteq V'$ is a dominating set such that $V_{\mathcal{F}} \subseteq S'$ and $|S'| = \mathrm{OPT} + |V_{\mathcal{F}}| + 3|E|$. Therefore, we have $\mathrm{OPT}' \leq \mathrm{OPT} + |V_{\mathcal{F}}| + 3|E|$.

In order to show $\mathrm{OPT}' \geq \mathrm{OPT} + |V_{\mathcal{F}}| + 3|E|$, let $S' \subseteq V'$ be an optimal solution of Dominating Set with Degree-4-Constraint in G', that is, $|S'| = \mathrm{OPT}'$. By using S', we will construct a vertex cover in G. By the definition of the problem, S' contains every vertex of degree at least 4, and hence we have $V_{\mathcal{F}} \subseteq S'$. If $x^e_i \in S'$ for some $e \in E$ and for some $i \in \{1, 2, 4, 5, 6, 7, 9, 10\}$, then we can replace x^e_i with w^e_i keeping the optimality. With this observation, we may assume that

$$x^e_i \notin S' \text{ for any } e \in E \text{ and for any } i \in \{1, 2, 4, 5, 6, 7, 9, 10\}. \tag{1}$$

[1] For example, if $\{z, z'\} = \{w_2, w_3\}$ in Fig. 1, then we have four edge disjoint paths $(w_2, w_1, x_1, V_{F'})$, (w_2, x_2, V_F), $(w_3, w_4, x_4, V_{F'})$, and $(w_3, w_8, w_9, x_9, V_F)$ from $\{z, z'\}$ to $V_{\mathcal{F}}$.

Furthermore, if $|S' \cap \{w_i^e \mid i \in \{1,\dots,10\}\}| \geq 4$ for some $e \in E$, then we can replace $S' \cap \{w_i^e \mid i \in \{1,\dots,10\}\}$ with $\{u, w_3^e, w_6^e, w_{10}^e\}$ or $\{w_1^e, w_5^e, w_8^e, v\}$ keeping the optimality. Since a dominating set has to contain at least three vertices in $\{w_i^e \mid i \in \{1,\dots,10\}\}$ for each $e \in E$ under the assumption (1), we may assume that

$$|S' \cap \{w_i^e \mid i \in \{1,\dots,10\}\}| = 3 \text{ for any } e \in E. \tag{2}$$

If there exists an edge $e = uv \in E$ such that $S' \cap \{u, v\} = \emptyset$, then S' cannot dominate the vertices in $\{w_i^e \mid i \in \{1,\dots,10\}\}$ by the assumptions (1) and (2). Therefore, since S' is a dominating set in G', $S' \cap V$ forms a vertex cover in G. This shows that G has a vertex cover of size $|S' \cap V| = \mathrm{OPT}' - |V_{\mathcal{F}}| - 3|E|$, which shows that $\mathrm{OPT}' \geq \mathrm{OPT} + |V_{\mathcal{F}}| + 3|E|$. □

Claims 3.2 and 3.3 show that the minimum vertex cover problem in planar 3-regular 3-connected graphs can be reduced to Dominating Set with Degree-4-Constraint in nearly 4-edge-connected planar graphs. This completes the proof for the case of $k = 4$ in Proposition 3.1.

We remark here that if a vertex u is on the boundary of some face F in G, then the face of G' containing both u and v_F is surrounded by a cycle of length at most 8. This shows that each face of G' is surrounded by a cycle of length at most 8. Therefore, we have the following corollary.

Corollary 3.4. Dominating Set with Degree-4-Constraint *is NP-hard even if the input graph is a nearly* 4*-edge-connected planar graph in which each face is surrounded by a cycle of length at most* 8*.*

3.2 Case of $k \in \{5, 6\}$

Suppose that we are given a graph $G = (V, E)$, which is planar, 3-regular, and 3-connected. Since a well-known theorem of Petersen [20] states that every 3-regular 2-connected graph has a perfect matching, G has a perfect matching $M \subseteq E$. By duplicating (resp. triplicating) the edges in M, we obtain a graph $\hat{G}$ that is planar, 4-regular (resp. 5-regular), and 3-connected. We fix a planar embedding of $\hat{G}$, where we note that $\hat{G}$ contains a face surrounded by two parallel edges. Note that $S \subseteq V$ is a vertex cover in G if and only if it is a vertex cover in $\hat{G}$. Thus, the problem of finding a minimum vertex cover in planar $(k-1)$-regular 3-connected graphs is NP-hard for $k \in \{5, 6\}$. In what follows, we reduce this problem to Dominating Set with Degree-k-Constraint in nearly k-edge-connected planar graphs for $k \in \{5, 6\}$.

Let $k \in \{5, 6\}$. Suppose we are given a graph $G_k = (V_k, E_k)$ that is planar, $(k-1)$-regular, and 3-connected. Let $\mathcal{F}_k$ denote the set of all the faces of G_k. We construct an instance of Dominating Set with Degree-k-Constraint from G_k as follows.

- For each face $F \in \mathcal{F}_k$, we create a new vertex v_F. Let $V_{\mathcal{F}_k} = \{v_F \mid F \in \mathcal{F}_k\}$.
- For each edge $e = uv \in E_k$, we execute the following (see Fig. 3).
  - We replace e with a path of length 11 connecting u and v. Its internal vertices are denoted by $w_1^e, \dots, w_{10}^e$ in this order from u to v. We add an edge connecting w_3^e and w_8^e.

– For $i = 1, 2, \ldots, 10$, we create a new vertex x_i^e and add an edge connecting w_i^e and x_i^e.

– For $i = 1, 2, 4, 5, 6, 7, 9, 10$, we create a new vertex y_i^e and add an edge connecting w_i^e and y_i^e.

– If $k = 6$, then for $i = 1, 2, \ldots, 10$, we create a new vertex z_i^e and add an edge connecting w_i^e and z_i^e.

– Among the two faces in $\mathcal{F}_k$ that are adjacent to e, we arbitrarily choose one face, say F, and denote the other face by F'.

– For $i = 1, 2, 3, 8, 9, 10$, we add $k-2$ parallel edges connecting x_i^e and v_F. For $i = 4, 5, 6, 7$, we add $k-2$ parallel edges connecting x_i^e and $v_{F'}$.

– For $i = 1, 2, \ldots, 10$, we add $k-2$ parallel edges connecting y_i^e and $v_{F'}$.

– If $k = 6$, then for $i = 1, 2, \ldots, 10$, we add $k-2$ parallel edges connecting z_i^e and $v_{F'}$.

The obtained graph is denoted by $G'_k = (V'_k, E'_k)$.

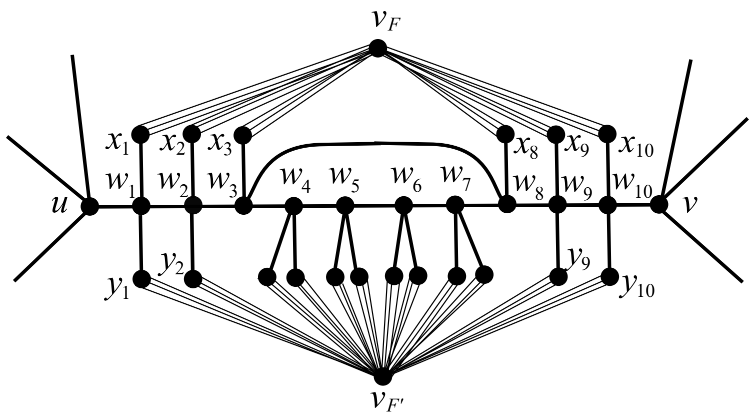

Figure 3: A gadget corresponding to an edge for the case of $k = 5$, where w_i^e, x_i^e, and y_i^e are simply denoted by w_i, x_i, and y_i, respectively.

In a similar way to Claim 3.2, we can show the following claim.

Claim 3.5. *For $k \in \{5, 6\}$, the graph G'_k defined as above is nearly k-edge-connected and planar.*

Proof. We can easily see that G'_k is planar and the minimum degree is $k-1$. In what follows, we show that $|\delta_{G'_k}(X)| \leq k-1$ implies $|X| \leq 1$ or $|V'_k \setminus X| \leq 1$ for any $X \subseteq V'_k$. Let $X \subseteq V'_k$ be a vertex set with $|\delta_{G'_k}(X)| \leq k-1$.

In G_k, suppose that two faces F and F' share an edge $e \in E_k$. Since the gadget corresponding to e contains six edge-disjoint paths connecting v_F and $v_{F'}$, v_F and $v_{F'}$ are both contained in X or both contained in $V'_k \setminus X$. By applying this argument for each pair of adjacent faces repeatedly, we have either $V_{\mathcal{F}} \subseteq X$ or $V_{\mathcal{F}} \subseteq V'_k \setminus X$. By changing the roles of X and $V'_k \setminus X$ if necessary, we may assume that $V_{\mathcal{F}} \subseteq V'_k \setminus X$, and our objective is to show that $|X| \leq 1$.

For any edge $zz' \in E'_k$ with $z, z' \in V'_k \setminus V_{\mathcal{F}_k}$, we can easily see by a case analysis that G'_k contains k edge-disjoint paths from $\{z, z'\}$ to $V_{\mathcal{F}_k}$, which shows that X contains at most one of z and z', because $|\delta_{G'_k}(X)| \leq k-1$. Thus, X is an independent set, and hence $|\delta_{G'_k}(X)| = \sum_{z \in X} |\delta_{G'_k}(z)|$. Since $|\delta_{G'_k}(z)| = k-1$ for any $z \in V'_k \setminus V_{\mathcal{F}_k}$, we have $|X| \leq 1$. This completes the proof. □

We can also obtain the following claim in the same as Claim 3.3.

Claim 3.6. *For $k \in \{5, 6\}$, let OPT_k be the size of a minimum vertex cover in G_k, and let OPT'_k be the optimal value of* Dominating Set with Degree-k-Constraint *in G'_k. Then,*

$$\mathrm{OPT}'_k = \mathrm{OPT}_k + |V_{\mathcal{F}_k}| + 3|E_k|.$$

Proof. In order to show $\mathrm{OPT}'_k \leq \mathrm{OPT}_k + |V_{\mathcal{F}_k}| + 3|E_k|$, let $S \subseteq V_k$ be a minimum vertex cover in G_k, that is, $|S| = \mathrm{OPT}_k$. By using S, we will construct a feasible solution of Dominating Set with Degree-k-Constraint in G'_k. We first add $V_{\mathcal{F}_k}$ to S. Then, for each edge $e = uv \in E_k$, we add w^e_3, w^e_6, and w^e_{10} to S if $u \in S$, and we add w^e_1, w^e_5, and w^e_8 to S otherwise. The obtained vertex set $S' \subseteq V'_k$ is a dominating set such that $V_{\mathcal{F}_k} \subseteq S'$ and $|S'| = \mathrm{OPT}_k + |V_{\mathcal{F}_k}| + 3|E_k|$. Therefore, we have $\mathrm{OPT}'_k \leq \mathrm{OPT}_k + |V_{\mathcal{F}_k}| + 3|E_k|$.

In order to show $\mathrm{OPT}'_k \geq \mathrm{OPT}_k + |V_{\mathcal{F}_k}| + 3|E_k|$, let $S' \subseteq V'_k$ be an optimal solution of Dominating Set with Degree-k-Constraint in G'_k, that is, $|S'| = \mathrm{OPT}'_k$. By using S', we will construct a vertex cover in G_k. By the definition of the problem, S' contains every vertex of degree at least k, and hence we have $V_{\mathcal{F}_k} \subseteq S'$. By the same argument as (1), if x^e_i, y^e_i or z^e_i is contained in S' for some $e \in E_k$ and for some $i \in \{1, \ldots, 10\}$, then it can be replaced with w^e_i keeping the optimality. Therefore, we may assume that

$$x^e_i, y^e_i, z^e_i \notin S' \text{ for any } e \in E_k \text{ and for any } i \in \{1, \ldots, 10\}. \tag{3}$$

Furthermore, by the same argument as (2), we may assume that

$$|S' \cap \{w^e_i \mid i \in \{1, \ldots, 10\}\}| = 3 \text{ for any } e \in E_k. \tag{4}$$

If there exists an edge $e = uv \in E_k$ such that $S' \cap \{u, v\} = \emptyset$, then S' cannot dominate the vertices in $\{w^e_i \mid i \in \{1, \ldots, 10\}\}$ by the assumptions (3) and (4). Therefore, since S' is a dominating set in G'_k, $S' \cap V_k$ forms a vertex cover in G_k. This shows that G_k has a vertex cover of size $|S' \cap V_k| = \mathrm{OPT}'_k - |V_{\mathcal{F}_k}| - 3|E_k|$, which shows that $\mathrm{OPT}'_k \geq \mathrm{OPT}_k + |V_{\mathcal{F}_k}| + 3|E_k|$. □

Claims 3.5 and 3.6 show that the minimum vertex cover problem in planar $(k-1)$-regular 3-connected graphs can be reduced to Dominating Set with Degree-k-Constraint in nearly k-edge-connected planar graphs for $k = 5, 6$. Thus, Dominating Set with Degree-k-Constraint in nearly k-edge-connected planar graphs is NP-hard for $k = 5, 6$. By combining this with Section 3.1, we obtain Proposition 3.1. □

We remark here that the graph G'_5 has additional properties as follows:

- The degree of each vertex of G'_5 is even.

- Each face of G'_5 is surrounded by a cycle of length at most 6.

Therefore, DOMINATING SET WITH DEGREE-5-CONSTRAINT is NP-hard even if the input graph is a nearly 5-edge-connected planar graph satisfying the above conditions.

Corollary 3.7. DOMINATING SET WITH DEGREE-5-CONSTRAINT *is NP-hard even if the input graph is a nearly* 5*-edge-connected planar graph in which each vertex has even degree and each face is surrounded by a cycle of length at most* 6*.*

4 Hardness of Minimum t-Spanner Problem

4.1 Proof of Theorem 1.1

The objective of this section is to show that MINIMUM t-SPANNER PROBLEM is NP-hard even if G is restricted to be planar for $t \in \{2, 3, 4\}$. To prove this, we reduce DOMINATING SET WITH DEGREE-$(t+2)$-CONSTRAINT in nearly $(t+2)$-edge-connected planar graphs to MINIMUM t-SPANNER PROBLEM in planar graphs. Suppose we are given a nearly $(t+2)$-edge-connected planar graph $G = (V, E)$ as an instance of DOMINATING SET WITH DEGREE-$(t+2)$-CONSTRAINT. We fix an embedding of G. Let $\mathcal{F}$ be the set of all the faces of G, and let $G^* = (V^*, E^*)$ be the dual graph of G, where V^* and E^* are the vertex set and the edge set of G^*, respectively (see e.g. [12] for duality of planar graphs). We note that V^* and E^* can be identified with $\mathcal{F}$ and E, respectively. For an edge $e \in E$ that is adjacent to two faces $F, F' \in \mathcal{F}$ in G, we say that an edge $e^* \in E^*$ *corresponds to* e if e^* connects F and F' in G^*.

We now show a relationship between DOMINATING SET WITH DEGREE-$(t+2)$-CONSTRAINT in G and a minimum t-spanner in G^*. We remark here that a dominating set in G corresponds to a set of faces in G^*, whereas a minimum t-spanner is a set of edges in G^*. The following proposition shows a relationship between these two objects that look completely unrelated.

Proposition 4.1. *Let* $G = (V, E)$ *be a nearly* $(t+2)$*-edge-connected planar graph. Let* OPT *be the optimal value of* DOMINATING SET WITH DEGREE-$(t+2)$-CONSTRAINT *in* G*, and let* OPT^* *be the number of edges of a minimum* t*-spanner in* G^**. Then,* $\mathrm{OPT}^* = \mathrm{OPT} - |V| + |E|$*.*

Proof. In order to show $\mathrm{OPT}^* \leq \mathrm{OPT} - |V| + |E|$, let $S \subseteq V$ be an optimal solution of DOMINATING SET WITH DEGREE-$(t+2)$-CONSTRAINT, that is, $|S| = \mathrm{OPT}$. By using S, we will construct a t-spanner in G^*. Since S is a dominating set, for any vertex $v \in V \setminus S$, there exists an edge $e_v \in \delta_G(v)$ that connects v and a vertex in S. Define $E' = \{e_v \mid v \in V \setminus S\}$ (see Fig. 4), and define an edge set $E^*_H \subseteq E^*$ as the edge subset of G^* corresponding to $E \setminus E'$. For any $v \in V \setminus S$, since $|\delta_G(v)| = t + 1$, the edge subset of E^* corresponding to $\delta_G(v) \setminus \{e_v\}$ forms a path of length t connecting the endpoints of e^*_v, where e^*_v is the edge in E^* that corresponds to e_v. Furthermore, this path is contained in $H^* = (V^*, E^*_H)$, since $\delta_G(v) \setminus \{e_v\} \subseteq E \setminus E'$ by the definition of E'. Thus, for any $v \in V \setminus S$, H^* contains a path of length t connecting the endpoints of e^*_v. This shows that H^* is a t-spanner in G^* by Lemma 2.1. Since

$$|E^*_H| = |E| - |E'| = |E| - (|V| - |S|) = |E| - |V| + \mathrm{OPT},$$

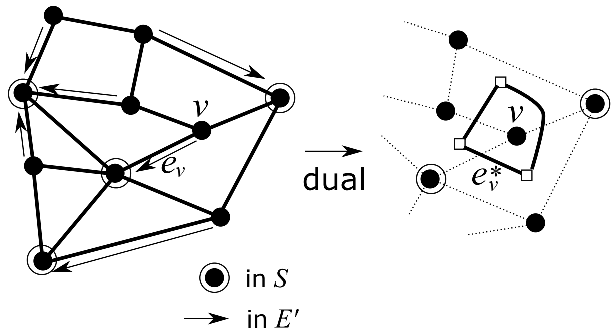

Figure 4: In the left figure, an edge e_v is represented by an arrow from v to a vertex in S. In the right figure, the edges in E^* corresponding to $\delta_G(v)$ are represented by solid lines.

we obtain $\mathrm{OPT}^* \leq \mathrm{OPT} - |V| + |E|$.

In order to show $\mathrm{OPT}^* \geq \mathrm{OPT} - |V| + |E|$, let $E^*_H \subseteq E^*$ be an edge set of a minimum t-spanner in G^*, that is, $|E^*_H| = \mathrm{OPT}^*$. By using E^*_H, we will construct a feasible solution of Dominating Set with Degree-$(t+2)$-Constraint in G. By Lemma 2.1, for each $e^* \in E^* \setminus E^*_H$, the subgraph $H^* = (V^*, E^*_H)$ of G^* contains a path P_{e^*} of length t connecting the endpoints of e^*. Since P_{e^*} and e^* form a cycle of length at most $t+1$, they correspond to a cut of size at most $t+1$ in G, which is denoted by $\delta_G(X)$ for some $X \subseteq V$. Since G is nearly $(t+2)$-edge-connected and $|\delta_G(X)| \leq t+1$, we have either $|X| = 1$ or $|V \setminus X| = 1$. By combining this with the fact that $\delta_G(X)$ contains the edge $e \in E$ corresponding to e^*, we can see that there exists an endpoint v_e of e such that $\delta_G(X) = \delta_G(v_e)$. Since P_{e^*} corresponds to $\delta_G(v_e) \setminus \{e\}$, we have $\delta_G(v_e) \setminus \{e\} \subseteq E_H$, where E_H is the subset of E corresponding to E^*_H. Define $V' = \{v_e \mid e \in E \setminus E_H\}$. Then, for any distinct edges $e, e' \in E \setminus E_H$, $v_{e'}$ is not an endpoint of e, because $\delta_G(v_{e'}) \cap (E \setminus E_H) = \{e'\}$. This shows that, for any $e \in E \setminus E_H$, e connects $v_e \in V'$ and a vertex in $V \setminus V'$, which means that $V \setminus V'$ is a dominating set in G. Since $|\delta_G(v_e)| = t+1$ holds for any $v_e \in V'$, $V \setminus V'$ is a feasible solution of Dominating Set with Degree-$(t+2)$-Constraint in G. Since

$$|V \setminus V'| = |V| - |E \setminus E_H| = |V| - (|E| - |E^*_H|) = |V| - |E| + \mathrm{OPT}^*,$$

we obtain $\mathrm{OPT}^* \geq \mathrm{OPT} - |V| + |E|$. □

This proposition shows that Dominating Set with Degree-$(t+2)$-Constraint in nearly $(t+2)$-edge-connected planar graphs can be reduced to Minimum t-Spanner Problem in planar graphs. By combining this with Proposition 3.1, we have that Minimum t-Spanner Problem is NP-hard even if G is restricted to be planar for $t \in \{2, 3, 4\}$. Since the NP-hardness for the case of $t \geq 5$ is shown in [3], this completes the proof of Theorem 1.1.

4.2 Degree Bounded Case

In this subsection, we consider the case with degree constraints and prove Theorems 1.2 and 1.3.

Recall that Corollary 3.4 shows that Dominating Set with Degree-4-Constraint is NP-hard even if the input graph is a nearly 4-edge-connected planar graph in which each face is surrounded by a cycle of length at most 8. This shows that the maximum degree of the dual graph G^* is at most 8. Therefore, Proposition 4.1 shows that Minimum 2-Spanner Problem is NP-hard in graphs of maximum degree at most 8, which completes the proof of Theorem 1.2.

We can apply a similar argument to Minimum 3-Spanner Problem. Corollary 3.7 shows that Dominating Set with Degree-5-Constraint is NP-hard even if the input graph is a nearly 5-edge-connected planar graph in which each vertex has even degree and each face is surrounded by a cycle of length at most 6. If each vertex has even degree in a graph $G = (V, E)$, then $|\delta_G(X)|$ is even for any $X \subseteq V$, which shows that the dual graph G^* of G contains no odd cycles. Thus, G^* is a bipartite graph whose maximum degree is at most 6. Therefore, Proposition 4.1 shows that Minimum 3-Spanner Problem is NP-hard in planar bipartite graphs whose maximum degree is at most 6. Furthermore, since bipartite graphs have no cycle of length 3, Lemma 2.1 shows that Minimum 4-Spanner Problem and Minimum 3-Spanner Problem are equivalent in bipartite graphs. Thus, we have Theorem 1.3.

5 An FPT Algorithm for the Parameterized Problem

In this section, we give a fixed-parameter algorithm for Parameterized Minimum t-Spanner Problem and prove Theorem 1.4. In our proof, we present an algorithm that converts a given instance to an equivalent smaller instance, where such an operation is called *kernelization* in the context of parameterized algorithms.

Since we can deal with each connected component separately, we may assume that $|E| = \Omega(|V|)$. For each $e \in E$, we compute a shortest path P_e in $G - e$ connecting the end vertices of e. If the length of P_e is at least $t + 1$, then there is no cycle of length at most $t + 1$ that contains e. By Lemma 2.1, this shows that a subgraph H is a t-spanner of G if and only if $H - e$ is a t-spanner of $G - e$. Therefore, we can remove e from G to obtain an equivalent smaller instance. By repeating this procedure, we obtain a graph in which the length of P_e is at most t for each $e \in E$.

This procedure can be implemented with running time $O(|V||E|)$ as follows. By applying the breadth first search from each vertex, we first compute P_e for every edge $e \in E$ in $O(|V||E|)$ time. Then, remove the edge set $F := \{e \in E \mid \text{length of } P_e \text{ is at least } t + 1\}$ from G. Since no edge in F is contained in cycles of length at most $t + 1$, by Lemma 2.1, we can remove F to obtain an equivalent smaller instance. We note that $E(P_e) \subseteq E \setminus F$ for any $e \in E \setminus F$, since $E(P_e) \cup \{e\}$ forms a cycle of length at most $t + 1$. That is, removing F does not affect P_e for $e \in E \setminus F$.

Therefore, we may assume that the length of P_e is at most t for each $e \in E$. Recall that our objective is to find an edge set $E' \subseteq E$ with $|E'| \geq k$ such that $H = (V, E \setminus E')$ is a t-spanner of G (if exists). In what follows, we divide the problem into two cases, and consider each separately.

We first consider the case when the obtained graph has at least $k^2t(t + 1)$ edges. In this case, we can find a desired edge set E' in $O(|V||E|)$ time, which is formally stated as follows.

Lemma 5.1. *Let $G = (V, E)$ be a graph with at least $k^2t(t+1)$ edges. Suppose that for each $e \in E$, $G - e$ contains a path P_e of length at most t connecting the end vertices of e. Then, in $O(|V||E|)$*

time, we can find an edge set $E' \subseteq E$ with $|E'| \geq k$ such that $H = (V, E \setminus E')$ is a t-spanner of G.

Proof. We first compute P_e for every $e \in E$, which can be done in $O(|V||E|)$ time. Let $E_0 = \emptyset$. For $i = 1, \dots, k$ in this order, we execute the following procedure.

- Let F_i be a set of kt edges in $E \setminus E_{i-1}$.
- Define $E_i = E_{i-1} \cup \{E(P_e) \cup \{e\} \mid e \in F_i\}$.

Then, we obtain a sequence of edge sets $E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots \subseteq E_k \subseteq E$. We note that we can choose F_i as above, since

$$\begin{aligned} |E \setminus E_{i-1}| &= |E| - \left| \bigcup_{j=1}^{i-1} \{E(P_e) \cup \{e\} \mid e \in F_j\} \right| \\ &\geq |E| - \sum_{j=1}^{i-1} \sum_{e \in F_j} (|E(P_e)| + 1) \\ &\geq k^2 t(t+1) - (k-1)kt(t+1) = kt(t+1) > kt \end{aligned}$$

holds for $i = 1, \dots, k$. This procedure can be executed in linear time.

Next, for $i = k, k-1, \dots, 1$ in this order, we pick up an edge e_i in $F_i \setminus \bigcup_{j=i+1}^{k} E(P_{e_j})$, which can be done in linear time. Note that this procedure is possible because $|F_i| = kt > (k-1)t \geq |\bigcup_{j=i+1}^{k} E(P_{e_j})|$. Let $E' = \{e_1, \dots, e_k\}$. By the choice of e_i, we have $e_i \notin E(P_{e_j})$ if $i < j$. Furthermore, if $i > j$, then

$$e_i \in F_i \subseteq E \setminus E_{i-1} \subseteq E \setminus E_j \subseteq E \setminus (E(P_{e_j}) \cup \{e_j\}),$$

which means that $e_i \neq e_j$ and $E(P_{e_j})$ does not contain e_i. Therefore, we have $e_i \neq e_j$ and $e_i \notin E(P_{e_j})$ for any distinct $i, j \in \{1, \dots, k\}$. This shows that $|E'| = k$ and $E(P_{e_i}) \subseteq E \setminus E'$ for any $i \in \{1, \dots, k\}$. Since the length of P_{e_i} is at most t for each i, $H = (V, E \setminus E')$ is a t-spanner of G by Lemma 2.1. □

We next consider the case when the obtained graph has less than $k^2 t(t+1)$ edges, i.e., $|E| < k^2 t(t+1)$. In this case, we check whether $H = (V, E \setminus E')$ is a t-spanner of G or not for every subset E' of E with $|E'| = k$. Since the number of possible choices of E' is at most $\binom{|E|}{k} = O(|E|^k)$ and we can check whether H is a t-spanner or not in $O(k|E|)$ time by Lemma 2.1, the total running time is $O(k|E|^{k+1}) = O(k(k^2 t(t+1))^{k+1})$. This completes the proof of Theorem 1.4.

6 Conclusion

In this paper, we showed the NP-hardness of Minimum t-Spanner Problem in planar graphs for $t \geq 2$, which was unknown for more than two decades. We also showed the NP-hardness of Minimum t-Spanner Problem for some degree-bounded cases. As in Table 1, there are several cases for which polynomial solvability is still unknown. For example, it is an interesting open question to determine the exact complexity of MINIMUM 2-SPANNER PROBLEM on graphs of bounded degree k with $5 \leq k \leq 7$. Since there are many variants of spanners such as additive spanners and (α, β)-spanners, it is also interesting to determine the complexity of variants of MINIMUM t-SPANNER PROBLEM.

We introduced a parameterized version of MINIMUM t-SPANNER PROBLEM, in which the number of removed edges is regarded as a parameter. We believe that this parameterization is natural and useful also in other problems in which we want to find a maximum edge/vertex set that can be removed under some conditions.

Acknowledgement

The author thanks the anonymous reviewers for their valuable comments.

References

[1] Amir Abboud and Greg Bodwin. The 4/3 additive spanner exponent is tight. *J. ACM*, 64(4):28:1–28:20, 2017.

[2] Surender Baswana and Telikepalli Kavitha. Faster algorithms for approximate distance oracles and all-pairs small stretch paths. In *Proceedings of the Forty-seventh Annual IEEE Symposium on Foundations of Computer Science (FOCS'06)*, pages 591–602, 2006.

[3] Ulrik Brandes and Dagmar Handke. NP-completeness results for minimum planar spanners. In *Proceedings of the 23rd International Workshop Graph-Theoretic Concepts in Computer Science (WG'97)*, pages 85–99, 1997.

[4] Leizhen Cai. NP-completeness of minimum spanner problems. *Discrete Applied Mathematics*, 48(2):187–194, 1994.

[5] Leizhen Cai and Derek G. Corneil. Tree spanners. *SIAM J. Discrete Math.*, 8:359–387, 1995.

[6] Leizhen Cai and Mark Keil. Spanners in graphs of bounded degree. *Networks*, 24(4):233–249, 1994.

[7] Shiri Chechik. New additive spanners. In *Proceedings of the Twenty-fourth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA'13)*, pages 498–512, 2013.

[8] Shiri Chechik and Christian Wulff-Nilsen. Near-optimal light spanners. In *Proceedings of the Twenty-seventh Annual ACM-SIAM Symposium on Discrete Algorithms (SODA'16)*, pages 883–892, 2016.

[9] Edith Cohen. Fast algorithms for constructing t-spanners and paths with stretch t. *SIAM Journal on Computing*, 28(1):210–236, 1998.

[10] Edith Cohen. Polylog-time and near-linear work approximation scheme for undirected shortest paths. *J. ACM*, 47(1):132–166, 2000.

[11] Lenore J. Cowen and Christopher G. Wagner. Compact roundtrip routing in directed networks. *Journal of Algorithms*, 50(1):79–95, 2004.

[12] Reinhard Diestel. *Graph Theory, 4th ed.* Springer, 2010.

[13] William Duckworth, Nicholas C Wormald, and Michele Zito. A PTAS for the sparsest 2-spanner of 4-connected planar triangulations. *Journal of Discrete Algorithms*, 1(1):67–76, 2003.

[14] Michael Elkin. Computing almost shortest paths. *ACM Trans. Algorithms*, 1(2):283–323, 2005.

[15] Michael R. Garey and David S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness.* W. H. Freeman & Co., New York, NY, USA, 1979.

[16] M.S. Madanlal, G. Venkatesan, and C. Pandu Rangan. Tree 3-spanners on interval, permutation and regular bipartite graphs. *Information Processing Letters*, 59(2):97–102, 1996.

[17] David Peleg and Alejandro A. Schäffer. Graph spanners. *J. Graph Theory*, 13(1):99–116, 1989.

[18] David Peleg and Jeffrey D. Ullman. An optimal synchronizer for the hypercube. *SIAM J. Computing*, 18(4):740–747, 1989.

[19] David Peleg and Eli Upfal. A trade-off between space and efficiency for routing tables. *J. ACM*, 36(3):510–530, 1989.

[20] Julius Petersen. Die Theorie der regulären graphs. *Acta Mathematica*, 15:193–220, 1891.

[21] Mikkel Thorup and Uri Zwick. Approximate distance oracles. *J. ACM*, 52(1):1–24, 2005.

[22] Ryuhei Uehara. NP-complete problems on a 3-connected cubic planar graph and their applications. Technical report, Tokyo Woman's Christian University, Technical Report TWCU-M-0004, 1996.

[23] G. Venkatesan, U. Rotics, M.S. Madanlal, J.A. Makowsky, and C.Pandu Rangan. Restrictions of minimum spanner problems. *Information and Computation*, 136(2):143–164, 1997.