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THE L_∞/L_2 HANKEL OPERATOR/NORM OF SAMPLED-DATA SYSTEMS*

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Abstract. This paper is concerned with the Hankel operator and the Hankel norm of sampled-data systems. Even though these systems are intrinsically periodically time-varying, no studies have taken this important feature into account in the treatment of the Hankel operator/norm. We characterize the Hankel operator/norm of sampled-data systems adequately under the treatment with the L_2 norm for the past input and the L_∞ norm for the future output. Such treatment still captures the essential issue on the periodicity but contributes to neat arguments compared with the case when the L_2 norm is also considered on the future output.

Key words. sampled-data systems, Hankel norm, disturbance rejection, L_2, L_∞

AMS subject classifications. 93C57, 93B28, 47N70, 93C05

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1. Introduction.

Most systems in engineering and science are dynamical systems, which are characterized as such systems whose present output is not determined solely by the present input but is affected by the past input as well. Hence, it is important to quantify the effect of the past input on the future output. Roughly speaking, the mapping between the past input and the future output is called the Hankel operator and its norm is called the Hankel norm [1]. Studies on the Hankel operator/norm have attracted much attention in control theory because of their relevance to the model reduction and related problems of dynamical systems [2, 3, 4, 5, 6]; when a dynamical system is approximated by another, the error system is also dynamical and one of the reasonable ways for assessing the effectiveness of the approximation is to evaluate the Hankel norm. It is often the case that both the past input and the future output for stable continuous-time systems are evaluated in the L_2 norm, but there is also a study that is interested in evaluating the output in the L_∞ norm [7]. In this alternative situation, it has been clarified for continuous-time linear time-invariant (LTI) systems that the Hankel norm equals the corresponding induced norm, where the latter is relevant to the input-output behavior with respect to the *future* input and the future output. It is also well-known that the above induced norm is further equal to the standard H_2 norm for the multi-input single-output (MISO) case [7, 8, 9] (and thus the relationship among the L_∞/L_2 Hankel/induced norms and H_2 norm, whose relevance to impulse responses is well known, is also well-understood even for the multioutput case). Hence, the Hankel norm in that situation is important also as a measure for disturbance rejection ability in a usual sense.

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This paper deals with a relevant topic for stable sampled-data systems. Specifically, we first assume that the continuous-time plant is LTI and the controller is also LTI in the discrete-time sense. We then assume that the disturbance input affecting the sampled-data system is a continuous-time signal and that the output to be evaluated is also a continuous-time signal. Under these assumptions, we extend the study of the Hankel operator for a continuous-time LTI system to such a sampled-data system. To our best knowledge, there exists only a single study in that direction for sampled-data systems [10]. This pioneering study, however, does not properly capture the quite essential feature of sampled-data systems in terms of their continuous-time input-output behavior; because of the periodic action of the sampler, the input-output characteristics are h-periodic, where h is the sampling period. Hence, it would matter how to determine the time instant at which the past and future should be separated when we develop a study on the Hankel operator/norm of sampled-data systems. Nevertheless, this very important time instant was fixed in that study to a sampling instant.

The present paper aims at developing more sophisticated and appropriate arguments on the Hankel operator/norm of sampled-data systems. In tackling such a study, we take a standpoint slightly different from that in [10] and evaluate the future output in the L_∞ norm rather than the L_2 norm. Thus, we actually consider what we call the L_∞/L_2 Hankel operator/norm of sampled-data systems. The authors believe that slightly modifying the direction in this way is not essential in amending the study in [10] in its treatment of the instant separating the past and future and that similar arguments can also be developed for the L_2/L_2 Hankel operator/norm dealt with in that study. Still, working in the modified direction is convenient because of the properties of the L_∞ space. Indeed, by working on what we call the overlap L_∞/L_2 Hankel operator, we can avoid introducing two finite-rank operators whose composition gives the mapping from the past input to the future output (where their explicit representations would become rather messy in sampled-data systems unless the past and future are separated at a sampling instant as in [10]). Furthermore, we can clarify the relationship between the modified direction and two other related studies; one is the aforementioned study on the L_∞/L_2 Hankel norm of *continuous-time* LTI systems [7] and the other on the L_∞/L_2-*induced norm* of sampled-data systems [11, 12] (including some relevance to the H_2 norm of sampled-data systems [12, 13, 14]).

The organization of this paper is as follows. Section 2 first revisits the study in [10] on the Hankel norm of sampled-data systems, raises an issue therein that is crucial and motivates the present study, and gives a brief sketch of the problem tackled in the present paper. The lifting treatment of sampled-data systems used in our theoretical development is also reviewed. Section 3 first gives the formal definitions of the L_∞/L_2 Hankel operator/norm and introduces the Hankel operator matrix playing a central role in this study. It is also argued that an approach that views the "past" and "future" to have an overlap by a single sampling interval is helpful for the following arguments as an intermediate step. The reasons why this paper deals with the L_∞/L_2 Hankel operator/norm instead of the usual L_2/L_2 Hankel operator/norm are also stated, and some preliminary study is provided as to when the L_∞/L_2 Hankel operator is well-definable (even though the L_∞/L_2 Hankel norm is always well-defined for stable sampled-data systems). Section 4 tackles the problem raised in section 2 and gives the main results on the characterization of the L_∞/L_2 Hankel operator/norm of sampled-data systems including the associated numerical computation methods. More importantly, the relevance of this computation method to determining the worst instant at which the past and future are separated (if it is well-definable) is also discussed. Section 5 gives a numerical example for the arguments developed in this paper and confirms the validity of the standpoint of this paper. Section 6 gives concluding remarks on the paper.

The notation in this paper is as follows. $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0$ implies $\mathbb{N} \cup \{0\}$. $\mathbb{R}^\nu$ denotes the set of real ν-vectors, whose 2-norm and ∞-norm are denoted by $|x|_2$ and $|x|_\infty$, respectively. For a vector function $z(\cdot)$ defined on $[\Theta, \infty)$, its $L_{\infty,\infty}[\Theta, \infty)$ norm and $L_{\infty,2}[\Theta, \infty)$ norm are defined as in [7] by

$$\|z(\cdot)\|_{\infty,p}^{[\Theta]} := \operatorname*{ess\,sup}_{\Theta \le t < \infty} |z(t)|_p, \quad p = 2, \infty, \tag{1}$$

if the right-hand side is well-defined. The function space of such $z(\cdot)$ on $[\Theta, \infty)$ endowed with the associated norm is denoted by $L_{\infty,p}[\Theta, \infty)$ $(p = 2, \infty)$, and the simplified notation $L_\infty[\Theta, \infty)$ (or even L_∞) is also used occasionally to mean $L_{\infty,\infty}[\Theta, \infty)$ and/or $L_{\infty,2}[0, \infty)$. Furthermore, we sometimes take the interval $[0, h)$ instead and refer to the function space $L_\infty[0, h)$. On the other hand, the $L_2(-\infty, \Theta)$ norm of a vector function $w(\cdot)$ defined on $(-\infty, \Theta)$ is denoted by

$$\|w(\cdot)\|_{2-}^{[\Theta]} := \left(\int_{-\infty}^{\Theta} w(t)^T w(t) dt \right)^{1/2}. \tag{2}$$

We also say $w \in L_2(-\infty, \Theta)$ even for w defined on $(-\infty, \infty)$ when $w(t) = 0,\ t \ge \Theta$. We also use the shorthand notation L_2 for $L_2(-\infty, \Theta)$.

This paper is a completely rewritten and substantially extended version of a very early conference version [15], e.g., with many more theorems (and all their proofs) covering a much larger scope as well as a numerical example.

2. Revisiting Hankel operators in sampled-data systems and lifting treatment of sampled-data systems. This section briefly revisits the study on the Hankel operator/norm of sampled-data systems [10] and then reveals a quite important missing issue in the study. Then, we state why this paper is interested in what we call the L_∞/L_2 Hankel operator/norm (whose precise definitions will be given in the following section). Finally, we review the lifting treatment of sampled-data systems [16, 17, 18] to facilitate the following arguments.

2.1. Revisiting Hankel operators in sampled-data systems. Many studies on Hankel operators for LTI systems have been conducted, in most of which the input in $L_2(-\infty, 0)$ and the output in $L_2[0, \infty)$ are regarded as the past input and future output, respectively. This paper is interested in the study of the Hankel operator/norm for sampled-data systems consisting of a continuous-time plant and a discrete-time controller connected through the ideal sampler and the zero-order hold. Such a study can be useful for model reduction, approximation, and disturbance rejection problems relevant to sampled-data systems, and one such typical problem is the digital redesign of a continuous-time controller in a continuous-time control system. Indeed, the pioneering work on the Hankel norm of sampled-data systems [10] was conducted with this kind of applications in mind, where the past input was taken from $L_2(-\infty, 0)$ and the future output was regarded as an element in $L_2[0, \infty)$ as in the continuous-time case.

Unfortunately, however, we must point out that a very important issue has been neglected in that study. First of all, we note that even though the plant itself was assumed to be LTI and the controller itself was also assumed to be LTI in the discrete-time sense, it does not mean that the dynamical characteristics of the sampled-data systems from the continuous-time external input to the continuous-time controlled output are LTI. Instead, because of the periodic sampling of signals, a very important feature of sampled-data systems is that their dynamical characteristics are periodically time-varying. Hence, unlike in the Hankel operator of LTI systems, it should matter significantly when to take the time instant that separates the past and future to define the Hankel operator/norm. Nevertheless, this very important instant was simply taken as a sampling instant without any specific reasoning in [10]. Even though this choice significantly facilitates the arguments, it fails to capture the aforementioned most significant feature of sampled-data systems. Instead, it is reasonable to consider that any time instant Θ during one sampling interval should be a candidate for the time instant separating the past and future. The purpose of our paper is exactly to pursue such a direction for more sophisticated and appropriate arguments on the Hankel operator/norm of sampled-data systems.

However, we actually tackle a slightly modified version of the problem by considering the L_∞ norm rather than the L_2 norm of the future output. In other words, we study what we call the L_∞/L_2 Hankel norm of sampled-data systems (and an operator associated with the L_∞/L_2 Hankel norm called the L_∞/L_2 Hankel operator, whose precise definition will become clear as our arguments proceed). The present paper confines itself to the L_∞/L_2 Hankel norm of sampled-data systems because, in the authors' view, its treatment is less complicated than the case for dealing with the L_2 norm of the future output (i.e., the L_2/L_2 Hankel norm as in [10]) when the past and future are to be separated at an arbitrary intersample instant (as stated in the introduction). Another reason why we deal with the L_∞/L_2 Hankel norm is that we can show its equivalence to the L_∞/L_2-induced norm [11, 12] as in the continuous-time case [7]. We will give some further words on the standpoint of our paper later in section 3.2.

2.2. Lifting treatment of sampled-data systems. Let us consider the internally stable sampled-data system Σ_{SD} shown in Figure 1 consisting of the continuous-time LTI generalized plant P and the discrete-time LTI controller Ψ together with the ideal sampler $\mathcal{S}$ and the zero-order hold $\mathcal{H}$ operating at sampling period h in a synchronous fashion. Solid lines and dashed lines are used in this figure to represent continuous-time signals and discrete-time signals, respectively. We assume that P and Ψ are described respectively by

$$P: \begin{cases} \dfrac{dx}{dt} = Ax + B_1 w + B_2 u, \\ z = C_1 x + D_{12} u, \\ y = C_2 x, \end{cases} \tag{3}$$

$$\Psi: \begin{cases} \psi_{k+1} = A_\Psi \psi_k + B_\Psi y_k, \\ u_k = C_\Psi \psi_k + D_\Psi y_k, \end{cases} \tag{4}$$

where $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^{n_w}$, $u(t) \in \mathbb{R}^{n_u}$, $z(t) \in \mathbb{R}^{n_z}$, $y(t) \in \mathbb{R}^{n_y}$, $\psi_k \in \mathbb{R}^{n_\Psi}$, $y_k = y(kh)$, and $u(t) = u_k$ $(kh \le t < (k+1)h)$.[1] It would be worth noting that we have assumed "$D_{11} = 0$" and "$D_{21} = 0$" in (3) and that these assumptions

[1] We remark that this property of $u(t)$ leads to the right continuity of z throughout the paper.

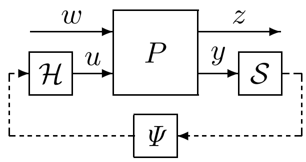

FIG. 1. *Sampled-data system* $\Sigma_{\rm SD}$.

are natural. This is because, even though the main topic of the present paper is to study the L_∞/L_2 Hankel operator/norm of the sampled-data system $\Sigma_{\rm SD}$, we are also partially interested in the relationship between this Hankel norm and the L_∞/L_2-induced norm[2] of the same $\Sigma_{\rm SD}$; the above assumptions are necessary (and sufficient by the stability of $\Sigma_{\rm SD}$) for its L_∞/L_2-induced norm to be bounded/well-defined (note that "$D_{21} \neq 0$" leads to sampling of signals in L_2, which is not well-behaved [19]).

As mentioned earlier, the continuous-time input-output behavior from the input w to the output z in the sampled-data system $\Sigma_{\rm SD}$ is h-periodic. The lifting technique [16, 17, 18] is often used to describe the behavior equivalently in a simpler way, i.e., in an LTI fashion, which is based on the lifted representation of continuous-time signals. Lifting yields from the continuous-time vector-valued function $f(\cdot)$ the sequence $\{\widehat{f}_k(\theta)\}$ of functions on $[0,h)$ given by

$$\widehat{f}_k(\theta) = f(kh+\theta) \quad (0 \leq \theta < h). \tag{5}$$

In the following, we assume that kh are the sampling instants. Obviously, this particularly means that time 0 is also a sampling instant. Thus, recalling the arguments in the preceding subsection, one might question why such an assumption could be acceptable. However, this is not a problem at all since we may simply refer to the instant separating the future and past as Θ (rather than 0), where we may assume $\Theta \in [0,h)$ without loss of generality. Hence, under the above assumption and with the lifted representations of w and z, we may simply follow the standard arguments: the sampled-data system $\Sigma_{\rm SD}$ can be described by

$$\Sigma_{\rm SD} : \begin{cases} \xi_{k+1} = \mathcal{A}\xi_k + \mathcal{B}\widehat{w}_k, \\ \widehat{z}_k \quad\ = \mathcal{C}\xi_k + \mathcal{D}\widehat{w}_k, \end{cases}$$

where we define $\xi_k := [x(kh)^T \ \psi_k^T]^T$, the matrix

$$\mathcal{A} = \begin{bmatrix} A_d + B_{2d}D_\Psi C_{2d} & B_{2d}C_\Psi \\ B_\Psi C_{2d} & A_\Psi \end{bmatrix} : \mathbb{R}^{n+n_\Psi} \to \mathbb{R}^{n+n_\Psi}, \tag{6}$$

and the operators

$$\mathcal{B} = J_\Sigma \mathbf{B}_1 : L_2[0,h) \to \mathbb{R}^{n+n_\Psi}, \tag{7}$$

$$\mathcal{C} = \mathbf{M}_1 C_\Sigma : \mathbb{R}^{n+n_\Psi} \to L_\infty[0,h), \tag{8}$$

$$\mathcal{D} = \mathbf{D}_{11} : L_2[0,h) \to L_\infty[0,h), \tag{9}$$

[2]Unlike the case with the Hankel norm, this induced norm is relevant to the input and output both on $[0,\infty)$. Furthermore, it would be worth remarking that this induced norm is *not affected* by how time 0 is defined (i.e., whether it is a sampling instant or an intersample instant).

where

$$A_d := \exp(Ah),\ B_{2d} := \int_0^h \exp(A\theta) B_2 d\theta,\ C_{2d} := C_2, \tag{10}$$

$$J_\Sigma := \begin{bmatrix} I \\ 0 \end{bmatrix} \in \mathbb{R}^{(n+n_\Psi)\times n},\ C_\Sigma := \begin{bmatrix} I & 0 \\ D_\Psi C_{2d} & C_\Psi \end{bmatrix}, \tag{11}$$

$$\mathbf{B}_1 w = \int_0^h \exp(A(h-\theta)) B_1 w(\theta) d\theta, \tag{12}$$

$$\left(\mathbf{M}_1 \begin{bmatrix} x \\ u \end{bmatrix}\right)(\theta) = M_1 \exp(A_2\theta) \begin{bmatrix} x \\ u \end{bmatrix} A_2 := \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix},\ M_1 := \begin{bmatrix} C_1 & D_{12} \end{bmatrix}, \tag{13}$$

$$(\mathbf{D}_{11} w)(\theta) = \int_0^\theta C_1 \exp(A(\theta-\tau)) B_1 w(\tau) d\tau. \tag{14}$$

Note that the matrix $\mathcal{A}$ is stable, i.e., has all its eigenvalues in the open unit disc by the stability assumption of Σ_{SD}.

3. Quasi L_∞/L_2 Hankel operators, the L_∞/L_2 Hankel norm and the overlap L_∞/L_2 Hankel operator.

3.1. Quasi L_∞/L_2 Hankel operators and the L_∞/L_2 Hankel norm. As mentioned in the preceding section, we consider separating the past and future at $\Theta \in [0,h)$ when we study how to define the Hankel norm of the sampled-data system Σ_{SD} appropriately. In this case, we consider the (past) input w belonging to $L_2(-\infty,\Theta)$ and the corresponding (future) output z viewed as an element in $L_{\infty,p}[\Theta,\infty)$ (where either $p=2$ or $p=\infty$, depending on the context). We also refer to the corresponding mapping between $w \in L_2(-\infty,\Theta)$ and $z \in L_{\infty,p}[\Theta,\infty)$ as the quasi L_∞/L_2 Hankel operator at Θ and denote it by $\mathbf{H}_p^{[\Theta]}$. Its norm defined as

$$\|\mathbf{H}_p^{[\Theta]}\|_{\infty,p/2-}^{[\Theta]} := \sup_{w\in L_2(-\infty,\Theta)} \frac{\|z\|_{\infty,p}^{[\Theta]}}{\|w\|_{2-}^{[\Theta]}} \tag{15}$$

is called the quasi $L_{\infty,p}/L_2$ Hankel norm (or simply the quasi L_∞/L_2 Hankel norm) at Θ (under $p=2$ or $p=\infty$). Instead of the left-hand side, introducing the shorthand notation $\|\mathbf{H}_p^{[\Theta]}\|$ would not be confusing.

In view of the discussions in section 2.1, the standpoint of this paper is to define the L_∞/L_2 Hankel norm $\|\Sigma_{\mathrm{SD}}\|_{\mathrm{H},p}$ of the sampled-data system Σ_{SD} as

$$\|\Sigma_{\mathrm{SD}}\|_{\mathrm{H},p} := \sup_{\Theta\in[0,h)} \|\mathbf{H}_p^{[\Theta]}\|, \quad p = 2,\infty. \tag{16}$$

Remark 3.1. Somewhat relevant arguments for the definition of the "l_2/l_2 Hankel norm" for discrete-time periodic systems can be found in [20].

As shown in the appendix, the h-periodicity of the input-output behavior of Σ_{SD} together with $D_{12}=0$ leads to

$$\|\mathbf{H}_p^{[\Theta]}\| \to \|\mathbf{H}_p^{[0]}\| \quad (\Theta \to h-0) \tag{17}$$

(primarily because z is continuous if $D_{12}=0$). Since the continuity of $\|\mathbf{H}_p^{[\Theta]}\|$ in $\Theta \in [0,h)$ is obvious, the above property immediately implies that $\sup_{\Theta\in[0,h)} \|\mathbf{H}_p^{[\Theta]}\|$ is always attained by some $\Theta = \Theta^\star \in [0,h)$ when $D_{12}=0$. We define $\mathbf{H}_p^\star := \mathbf{H}_p^{[\Theta^\star]}$

and call it *the* L_∞/L_2 Hankel operator of the sampled-data system $\varSigma_{\mathrm{SD}}$ with $D_{12}=0$. It should be noted, however, that there could exist more than one $\varTheta \in [0,h)$ eligible for the choice as $\varTheta^\star$ and thus $\mathbf{H}_p^\star$ may not actually be unique. Hence, even though the introduction of this terminology is convenient, it may not be fully rigorous. To avoid possible ambiguity relevant to the nonuniqueness of $\varTheta^\star$, we introduce the following terms for the subsequent arguments.

DEFINITION 3.2. *For the underlying $p=2$ or $p=\infty$, we say that $\varTheta^\star \in [0,h)$ is a critical instant if $\|\mathbf{H}_p^{[\varTheta^\star]}\| = \max_{\varTheta\in[0,h)} \|\mathbf{H}_p^{[\varTheta]}\|$. If a critical instant exists, we say that the L_∞/L_2 Hankel operator $\mathbf{H}_p^\star$ is well-definable.*

When $D_{12}\neq 0$, a critical $\varTheta^\star$ may not exist and thus $\mathbf{H}_p^\star$ may not be well-definable. The following arguments involve tackling the question about when the critical instants exist under $D_{12}\neq 0$ (and the relevant question on how the critical instants $\varTheta^\star$ can be determined), as well as characterizing the L_∞/L_2 Hankel norm $\|\varSigma_{\mathrm{SD}}\|_{\mathrm{H},p}$ in such a way that its computation can be carried out. Note that the L_∞/L_2 Hankel norm defined by (16) is entirely meaningful even when the critical instants $\varTheta^\star$ do not exist (i.e., when *the* L_∞/L_2 Hankel operator $\mathbf{H}_p^\star$ is not well-definable).

In tackling the above issues, the standpoint of the present paper is that we do not directly deal with the computation or characterization of $\|\mathbf{H}_p^{[\varTheta]}\|$ for each $\varTheta\in[0,h)$. Even though such treatment might appear indispensable because of the definition (16) for the L_∞/L_2 Hankel norm, this paper develops an alternative method via what we call the overlap L_∞/L_2 Hankel operator, which we believe is on one hand somewhat simpler after all, and more appropriate on the other hand in the theoretical study on the L_∞/L_2 Hankel norm as well as the existence of the critical instants $\varTheta^\star$ and *the* L_∞/L_2 Hankel operator.

Symbols with $\star$ in the following arguments imply that they are relevant to critical instants.

Remark 3.3. Although it may somewhat distract the attention of the reader, we are in a position to make a remark for lack of an alternative and better position. That is, this remark is devoted to some issues relevant to the introduction of $\varTheta\in[0,h)$, which is particularly related to the close connection between the L_∞/L_2 Hankel norm and the standard H_2 norm in continuous-time LTI systems and, moreover, how such a connection could be inherited by the sampled-data system $\varSigma_{\mathrm{SD}}$. To clarify the intention of such a remark, we begin by a series of known facts:

- (i-a) In continuous-time LTI systems, the L_∞/L_2 Hankel norm coincides with the L_∞/L_2 induced norm both for $p=2,\infty$ [7].
- (i-b) In MISO continuous-time LTI systems (for which no distinction is necessary about $p=2$ or $p=\infty$), the L_∞/L_2 induced norm coincides with the standard H_2 norm [7, 8, 9].
- (i-c) Hence, for MISO LTI systems, the H_2 norm could also be redefined equivalently through the viewpoint of the L_∞/L_2 Hankel norm (and if one wishes further, it is straightforward to redefine the H_2 norm equivalently in such a way that multioutput systems are also covered through their decomposition to MISO subsystems).

To our best knowledge, however, the H_2 norm has never been defined on the sampled-data system $\varSigma_{\mathrm{SD}}$ through the viewpoint of the L_∞/L_2 Hankel norm. Regarding this issue, our study obviously motivates us to proceed in a new direction as follows (*where we restrict our attention to the MISO case for simplicity* although the multioutput case can be covered by somewhat modifying the arguments as in the continuous-time case). This is because if we regard continuous-time LTI systems as a special case of $\varSigma_{\rm SD}$ without the discrete-time controller $\varPsi$, then we readily see that the quasi L_∞/L_2 Hankel norms are constant for $\varTheta \in [0,h)$ and coincide with the H_2 norm:

(ii-a) We may define the L_∞/L_2 Hankel norm (which is the supremum of the quasi L_∞/L_2 Hankel norms for $\varTheta \in [0,h)$) as a new H_2 norm for (MISO) sampled-data systems.

(ii-b) Alternatively, we may also define the root-mean-square (RMS) of the quasi L_∞/L_2 Hankel norms for $\varTheta \in [0,h)$) as yet another H_2 norm for (MISO) sampled-data systems.

In connection to the above two possible new definitions of the H_2 norm for sampled-data systems, it would be quite interesting to note the following:

(iii-a) The current most standard definition of the H_2 norm for the sampled-data system $\varSigma_{\rm SD}$ in [14] is somewhat similar to the second new definition above in its RMS-type of arguments. More precisely, the impulse input at $\varTheta \in [0,h)$ applied to each input channel is considered, the sum of the squared L_2 norms for the corresponding responses is taken, and then the square root of the average of the sums taken over all $\varTheta \in [0,h)$ is defined as the H_2 norm for $\varSigma_{\rm SD}$ in [14].

(iii-b) On the other hand, the first new definition above is similar to another definition of the H_2 norm for $\varSigma_{\rm SD}$ introduced recently in [12] in its sup-type of arguments. More precisely, the impulse input at $\varTheta \in [0,h)$ applied to each input channel is considered, the square root of the sum of the squared L_2 norms for the corresponding responses is taken, and then the supremum of all the resulting values for $\varTheta \in [0,h)$ is defined as an alternative H_2 norm for $\varSigma_{\rm SD}$ in [12]. This definition might be called the third definition after the first one that only considered the impulse response for $\varTheta = 0$ in [13] and the second one in [14] (i.e., the standard one stated above).

With all such qualitative similarities, however, we can easily confirm even through numerical examples of single-input single-output sampled-data systems that the fourth definition (ii-a) is different from the third one in (iii-b) and the fifth definition (ii-b) is different from the standard one of (iii-a) in [14]. In particular, all five definitions for the H_2 norm of $\varSigma_{\rm SD}$ are different, in general (see also some relevant arguments in [12]). Yet the two new definitions (ii-a) and (ii-b) through the quasi L_∞/L_2 Hankel norms for $\varTheta \in [0,h)$ may also capture some important aspects in the studies of sampled-data systems and thus it may be an interesting topic to pursue such directions as alternative H_2 problems of sampled-data systems.

Remark 3.4. To some readers, the RMS treatment in (ii-b) in the above remark may sound more appropriate *as the L_∞/L_2 Hankel norm itself than the one through supremum adopted in this paper.* The basic standpoint of the authors, however, is that the Hankel norm in the continuous-time LTI systems is more like an induced norm (rather than the H_2 norm, which is more like the Frobenius norm and is not a usual type of induced norm) in the sense that the associated worst input may be discussed; it is the supremum-type of treatment that allows us to proceed to a similar direction and thus the study of critical instants and *the* L_∞/L_2 Hankel operator.

3.2. Overlap L_∞/L_2 Hankel operator and its relevance to the L_∞/L_2 Hankel norm. Before we tackle the problem of characterizing the L_∞/L_2 Hankel norm and critical instants (if they exist) of the sampled-data system $\varSigma_{\rm SD}$ in the following section, we introduce an infinite-dimensional operator matrix useful for such a study. For convenience in the overall arguments, we first consider a somewhat artificial alternative viewpoint about the "past" and "future," once putting aside the fact that the past and future should be separated at an arbitrary instant $\Theta \in [0, h)$ after all. That is, we consider for the time being the situation in which the "past" about the input and the "future" about the output have an overlap by the time interval $[0, h)$. We call the associated operator the overlap L_∞/L_2 Hankel operator. It corresponds to considering the following relation between the lifted representation $\{\widehat{w}_k\}_{k=-\infty}^{0}$ of the "past" input in $L_2(-\infty, h)$ and that of the "future" output in $L_\infty[0, \infty)$, i.e., $\{\widehat{z}_k\}_{k=0}^{\infty}$, assuming that $x(-\infty) = 0$, $\psi_{-\infty} = 0$ and $w(t) = 0,\ t \geq h$:

$$\begin{bmatrix} \widehat{z}_0 \\ \widehat{z}_1 \\ \widehat{z}_2 \\ \widehat{z}_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} \mathcal{D} & \mathcal{CB} & \mathcal{CAB} & \mathcal{CA}^2\mathcal{B} & \dots \\ \mathcal{CB} & \mathcal{CAB} & \mathcal{CA}^2\mathcal{B} & \dots & \\ \mathcal{CAB} & \mathcal{CA}^2\mathcal{B} & \ddots & & \\ \mathcal{CA}^2\mathcal{B} & \vdots & & & \\ \vdots & & & & \end{bmatrix} \begin{bmatrix} \widehat{w}_0 \\ \widehat{w}_{-1} \\ \widehat{w}_{-2} \\ \widehat{w}_{-3} \\ \vdots \end{bmatrix}. \tag{18}$$

We denote by $\mathcal{M}$ the operator matrix on the right-hand side (which quite naturally has the so-called Hankel structure).

We make some comments on why we consider this somewhat artificial operator and why this paper is confined to the L_∞/L_2 Hankel norm rather than the L_2/L_2 Hankel norm.

1. As it turns out, we can show that the norm of this overlap L_∞/L_2 Hankel operator (which we call the overlap L_∞/L_2 Hankel norm) coincides with the L_∞/L_2 Hankel norm, which is precisely what we are actually interested in (i.e., defined by (16)).
2. We can further show that the overlap L_∞/L_2 Hankel norm (and thus the L_∞/L_2 Hankel norm, too) equals the L_∞/L_2-induced norm of Σ_{SD}, where the computation methods for the induced norm are already known [11, 12].
3. A study on the existence of the critical instants $\Theta^\star$ (or *the* L_∞/L_2 Hankel operator) through dealing with the overlap L_∞/L_2 Hankel operator is possible.
4. All these issues are strongly related to the properties of the L_∞ space; in some mathematically less strict words, the norm of a function in this space is (almost) determined solely by its instantaneous value at some specific time instant, and this property is helpful in many aspects. For example, we can consider a simple case when the overlap L_∞/L_2 Hankel operator is norm-attaining (i.e., the worst input exists), and this further helps us study a condition for the existence of the critical instants $\Theta^\star$.
5. By the properties of L_∞, the overlap L_∞/L_2 Hankel norm can be considered by rowwise treatment of the operator matrix $\mathcal{M}$. This leads to a neat representation for the L_∞/L_2 Hankel norm, which in turn is also helpful in determining the critical instants $\Theta^\star$ and *the* L_∞/L_2 Hankel operator (if the former exist and thus the latter is well-definable).
6. The arguments for the L_2/L_2 Hankel norm are expected to become less neat because we cannot exploit the above specific properties of the L_∞ space. Even though we believe that somewhat parallel arguments can be developed for the L_2/L_2 Hankel norm of sampled-data systems, such arguments will be rather different from those in this paper and will be treated in our future study.

In the rest of this section, we give more explicit results about some of the first five items while the remaining items will be discussed later in section 4.

Let us rewrite (18) as $\widehat{z} = \mathcal{M}\widehat{w}$ by defining $\widehat{w} := [\widehat{w}_0^T, \widehat{w}_{-1}^T, \widehat{w}_{-2}^T, \ldots]^T$ and $\widehat{z} := [\widehat{z}_0^T, \widehat{z}_1^T, \widehat{z}_2^T, \ldots]^T$. We further define their norms by

$$\|\widehat{w}\|_{2-} := \left(\sum_{k=-\infty}^{0} \|\widehat{w}_k\|_2^2\right)^{1/2}, \quad \|\widehat{z}\|_{\infty,p} := \sup_{k\in\mathbb{N}_0} \|\widehat{z}_k\|_{\infty,p} \quad (p = 2, \infty), \tag{19}$$

where $\|\widehat{w}_k\|_2$ is defined as the $L_2[0,h)$ norm of $\widehat{w}_k$ and $\|\widehat{z}_k\|_{\infty,p}$ is used in the sense corresponding to the right-hand side of (1) *with the interval of* t *replaced by* $[0,h)$. We readily see that the norm $\|\widehat{w}\|_{2-}$ coincides with $\|w\|_{2-}^{[h]}$ defined in (2), while the norm $\|\widehat{z}\|_{\infty,p}$ coincides with $\|z\|_{\infty,p}^{[0]}$ defined in (1). This is nothing but the well-known fact that lifting w and z preserves norms. Since $\mathcal{M}$ is the lifting-based matrix representation of the overlap L_∞/L_2 Hankel operator defined as the mapping from $w \in L_2(-\infty,h)$ to $z \in L_{\infty,p}[0,\infty)$ (where either $p = 2$ or $p = \infty$ is assumed depending on the context), which we denote by $\mathbf{H}_p^{[0,h)}$, we readily see that the overlap L_∞/L_2 Hankel norm is given by

$$\|\mathbf{H}_p^{[0,h)}\| = \|\mathcal{M}\| = \sup_{w\in L_2(-\infty,h)} \frac{\|\widehat{z}\|_{\infty,p}}{\|\widehat{w}\|_{2-}}. \tag{20}$$

In computing $\|\mathcal{M}\|$, it is crucial to note that each row in $\mathcal{M}$ is a shifted and extended version of the next row. Hence, $\|\mathcal{M}\|$ coincides with the norm (defined in an obvious way) of the first row of $\mathcal{M}$, which we denote by $\mathcal{F}$:

$$\mathcal{F} := \begin{bmatrix} \mathcal{D} & \mathcal{C}\mathcal{B} & \mathcal{C}\mathcal{A}\mathcal{B} & \mathcal{C}\mathcal{A}^2\mathcal{B} & \cdots \end{bmatrix}. \tag{21}$$

It is quite important to note that the above operator $\mathcal{F}$ has actually appeared in a relevant study [12] on the sampled-data system Σ_{SD} about its induced norm from $L_2[0,\infty)$ to $L_{\infty,p}[0,\infty)$. More precisely, it has been shown that the L_∞/L_2-induced norm of Σ_{SD} is also given by the norm of $\mathcal{F}$ (under $p = 2$ or $p = \infty$). This implies that the overlap L_∞/L_2 Hankel norm coincides with the L_∞/L_2-induced norm. The computation methods for the latter have been already known [11, 12], which will be reviewed in section 4 (see Proposition 4.1) after we show the following theorem asserting that the L_∞/L_2 Hankel norm actually coincides with the overlap L_∞/L_2 Hankel norm. Hence, it obviously turns out that we can have a method for computing the L_∞/L_2 Hankel norm numerically (by computing the L_∞/L_2-induced norm instead). In the standpoint of our paper, however, we are less interested in the numerical computation method itself and are more interested in whether and how the method can be related to answering our fundamental question about the existence (and characterization) of the critical instants $\Theta = \Theta^\star$.

We first give the following theorem, which we claimed in the above arguments.

THEOREM 3.5. *The L_∞/L_2 Hankel norm $\|\Sigma_{\mathrm{SD}}\|_{\mathrm{H},p}$ satisfies*

$$\|\Sigma_{\mathrm{SD}}\|_{\mathrm{H},p} = \|\mathbf{H}_p^{[0,h)}\|, \quad p = 2, \infty. \tag{22}$$

Proof. By considering the "inclusion relation" between the domains of $\mathbf{H}_p^{[\Theta]}$ and $\mathbf{H}_p^{[0,h)}$ as well as that of their co-domains, it is obvious that $\|\mathbf{H}_p^{[0,h)}\| \geq \|\mathbf{H}_p^{[\Theta]}\|\ \forall \Theta \in [0,h)$. Hence,

$$\|\mathbf{H}_p^{[0,h)}\| \geq \sup_{\Theta\in[0,h)} \|\mathbf{H}_p^{[\Theta]}\|. \tag{23}$$

It also follows from the definition of $\|\mathbf{H}_p^{[0,h)}\|$ that for each $\varepsilon > 0$, there exists $w \in L_2(-\infty, h)$ such that $\|w\|_{2-}^{[h]} \leq 1$ and the corresponding output $z \in L_{\infty,p}[0,\infty)$ satisfies $\|z\|_{\infty,p}^{[0]} \geq \|\mathbf{H}_p^{[0,h)}\| - \varepsilon$. Hence, there exists Θ such that $|z(\Theta)|_p \geq \|\mathbf{H}_p^{[0,h)}\| - \varepsilon$, where $\Theta \in [0,h)$ may be assumed without loss of generality; if such Θ satisfies $\Theta \geq h$ for w, then it suffices to replace the input w in $L_2(-\infty, h)$ with another obtained by advancing the original w by an integer multiple of the sampling period h until the new Θ belongs to $[0,h)$ (by noting the h-periodicity of the input-output relation of $\Sigma_{\rm SD}$).

For notational clarity, let us suppose that an arbitrary $\varepsilon > 0$ is taken and

$$|z^\bullet(\Theta^\bullet)|_p \geq \|\mathbf{H}_p^{[0,h)}\| - \varepsilon, \quad \Theta^\bullet = \Theta^\bullet(\varepsilon) \in [0,h), \tag{24}$$

where $w^\bullet \in L_2(-\infty, h)$ and $\|w^\bullet\|_{2-}^{[h]} \leq 1$. If we note the causality of $\Sigma_{\rm SD}$, we may assume without loss of generality that $w^\bullet$ satisfying the above inequality actually satisfies $w^\bullet(t) = 0,\ t \geq \Theta^\bullet$, i.e., $w^\bullet \in L_2(-\infty, \Theta^\bullet)$. If we also regard the corresponding output $z^\bullet$ as an element in $L_{\infty,p}[\Theta^\bullet, -\infty)$ accordingly, it is equivalent to considering the quasi $L_{\infty,p}/L_2$ Hankel operator at $\Theta^\bullet$. Since $z^\bullet$ is right continuous, it follows from the definition of $\|\mathbf{H}_p^{[\Theta^\bullet]}\|$ that $\|\mathbf{H}_p^{[\Theta^\bullet]}\| \geq |z^\bullet(\Theta^\bullet)|_p$, which further implies

$$\|\mathbf{H}_p^{[\Theta^\bullet]}\| \geq \|\mathbf{H}_p^{[0,h)}\| - \varepsilon \tag{25}$$

by (24). Since $\varepsilon > 0$ is arbitrary, we see from (23) that

$$\sup_{\Theta \in [0,h)} \|\mathbf{H}_p^{[\Theta]}\| = \|\mathbf{H}_p^{[0,h)}\|. \tag{26}$$

This completes the proof by the definition (16) of the L_∞/L_2 Hankel norm. ◻

The following result, which is an extension of the parallel results for continuous-time LTI systems [7], is a direct consequence from what was discussed before the above theorem.

COROLLARY 3.6. *The L_∞/L_2 Hankel norm $\|\Sigma_{\rm SD}\|_{{\rm H},p}$ coincides with the L_∞/L_2-induced norm $\|\Sigma_{\rm SD}\|_{L_{\infty,p}/L_2}$ for $p = 2, \infty$.*

Remark 3.7. We can readily show that the norm of a "generalized overlap Hankel operator" of any sort such that the corresponding "past" and "future" intervals have an overlap that contains a critical instant is also equal to the induced norm. This fact can be regarded as an extension of a parallel result in [7] for continuous-time *LTI* systems to sampled-data systems, which are periodic.

3.3. Preliminary existence study on the critical instants $\Theta^\star$ and the L_∞/L_2 Hankel operator. Very roughly speaking, the arguments in the above subsection indicate that a critical instant $\Theta = \Theta^\star$ would be given by $\lim_{\varepsilon \to +0} \Theta^\bullet(\varepsilon)$. This paper aims at providing somewhat more rigorous arguments that can be developed only through the definitions of the L_∞/L_2 Hankel norm, the critical instants $\Theta^\star$, and *the* L_∞/L_2 Hankel operator; we will return to a seemingly similar topic in section 4 after we provide an explicit computation method for the L_∞/L_2 Hankel norm in Proposition 4.1, but the standpoints here and there are entirely different. As mentioned above, we first confine ourselves to the existence study of the critical instants $\Theta^\star$ (or *the* L_∞/L_2 Hankel operator) from a somewhat theoretical aspect rather than from an aspect of numerically characterizing the critical instants $\Theta^\star$ (if they exist).

For this purpose, we first proceed with our arguments under the assumption that the overlap L_∞/L_2 Hankel operator is norm-attaining (i.e., it has the worst input denoted by $w^\bullet \in L_2(-\infty,h)$). Without loss of generality, we assume $\|w^\bullet\|_{2-}^{[h]}=1$, in which case the corresponding output $z^\bullet$ satisfies $\|z^\bullet\|_{\infty,p}^{[0]} = \|\mathbf{H}_p^{[0,h)}\|$. We further consider the case when this $z^\bullet$ is also norm-attaining, by which we mean that there exists $\Theta^\bullet \geq 0$ such that $\|z^\bullet\|_{\infty,p}^{[0]} = |z^\bullet(\Theta^\bullet)|_p$, i.e.,

$$|z^\bullet(\Theta^\bullet)|_p = \|\mathbf{H}_p^{[0,h)}\|, \tag{27}$$

where we may assume without loss of generality that $\Theta^\bullet \in [0,h)$ (for the same reasoning as before, which also shows that $w^\bullet$ satisfying this equality may actually be assumed to belong to $L_2(-\infty,\Theta^\bullet)$ and satisfy $\|w^\bullet\|_{2-}^{[\Theta^\bullet]}=1$ without loss of generality). Since $z^\bullet$ is right continuous, it follows that $\|\mathbf{H}_p^{[\Theta^\bullet]}\| \geq |z^\bullet(\Theta^\bullet)|_p$ and thus $\|\mathbf{H}_p^{[\Theta^\bullet]}\| \geq \|\mathbf{H}_p^{[0,h)}\|$ by (27). This together with (23) implies that $\|\mathbf{H}_p^{[0,h)}\| = \max_{\Theta\in[0,h)} \|\mathbf{H}_p^{[\Theta]}\| = \|\mathbf{H}_p^{[\Theta^\bullet]}\|$ under the two assumptions in the above arguments.

Summarizing the above leads to the following consequences about the existence of a critical instant $\Theta=\Theta^\star$ (or whether *the* L_∞/L_2 Hankel operator is well-definable).

(i) If the overlap L_∞/L_2 Hankel operator is norm-attaining and if the output for the corresponding worst input is norm-attaining, then *the* L_∞/L_2 Hankel operator is well-definable.

(ii) Under the assumptions in (i), a critical instant $\Theta=\Theta^\star$ is given by $\Theta^\bullet$, where $\Theta^\bullet$ is determined from the output of the overlap L_∞/L_2 Hankel operator for its worst input through (27).

In connection with the above consequences, we offer the following lemma.

LEMMA 3.8. *If a critical instant $\Theta=\Theta^\star$ exists for the underlying p, then the overlap L_∞/L_2 Hankel operator $\mathbf{H}_p^{[0,h)}$ is norm-attaining.*

What this lemma implies is that the first assumption in (i) above is actually a necessary condition for the existence of a critical instant $\Theta^\star$ (or equivalently, the well-definablily of *the* L_∞/L_2 Hankel operator). In other words, as long as our interest lies in discussing when the critical instants $\Theta^\star$ do exist, the first assumption in (i) by no means leads to loss of generality. The proof of this lemma is given below.

Proof. It suffices to show that assuming the existence of a critical instant $\Theta^\star$ and assuming that $\mathbf{H}_p^{[0,h)}$ is *not* norm-attaining lead to contradiction. Indeed, if $\mathbf{H}_p^{[0,h)}$ is not norm-attaining, the output z for its input $w\in L_2(-\infty,h)$ satisfies $\|z\|_{\infty,p}^{[0]} < \|\mathbf{H}_p^{[0,h)}\|$ whenever $\|w\|_{2-}^{[h]} \leq 1$. By arguments similar to what we have repeatedly used, it then follows for every $\Theta\in[0,h)$ that the output z viewed as an element of $L_{\infty,p}[\Theta,\infty)$ satisfies $\|z\|_{\infty,p}^{[\Theta]} < \|\mathbf{H}_p^{[0,h)}\|$ for every $w\in L_2(-\infty,\Theta)$ with $\|w\|_{2-}^{[\Theta]}\leq 1$. Since this is also true for every critical instant $\Theta^\star$, it implies from (16) and (22) that $\mathbf{H}_p^{[\Theta^\star]}$ is not norm-attaining. This consequence, however, contradicts the fact that $\mathbf{H}_p^{[\Theta^\star]}$ is a bounded finite-rank operator represented through $[x(\Theta^\star)^T, u_0^T, \xi_0^T]^T$ as the composition of a finite-rank operator $L_2(-\infty,\Theta^\star)\to\mathbb{R}^{n+n_u+n_\Psi}$ and a finite-rank operator $\mathbb{R}^{n+n_u+n_\Psi}\to L_\infty[\Theta^\star,\infty)$. This completes the proof. ∎

The implication of the above theoretical arguments will become clearer in the following section, where a numerical procedure for the L_∞/L_2-induced norm of Σ_{SD} [12] is reviewed (which is nothing but a computation method for the L_∞/L_2 Hankel norm by Corollary 3.6) and is related to determining the critical instants $\Theta^\star$ and *the* L_∞/L_2 Hankel operator (if the former exist and thus the latter is well-definable).

4. Existence study on the critical instants $\Theta^\star$ and their characterization through a computation method of the L_∞/L_2 Hankel norm. By Corollary 3.6, the L_∞/L_2 Hankel norm $\|\Sigma_{\rm SD}\|_{{\rm H},p}$ can be computed by applying the computation method for the $L_{\infty,p}/L_2$-induced norm in [12]. This section is devoted to discussing how this computation method can be used in combination with our preceding arguments to study when the critical instants $\Theta = \Theta^\star$ exist and how they (and thus *the* L_∞/L_2 Hankel operator) can be determined if they exist.

4.1. Computation method of the L_∞/L_2 Hankel norm and its interpretation. We begin by reviewing the computation method in [12] for the $L_{\infty,p}/L_2$-induced norm (and thus the L_∞/L_2 Hankel norm $\|\Sigma_{\rm SD}\|_{{\rm H},p}$) given as follows, where $\mu_p(\cdot)$ denotes the maximum eigenvalue and maximum diagonal entry of a positive semidefinite symmetric matrix for $p=2$ and $p=\infty$, respectively.

PROPOSITION 4.1. *The L_∞/L_2 Hankel norm $\|\Sigma_{\rm SD}\|_{{\rm H},p}$ defined by* (16) *is given by*

$$\|\Sigma_{\rm SD}\|_{{\rm H},p} = \sup_{\theta\in[0,h)} \mu_p^{1/2}(F(\theta)), \quad p=2,\infty, \tag{28}$$

where the matrix $F(\theta)$ is defined by

$$F(\theta) := \int_0^\theta D_\theta(\tau)D_\theta^T(\tau)d\tau + \sum_{k=0}^\infty \int_0^h (C_\theta \mathcal{A}^k B_h(\tau))(C_\theta \mathcal{A}^k B_h(\tau))^T d\tau \tag{29}$$

with the matrix functions

$$B_h(\tau) = J_\Sigma \exp(A(h-\tau))B_1, \quad D_\theta(\tau) = C_1 \exp(A(\theta-\tau))B_1 \mathbf{1}(\theta-\tau) \tag{30}$$

(with $\mathbf{1}(\cdot)$ being the unit step function) and the matrix

$$C_\theta = M_1 \exp(A_2\theta)C_\Sigma. \tag{31}$$

An explicit computation method for the infinite series (29) based on a Lyapunov equation is available but we do not touch on further details because this paper is not quite interested in the numerical computation itself. Instead, we only remark that the above proposition[3] was a consequence—under appropriate rephrased interpretation to match the present situation on the treatment of the overlap L_∞/L_2 Hankel operator and the L_∞/L_2 Hankel norm—of considering the maximization of $|z(\theta)|_p$ with respect to $w \in L_2(-\infty,h)$ (or $w\in L_2(-\infty,\theta)$, equivalently) for each $\theta\in[0,h)$. We suppress reviewing all such arguments because we do not need any details at all. In fact, we simply state that a more precise description of the above interpretation is that

$$\mu_p^{1/2}(F(\theta)) = \sup_{w\in L_2(-\infty,\theta)} \frac{|z(\theta)|_p}{\|w\|_{2-}^{[\theta]}}, \quad p=2,\infty. \tag{32}$$

Note that no time instant separating the past and future is relevant to the above arguments, which is why the symbol θ is used rather than Θ. Another issue that might be worth stating is that if $\Sigma_{\rm SD}$ is actually a continuous-time LTI system as a special case, then $F(\theta)$ is a constant function.

[3]It was derived in the context of the L_∞/L_2-induced norm of $\Sigma_{\rm SD}$ through the use of the triangle inequality and the continuous-time/discrete-time Cauchy–Schwarz inequalities.

One might wonder from (16) and (28) whether $\mu_p^{1/2}(F(\theta)) = \|\mathbf{H}_p^{[\Theta]}\|$ if $\theta = \Theta$. However, this is not the case, in general. In fact, by regarding θ as the instant separating the past and future, it is easy to see from (32) that

$$\|\mathbf{H}_p^{[\theta]}\| \geq \mu_p^{1/2}(F(\theta)) \ \forall\theta \in [0,h) \tag{33}$$

because the output z of $\mathbf{H}_p^{[\theta]}$ is right continuous and thus satisfies $\|z\|_{\infty,p}^{[\theta]} \geq |z(\theta)|_p$. The inequality (33) plays a key role in the derivation of the following theorem, which holds for $p = 2$ as well as $p = \infty$.

THEOREM 4.2. *For the underlying $p = 2$ or $p = \infty$, suppose that $\mu_p^{1/2}(F(\theta))$, $\theta \in [0,h)$, is maximum-attaining (in the sense that there exists $\theta^\star \in [0,h)$ such that $\mu_p(F(\theta^\star)) = \max_{\theta\in[0,h)} \mu_p(F(\theta))$). For each such $\theta^\star$, let $\Theta^\star = \theta^\star$. Then, $\Theta^\star$ is a critical instant relevant to $\|\Sigma_{\mathrm{SD}}\|_{\mathrm{H},p}$. In other words, the above hypothesis ensures that the L_∞/L_2 Hankel operator $\mathbf{H}_p^\star$ is well-definable and the quasi L_∞/L_2 Hankel operator $\mathbf{H}_p^{[\Theta^\star]}$ is indeed one possible choice for the L_∞/L_2 Hankel operator.*

Proof. By the assumption on $\theta^\star$, (28) implies that $\|\Sigma_{\mathrm{SD}}\|_{\mathrm{H},p} = \mu_p^{1/2}(F(\theta^\star))$. It then follows that equality must hold in (33) when $\theta = \theta^\star$; this is because otherwise we are led to $\|\mathbf{H}_p^{[\theta^\star]}\| > \sup_{\theta\in[0,h)} \|\mathbf{H}_p^{[\theta]}\|$ by (16), which is an obvious contradiction. Hence, we have $\|\mathbf{H}_p^{[\theta^\star]}\| = \mu_p^{1/2}(F(\theta^\star)) = \|\Sigma_{\mathrm{SD}}\|_{\mathrm{H},p}$ by (28). This completes the proof. ∎

The above theorem implies that under its hypothesis, *the* L_∞/L_2 Hankel operator $\mathbf{H}_p^\star$ is well-definable and a critical instant $\Theta = \Theta^\star$ can be detected through the curve of $\mu_p^{1/2}(F(\theta))$, $\theta \in [0,h)$. We can confirm the following result (already stated in section 3.1 in an informal fashion by referring to the completely independent arguments in the appendix) by noting the continuity of $F(\theta)$ on $[0,h)$; we can show through manipulating (29) that $\lim_{\theta\to h-0} F(\theta) = F(0)$ if $D_{12} = 0$, whose details are omitted, and thus $\mu_p^{1/2}(F(\theta))$, $\theta \in [0,h)$, is maximum-attaining then.

COROLLARY 4.3. *The L_∞/L_2 Hankel operator $\mathbf{H}_p^\star$ is well-definable if $D_{12} = 0$ ($p = 2,\infty$).*

4.2. Further study on the existence of the critical instants $\Theta^\star$ and their detection through $\mu_p^{1/2}(F(\theta))$. The arguments in the preceding subsection show that if $\mu_p^{1/2}(F(\theta))$ is maximum-attaining, *the* L_∞/L_2 Hankel operator $\mathbf{H}_p^\star$ is well-definable and one of (or a subset of) the critical instants $\Theta^\star$ will be reflected on the curve of $\mu_p^{1/2}(F(\theta))$ as $\arg\max_{\theta\in[0,h)} \mu_p^{1/2}(F(\theta))$. However, it is not clear whether $\mu_p^{1/2}(F(\theta))$ is maximum-attaining whenever the critical instants $\Theta^\star$ exist and thus the L_∞/L_2 Hankel operator $\mathbf{H}_p^\star$ is well-definable, or alternatively, whether every critical instant $\Theta^\star$, if it exists, can always be detected as $\arg\max_{\theta\in[0,h)} \mu_p^{1/2}(F(\theta))$. This subsection is devoted to arguments relevant to such issues. To this end, our preliminary theoretical arguments in section 3.3, which were completely indifferent to the quantity $\mu_p^{1/2}(F(\theta))$, will now be helpful.

We begin by the following assumption.

Assumption 4.4. A critical instant exists for the underlying p. The output z of the associated L_∞/L_2 Hankel operator $\mathbf{H}_p^\star$ for its worst input is norm-attaining (i.e., there exists t no smaller than the critical instant such that the norm of z equals $|z(t)|_p$).

Remark 4.5. The existence of the worst input in the above assumption has been ensured in the proof of Lemma 3.8.

Let us recall the arguments in section 3.3 under Assumption 4.4 (which ensures by Lemma 3.8 and (26) that the assumptions in section 3.3 are satisfied). Then, by (22), it follows from (27) that

$$|z^\bullet(\Theta^\bullet)|_p = \|\Sigma_{\mathrm{SD}}\|_{\mathrm{H},p}, \tag{34}$$

where $z^\bullet$ is the output for $w^\bullet \in L_2(-\infty, \Theta^\bullet)$ with $\|w^\bullet\|_{2-}^{[\Theta^\bullet]} = 1$ in the arguments of section 3.3. On the other hand, if we consider the input w other than $w^\bullet$ (and the corresponding output z for the input w), we see from (32) that

$$\mu_p^{1/2}(F(\Theta^\bullet)) \geq |z^\bullet(\Theta^\bullet)|_p. \tag{35}$$

Hence, by (34), we have

$$\mu_p^{1/2}(F(\Theta^\bullet)) \geq \|\Sigma_{\mathrm{SD}}\|_{\mathrm{H},p}. \tag{36}$$

This together with (28) implies that if we let $\theta^\star = \Theta^\bullet$, then $\mu_p^{1/2}(F(\theta^\star))$ must coincide with $\|\Sigma_{\mathrm{SD}}\|_{\mathrm{H},p}$, and that the right-hand side of (28) is attained as the maximum over $\theta \in [0, h)$ (at $\theta = \theta^\star$). As a side remark, it is obvious from Theorem 4.2 that $\theta^\star$ is a critical instant (i.e., $\mathbf{H}_p^{[\theta^\star]}$ is one possible choice for the L_∞/L_2 Hankel operator $\mathbf{H}_p^\star$).

Remark 4.6. Let $\Theta^\star$ be a critical instant. If one could establish the claim that the output $z^\star$ of $\mathbf{H}_p^{[\Theta^\star]}$ for its worst input attains its norm at $\Theta^\star$ (i.e., the claim that $\|z^\star\|_{\infty,p}^{[\Theta^\star]} = |z^\star(\Theta^\star)|_p$), then we can see that $\Theta^\bullet$ in (34) is given by $\Theta^\star$, which in turn would immediately lead to the consequence that the maximum of $\mu_p^{1/2}(F(\theta))$ is attained at $\theta = \Theta^\star$. Unfortunately, however, it is an open question at the moment whether one can indeed establish the above claim (even though we can easily see that $\Theta^\bullet \geq \Theta^\star$). Thus, the preceding arguments do not necessarily imply that every critical instant is reflected on the curve of $\mu_p^{1/2}(F(\theta))$, $\theta \in [0, h)$, as a maximum-attaining point. More precisely, the arguments only show that existence of a critical instant ensures that at least one of the critical instants is reflected on the curve of $\mu_p^{1/2}(F(\theta))$, $\theta \in [0, h)$ as a maximum-attaining point. The above possible claim is actually closely related with how strong the hypothesis would be in Theorem 4.9 given below.

Summarizing the above arguments under Assumption 4.4 leads to the following theorem, which corresponds to a sort of converse for Theorem 4.2 and gives a sufficient condition for at least one critical instant $\Theta^\star$ to be reflected on the curve of $\mu_p^{1/2}(F(\theta))$, $\theta \in [0, h)$ as a maximum-attaining point.

THEOREM 4.7. *If Assumption* 4.4 *is satisfied in the sampled-data system* Σ_{SD}*, then* $\mu_p^{1/2}(F(\theta))$*,* $\theta \in [0, h)$ *is maximum-attaining (i.e., there exists* $\theta^\star \in [0, h)$ *such that* $\mu_p^{1/2}(F(\theta^\star)) = \max_{\theta \in [0,h)} \mu_p^{1/2}(F(\theta))$*). Every such* $\theta^\star$ *is a critical instant.*

Remark 4.8. (i) A sufficient condition for Assumption 4.4 to be satisfied is that $D_{12} = 0$. This is because the L_∞/L_2 Hankel operator $\mathbf{H}_p^\star$ is ensured to be well-definable by Corollary 4.3 in that case, and the output for its worst input (ensured to exist; recall Remark 4.5) is norm-attaining then; to see the latter, note that the output z of the sampled-data system Σ_{SD} is always continuous if $D_{12} = 0$ and that the output of $\mathbf{H}_p^\star$ tends to 0 as $t \to \infty$ by the internal stability of Σ_{SD}. (ii) However, asserting *as a corollary to the above theorem* the fact that $\mu_p^{1/2}(F(\theta)),\ \theta \in [0,h)$, is maximum-attaining if $D_{12}=0$ is inappropriate because such reasoning leads to circular arguments. This is because the maximum-attaining property has already been used regardless of the two possible standpoints as to the derivation of Corollary 4.3 (recall the descriptions preceding the corollary). (iii) Aside from this, however, it is obvious from Theorem 4.2 that if $D_{12}=0$, then a critical instant $\Theta^\star$ can be detected from the curve of $\mu_p^{1/2}(F(\theta)),\ \theta \in [0,h)$, as $\arg\max_{\theta\in[0,h)} \mu_p^{1/2}(F(\theta))$.

We can also show the following theorem, which does not depend on Assumption 4.4.

THEOREM 4.9. *For the underlying p, suppose that the sampled-data system $\Sigma_{\rm SD}$ has a critical instant $\Theta^\star$ but that no $\underline{\Theta} \in [0,h)$ exists such that every $\Theta \in [\underline{\Theta},h)$ is a critical instant. Then, $\mu_p^{1/2}(F(\theta)),\ \theta\in[0,h)$, is maximum-attaining, where every $\theta^\star$ such that $\mu_p^{1/2}(F(\theta^\star)) = \max_{\theta\in[0,h)} \mu_p^{1/2}(F(\theta))$ is a critical instant.*

Proof. In view of Theorem 4.7, it suffices to consider the case when Assumption 4.4 is not satisfied. Under the hypothesis of the present theorem, this is equivalent to assuming that for each $\mathbf{H}_p^\star$ corresponding to a critical instant $\Theta^\star$, the output $z^\star$ for its worst input $w^\star$ is not norm-attaining. Here, by the internal stability of $\Sigma_{\rm SD}$, the output $z^\star$ can fail to be norm-attaining only in connection with its discontinuity. Since such discontinuity can occur only at sampling instants, it follows that we have only to consider the case when $\|z^\star\|_{\infty,p}^{[\Theta^\star]} = \max_{k\in\mathbb{N}} \lim_{t\to kh-0} |z^\star(t)|_p$ (and the right-hand side exceeds $|z^\star(\Theta^\star)|_p$). Now, for each $\Theta \in [\Theta^\star,h)$, it is obvious that the worst input $w^\star \in L_2(-\infty,\Theta^\star)$ for $\mathbf{H}_p^\star$ viewed as an element in $L_2(-\infty,\Theta)$ yields the output z identical to $z^\star$ on $[\Theta,\infty)$ if it is applied to $\mathbf{H}_p^{[\Theta]}$. Since $kh > \Theta,\ k\in\mathbb{N}$, it follows that

$$
\begin{aligned}
\|\mathbf{H}_p^{[\Theta]}\| &\geq \max_{k\in\mathbb{N}} \lim_{t\to kh-0} |z(t)|_p / \|w\|_{2-}^{[\Theta]} \\
&= \max_{k\in\mathbb{N}} \lim_{t\to kh-0} |z^\star(t)|_p / \|w^\star\|_{2-}^{[\Theta^\star]} \\
&= \|\Sigma_{\rm SD}\|_{{\rm H},p}. \qquad (37)
\end{aligned}
$$

This implies $\|\mathbf{H}_p^{[\Theta]}\| = \|\Sigma_{\rm SD}\|_{{\rm H},p}, \forall\Theta\in[\Theta^\star,h)$ by (16), which contradicts the hypothesis of the theorem. This completes the proof of the first assertion, while the second assertion follows from Theorem 4.2. ∎

In view of Theorem 4.2, we readily obtain the following corollary for $p=2,\infty$.

COROLLARY 4.10. *Suppose that the sampled-data system $\Sigma_{\rm SD}$ has either finitely or countably many critical instants for the underlying p. Then, $\mu_p^{1/2}(F(\theta)),\ \theta\in[0,h)$, is maximum-attaining, where every $\theta^\star$ such that $\mu_p^{1/2}(F(\theta^\star)) = \max_{\theta\in[0,h)} \mu_p^{1/2}(F(\theta))$ is a critical instant. In particular, if $\Sigma_{\rm SD}$ has a unique critical instant $\Theta^\star$ for the underlying p, then there exists a unique $\theta^\star \in [0,h)$ such that $\mu_p^{1/2}(F(\theta^\star)) = \max_{\theta\in[0,h)} \mu_p^{1/2}(F(\theta))$, where $\theta^\star = \Theta^\star$.*

5. Numerical example. This section gives a numerical example for the arguments developed in this paper. Let us consider the sampled-data system $\Sigma_{\rm SD}$ consisting of P given by

TABLE 1
Numerical results for $\|\mathbf{H}_p^\star\| = \|\mathbf{H}_p^{[\Theta^\star]}\|$ *and the critical instant* $\Theta^\star$ *together with* $\|\mathbf{H}_p^{[0]}\|$.

	$p=2$	$p=\infty$
$\|\mathbf{H}_p^\star\| = \|\mathbf{H}_p^{[\Theta^\star]}\|$	4.3016	3.7870
$\Theta^\star$	0.806	0.728
$\|\mathbf{H}_p^{[0]}\|$	4.0629	3.5644

$$
\begin{aligned}
&A = \begin{bmatrix} -3 & 2 \\ -4 & 2 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 & 0.5 \\ -1 & 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix},\\
&C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix},\ D_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
\tag{38}
$$

and Ψ given by

$$
\begin{aligned}
&A_\Psi = \begin{bmatrix} -4.0534 & 4.0721 \\ -3.0738 & 3.0880 \end{bmatrix},\ B_\Psi = \begin{bmatrix} 0.0570 \\ 0.0432 \end{bmatrix},\\
&C_\Psi = \begin{bmatrix} -0.6349 & 0.6378 \end{bmatrix},\ D_\Psi = -0.2374,
\end{aligned}
\tag{39}
$$

where the sampling period is $h = 2$.

The computation through (29) shows that $\mu_p^{1/2}(F(\theta)),\ \theta \in [0,h)$, is maximum-attaining for both $p = 2$ and $p = \infty$. More precisely, the maximum is attained by a unique $\theta = \theta^\star \in [0,h)$, where $\theta^\star = 0.806$ for $p = 2$ and $\theta^\star = 0.728$. By Theorem 4.2, each $\theta^\star$ is a critical instant for the associated p, and the quasi L_∞/L_2 Hankel norm at $\Theta^\star = \theta^\star$ is as shown in Table 1 (which is nothing but the value of $\mu_p^{1/2}(F(\theta^\star))$ by Proposition 4.1). This table also shows the value of $\|\mathbf{H}_p^{[0]}\|$ for each p, which can be computed as $\sup_{\theta\in[0,h)} \mu_p^{1/2}(F_0(\theta))$, where $F_0(\theta)$ is defined as the second term in the right-hand side of (29). We skip the details for the derivation of this representation, but we remark that $\|\mathbf{H}_p^{[0]}\|$ equals the norm of the operator obtained by removing $\mathcal{D}$ from $\mathcal{F}$ in (21). This should not be hard to see because $\widehat{w}_0$ becomes irrelevant (in other words, it becomes 0) when we consider the quasi L_∞/L_2 Hankel operator at $\Theta = 0$. This is why the first term on the right-hand side of (29) should be dropped in the computation of $\|\mathbf{H}_p^{[0]}\|$.

We see from Table 1 that (i) the L_∞/L_2 Hankel norms are different for $p = 2$ and $p = \infty$, and (ii) $\|\mathbf{H}_p^\star\| > \|\mathbf{H}_p^{[0]}\|$ and thus $\Theta = 0$ is *not* a critical instant for each p (i.e., $\Theta^\star \in (0,h)$). The second observation clearly indicates that the standpoint of this paper taking an arbitrary $\Theta \in [0,h)$ as a possible instant for separating the past and future is indeed adequate in the definition of the L_∞/L_2 Hankel operator/norm of sampled-data systems (which is believed to be the case also in the L_2/L_2 Hankel operator/norm studied in [10]).

6. Conclusion. This paper introduced appropriate definitions of the Hankel operator/norm of sampled-data systems with their periodic input-output characteristics taken into account. That is, we took the standpoint that the past and future should be separated at an appropriate instant in a sampling interval when we are to consider the worst effect of the past input on the future output. Theoretical results were provided for the characterization of the Hankel operator/norm of sampled-data systems together with an associated numerical computation method for the L_∞/L_2 Hankel norm. More importantly, the relevance of this computation method to determining the worst instant at which the past and future are separated as well as *the* L_∞/L_2 Hankel operator (if they are well-definable) was also discussed. Finally, a numerical example was studied, by which the validity of the standpoint of this paper was confirmed.

Regarding possible extension of the arguments in this paper, it is straightforward to extend the results in section 3 to the quasi L_∞/L_∞ Hankel norm/operator, *the* L_∞/L_∞ Hankel norm/operator, and the overlap L_∞/L_∞ Hankel operator norm in an entirely parallel fashion (as long as "$D_{11} = 0$" and "$D_{21} = 0$"). However, it would be a challenging issue to study whether parallel arguments can be developed also for the arguments in section 4 in the L_∞/L_∞ setting. This can be explained as follows: the L_∞ induced norm analysis [21, 22] is a rather involved problem and is much harder than the H_2 norm analysis, for which closed-form representations are well-known not only for continuous-time systems but also for LTI sampled-data systems (e.g., [12, 14]). As discussed in Remark 3.4, the H_2 norm is closely related to the L_∞/L_2 induced norm and thus the L_∞/L_2 Hankel norm too, and such a connection seems to be deeply related to the fact that the L_∞/L_2 Hankel norm is also numerically computable through $F(\theta)$ in (29), which played a central role in the arguments in section 4. However, even with our recent renewed interest [23, 24, 25], no closed-form expression for the L_∞ induced norm is available and only asymptotically exact approximate computation methods are known. This circumstance prohibits us from developing parallel arguments in a straightforward fashion. It is nevertheless believed that somewhat parallel arguments can be developed for the usual L_2/L_2 Hankel operator/norm under similar and thus more appropriate treatment than that in [10], but the details will be reported elsewhere.

On the other hand, this paper did not deal with the computation/characterization of the quasi L_∞/L_2 Hankel norm at Θ (i.e., $\|\mathbf{H}_p^{[\Theta]}\|$) for each $\Theta \in [0, h)$ itself, even though *the* L_∞/L_2 Hankel norm is defined through $\|\mathbf{H}_p^{[\Theta]}\|$ as in (16), where Θ is the instant at which the past and future are separated. This is because we developed an alternative and indirect method for the computation of the L_∞/L_2 Hankel norm through what we call the overlap L_∞/L_2 Hankel norm (whose norm is shown to coincide with the L_∞/L_2-induced norm, for which we already have a numerical computation method [12]). Nonetheless, characterizing $\|\mathbf{H}_p^{[\Theta]}\|$ for each $\Theta \in [0, h)$ would be meaningful and important, and it is indeed possible [26]. However, we do not touch on this issue in this paper because it is mostly a numerical computation issue and its arguments have quite a different characteristic in the sense that they will be more like a continuation of the study in [12] rather than further enrichment of the present study tackling the theoretical interests in characterizing the worst instant for separating the past and future.

Appendix A. Proof of (17) under $D_{12} = 0$. This appendix is devoted to the derivation of (17) for $p = 2$ and $p = \infty$. Thus, we assume that $D_{12} = 0$ throughout this appendix and note that $\mathbf{H}_p^{[\Theta]}$ can readily be defined also for $\Theta \in [-h, 0)$. Hence, by h-periodicity, we aim at establishing the following instead:

$$\|\mathbf{H}_p^{[\Theta]}\| \to \|\mathbf{H}_p^{[0]}\| \quad (\Theta \to -0). \tag{40}$$

To this end, we begin by evaluating $|x(0) - x(\theta)|_p$ for $\theta \in [-h, 0)$ as follows.

LEMMA A.1. *Let $x(-\infty) = 0$, $\psi_{-\infty} = 0$, $w \in L_2(-\infty, 0)$, and $\|w\|_{2-}^{[0]} \le 1$. Then, for each $\varepsilon > 0$, there exists $\Theta \in [-h, 0)$ independent of w such that the corresponding $x(t)$ in the sampled-data system Σ_{SD} satisfies $|x(0) - x(\theta)|_p \le \varepsilon$ $\forall \theta \in [\Theta, 0)$ $(p = 2, \infty)$.*

Proof. Since $|\cdot|_p,\ p=2,\infty$, are equivalent norms due to finite-dimensionality, it suffices to consider only the case of $p=2$; hence, $|\cdot|_2$ together with the associated induced matrix norm will be denoted simply by $|\cdot|$ in this proof. For notational clarity, we denote the corresponding $x(-h)$ and $u(-h)$ in $\Sigma_{\rm SD}$ by $x_w(-h)$ and $u_w(-h)$, respectively. Since

$$x(\theta)=\exp(A(\theta+h))x_w(-h)+\int_{-h}^{\theta}\exp(A(\theta-t))B_1w(t)dt \\ +\int_{-h}^{\theta}\exp(A(\theta-t))B_2dtu_w(-h) \tag{41}$$

for $\theta\in[-h,0)$, we have

$$\begin{aligned} x(0)-x(\theta)&=(I-\exp(A\theta))\exp(Ah)x_w(-h)\\ &\quad+\int_{\theta}^{0}\exp(-At)B_1w(t)dt+(I-\exp(A\theta))\int_{-h}^{\theta}\exp(-At)B_1w(t)dt\\ &\quad+\int_{\theta}^{0}\exp(-At)B_2dtu_w(-h)+(I-\exp(A\theta))\int_{-h}^{\theta}\exp(-At)B_2dtu_w(-h). \end{aligned}\tag{42}$$

Hence,

$$\begin{aligned} |x(0)-x(\theta)|&\le|I-\exp(A\theta)|\cdot|\exp(Ah)|\cdot|x_w(-h)|\\ &\quad+\left|\int_{\theta}^{0}\exp(-At)B_1w(t)dt\right|+|I-\exp(A\theta)|\cdot\left|\int_{-h}^{\theta}\exp(-At)B_1w(t)dt\right|\\ &\quad+\left|\int_{\theta}^{0}\exp(-At)B_2dt\right|\cdot|u_w(-h)|\\ &\quad+|I-\exp(A\theta)|\cdot\left|\int_{-h}^{\theta}\exp(-At)B_2dt\right|\cdot|u_w(-h)|, \end{aligned}\tag{43}$$

where

$$\left|\int_{\theta}^{0}\exp(-At)B_1w(t)dt\right|\le\left(\int_{\theta}^{0}|\exp(-At)B_1|^2dt\right)^{1/2}\to 0\quad(\theta\to-0)\tag{44}$$

by the Cauchy–Schwarz inequality. Similarly, we readily see that $|\int_{-h}^{\theta}\exp(-At)B_1w(t)dt|$ and $|\int_{-h}^{\theta}\exp(-At)B_2dt|$ have upper bounds independent of $w\in L_2(-\infty,0)$, $\|w\|_{2-}^{[0]}\le 1$, and θ, while $\lim_{\theta\to-0}|\int_{\theta}^{0}\exp(-At)B_2dt|=0$ is obvious. Since $x_w(-h)$ and $u_w(-h)$ are uniformly bounded for $w\in L_2(-\infty,0)$, $\|w\|_{2-}^{[0]}\le 1$, by the stability of $\Sigma_{\rm SD}$, this completes the proof. ∎

We use the above lemma to establish the following results.

PROPOSITION A.2. $\|\mathbf{H}_p^{[\Theta]}\|,\Theta\in[-h,0]$ *is upper semicontinuous at* $\Theta=0$ *for* $p=2,\infty$ *if* $D_{12}=0$*:*

$$\limsup_{\Theta\to-0}\|\mathbf{H}_p^{[\Theta]}\|\le\|\mathbf{H}_p^{[0]}\|.\tag{45}$$

PROPOSITION A.3. $\|\mathbf{H}_p^{[\Theta]}\|,\Theta\in[-h,0]$ *is lower semicontinuous at* $\Theta=0$ *for* $p=2,\infty$ *if* $D_{12}=0$*:*

$$\liminf_{\Theta\to-0}\|\mathbf{H}_p^{[\Theta]}\|\ge\|\mathbf{H}_p^{[0]}\|.\tag{46}$$

Hence, we readily have the following result, establishing (40).

THEOREM A.4. $\|\mathbf{H}_p^{[\Theta]}\|$, $\Theta \in [-h, 0]$ *is continuous at* $\Theta = 0$ *for* $p = 2, \infty$ *if* $D_{12} = 0$.

Our remaining task is to prove the above two propositions. We begin by introducing a slightly modified notion of the quasi L_∞/L_2 Hankel norm at Θ (by modifying (15)). More precisely, we introduce the norm notation

$$\|\mathbf{H}_p\|^{[\Theta,\infty)/(-\infty,0)} := \sup_{w \in L_2(-\infty,0)} \frac{\|z\|_{\infty,p}^{[\Theta]}}{\|w\|_{2-}^{[0]}} \tag{47}$$

whose precise meanings (particularly about the left-hand side) will be described in the following; we begin by recalling that the left-hand side of (15) is also denoted simply by $\|\mathbf{H}_p^{[\Theta]}\|$ as noted below this equation, but let us further consider introducing yet another equivalent notation $\|\mathbf{H}_p\|^{[\Theta]}$. The idea behind this alternative notation is to avoid possible confusion stemming from the fact that the norm symbol $\|\cdot\|$ in $\|\mathbf{H}_p^{[\Theta]}\|$ actually depends on the quantity inside it; in fact, the value of Θ in $\mathbf{H}_p^{[\Theta]}$ is what determines the norm $\|\cdot\|$ precisely. Instead of this situation, the new alternative notation $\|\mathbf{H}_p\|^{[\Theta]}$ is meant for being quite specific with respect to the choice of the norm, but with the sacrifice of introducing possible ambiguity instead about what the operator $\mathbf{H}_p$ means. The standpoint in the notation $\|\mathbf{H}_p\|^{[\Theta]}$, however, is that $\mathbf{H}_p$ is regarded as a kind of Hankel operator in which the "past" and "future" are regarded to have a point of contact only at the instant Θ. That is, the norm symbol $\|\cdot\|^{[\Theta]}$ immediately implies that $\|\mathbf{H}_p\|^{[\Theta]}$ is actually shorthand notation for $\|\mathbf{H}_p^{[\Theta]}\|^{[\Theta]}$, the norm of the quasi L_∞/L_2 operator $\mathbf{H}_p^{[\Theta]}$. The notation on the left-hand side of (47) is meant for a similar sense, where $\mathbf{H}_p$ therein is regarded as a kind of Hankel operator[4] in which the "past" is defined as the interval $(-\infty, 0)$, while the "future" is defined as the interval $[\Theta, \infty)$. Considering somewhat different Hankel type of operators simultaneously is convenient in the following arguments, where all these operators are simply denoted by $\mathbf{H}_p$ whose distinctions will be made by the different norm symbols surrounding them.

Proof of Proposition A.2. Since $(-\infty, \Theta) \subset (-\infty, 0)$ for $\Theta \in [-h, 0)$, it is obvious that

$$\|\mathbf{H}_p^{[\Theta]}\| = \|\mathbf{H}_p\|^{[\Theta]} = \|\mathbf{H}_p\|^{[\Theta,\infty)/(-\infty,\Theta)} \le \|\mathbf{H}_p\|^{[\Theta,\infty)/(-\infty,0)} \quad \forall \Theta \in [-h, 0). \tag{48}$$

Here, we aim at comparing the rightmost $\|\mathbf{H}_p\|^{[\Theta,\infty)/(-\infty,0)}$ with $\|\mathbf{H}_p^{[0]}\| = \|\mathbf{H}_p\|^{[0]} = \|\mathbf{H}_p\|^{[0,\infty)/(-\infty,0)}$. Since the difference comes only from that for the intervals on which the outputs are evaluated, it is useful to evaluate $|z(\theta)|_p$ ($\theta \in [\Theta, 0)$) under $w \in L_2(-\infty, 0)$ and $\|w\|_{2-} \le 1$, where $|C_1|_p$ denotes the norm of the matrix C_1 induced from the vector p-norm. We readily have

$$\begin{aligned} |z(\theta)|_p &= |C_1 x(\theta)|_p \\ &\le |C_1 x(0)|_p + |C_1(x(\theta) - x(0))|_p \\ &\le |z(0)|_p + |C_1|_p \varepsilon \quad (\forall \theta \in [\Theta, 0)). \end{aligned} \tag{49}$$

[4]If $\Theta = -h$, the operator is essentially nothing but the overlap L_∞/L_2 Hankel operator introduced in section 3.2.

Note that $\Theta \in [-h, 0)$ (dependent on ε) satisfying the last inequality is ensured to exist by Lemma A.1. This inequality leads to

$$\begin{aligned} \|\mathbf{H}_p\|^{[\Theta,\infty)/(-\infty,0)} &= \max\left(\|\mathbf{H}_p\|^{[\Theta,0)/(-\infty,0)}, \|\mathbf{H}_p\|^{[0,\infty)/(-\infty,0)}\right) \\ &= \max\left(\sup_{w\in L_2(-\infty,0)} \frac{\sup_{\theta\in[\Theta,0)}|z(\theta)|_p}{\|w\|_{2-}^{[0]}}, \|\mathbf{H}_p\|^{[0]}\right) \\ &\le \max\left(|C_1|_p\varepsilon + \sup_{w\in L_2(-\infty,0)} \frac{|z(0)|_p}{\|w\|_{2-}^{[0]}}, \|\mathbf{H}_p\|^{[0]}\right) \\ &\le \max\left(|C_1|_p\varepsilon + \|\mathbf{H}_p\|^{[0]}, \|\mathbf{H}_p\|^{[0]}\right) \\ &= |C_1|_p\varepsilon + \|\mathbf{H}_p^{[0]}\|. \end{aligned} \tag{50}$$

Hence, by (48),

$$\|\mathbf{H}_p^{[\Theta]}\| \le |C_1|_p\varepsilon + \|\mathbf{H}_p^{[0]}\|. \tag{51}$$

Recall that $\Theta \in [-h, 0)$ in this inequality is such that (49) holds, whose existence was ensured by Lemma A.1. Hence, it is obvious that (51) holds for every $\Theta \in [\Theta_0, 0)$ for some $\Theta_0 = \Theta_0(\varepsilon) \in [-h, 0)$. This immediately implies that

$$\limsup_{\Theta\to-0} \|\mathbf{H}_p^{[\Theta]}\| \le \|\mathbf{H}_p^{[0]}\| + |C_1|_p\varepsilon. \tag{52}$$

Since $\varepsilon > 0$ is arbitrary, we have established the assertion of the proposition.

Proof of Proposition A.3. Let $w^\star \in L_2(-\infty, 0)$ be the worst input[5] for the quasi L_∞/L_2 Hankel operator $\mathbf{H}_p^{[0]}$ at $\Theta = 0$ attaining its norm $\|\mathbf{H}_p^{[0]}\|$ such that $\|w^\star\|_{2-}^{[0]} = 1$. Let us take an arbitrary $\theta \in [-h, 0)$ and let us define $w_\theta^\star \in L_2(-\infty, \theta)$ by the truncation of $w^\star$ after $t = \theta$ (i.e., $w_\theta^\star(t) = 0,\ t \in [\theta, 0)$); note that $\|w_\theta^\star\|_{2-}^{[\theta]} \le 1$. Let $x(-\infty) = 0$ and $\psi_{-\infty} = 0$ and let us denote $x(0)$ in the sampled-data system Σ_{SD} for the input $w^\star$ by $x^\star(0)$ and that for the input $w_\theta^\star$ by $x_\theta^\star(0)$. Then, by arguments similar to Lemma A.1, we can establish the following claim for $p = 2, \infty$:

> For each $\varepsilon > 0$, there exists $\Theta \in [-h, 0)$ such that $|x^\star(0) - x_\theta^\star(0)|_p \le \varepsilon\ \forall\theta \in [\Theta, 0)$.

Note that $|x^\star(0) - x_\theta^\star(0)|_p \le \varepsilon$ implies that $|u_0^\star - u_{0\theta}^\star|_p \le |D_\Psi C_2|_p\varepsilon$ by the third equation of (3) and the second equation of (4), where $u_0^\star$ and $u_{0\theta}^\star$ denote u_0 under the input $w^\star$ and the input $w_\theta^\star$, respectively.

Let us define $T \ge 0$ by

$$\|\mathbf{H}_p^{[0]} w^\star\|_{\infty,p}^{[0]} = |(\mathbf{H}_p^{[0]} w^\star)(T)|_p, \tag{53}$$

which is well-defined[6] by the continuity of z ensured by $D_{12} = 0$ together with stability of Σ_{SD} ensuring $z(t) \to 0$ as $t \to \infty$. Since $D_{12} = 0$, we have $z(T) = C_1 \exp(AT)x(0) + G_T u_0$ when $w(t) = 0,\ t \ge 0$, where $G_T := C_1 \int_0^T \exp(A(T-t))B_2 dt$. Hence, it follows from $\|w_\theta^\star\|_{2-}^{[\theta]} \le 1$ that

[5] See the proof of Lemma 3.8 for the existence of the worst input.

[6] We assume $T \in [0, h)$ without loss of generality (see the proof of Theorem 3.5 for essentially the same arguments).

$$\begin{aligned}\|\mathbf{H}_p^{[\theta]}\| &\geq \big|C_1\exp(AT)x_\theta^\star(0)+G_T u_{0\theta}^\star\big|_p\\ &\geq \big|C_1\exp(AT)x^\star(0)+G_T u_0^\star\big|_p - \big|C_1\exp(AT)(x^\star(0)-x_\theta^\star(0))\big|_p\\ &\quad - \big|G_T(u_0^\star-u_{0\theta}^\star)\big|_p\\ &= \big|(\mathbf{H}_p^{[0]}w^\star)(T)\big|_p - \big|C_1\exp(AT)(x^\star(0)-x_\theta^\star(0))\big|_p - \big|G_T(u_0^\star-u_{0\theta}^\star)\big|_p. \end{aligned} \tag{54}$$

Since $|(\mathbf{H}_p^{[0]}w^\star)(T)|_p = \|\mathbf{H}_p^{[0]}\|$ by the definitions of $w^\star$ and T, it follows from the above claim that

$$\|\mathbf{H}_p^{[\theta]}\| \geq \|\mathbf{H}_p^{[0]}\| - (|C_1\exp(AT)|_p + |G_T|_p|D_\Psi C_2|_p)\varepsilon \quad \forall\theta\in[\Theta(\varepsilon),0). \tag{55}$$

Hence, we have

$$\liminf_{\theta\to -0}\|\mathbf{H}_p^{[\theta]}\| \geq \|\mathbf{H}_p^{[0]}\| - (|C_1\exp(AT)|_p + |G_T|_p|D_\Psi C_2|_p)\varepsilon. \tag{56}$$

Since $\varepsilon > 0$ is arbitrary and T is independent of ε, we have established the assertion of the proposition (after rewriting θ as Θ).

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