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Heterotic string field theory with cyclic L_∞ structure

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We construct a complete heterotic string field theory that includes both the Neveu–Schwarz and Ramond sectors. We give a construction of general string products, which realizes a cyclic L_∞ structure and thus provides with a gauge-invariant action in the homotopy algebraic formulation. Through a map of the string fields, we also give the Wess–Zumino–Witten-like action in the large Hilbert space, and verify its gauge invariance independently.

Subject Index B28

1. Introduction

There are three main formulations of superstring field theories: the formulation based on a homotopy algebraic structure in the small Hilbert space [1,2], the WZW[1]-like formulation in the large Hilbert space [3,4], and Sen's formulation with an extra free string field [5,6], each of which has both advantages and disadvantages. They are complementary and worth studying independently. In this paper we focus on the former two formulations since they are not yet fully established for all the superstring field theories, while Sen's formulation is.

In these formulations important progress has recently been made: a complete gauge-invariant action for the open superstring field theory including both the Neveu–Schwarz (NS) and Ramond (R) sectors was constructed first in the WZW-like formulation [7] and soon afterwards in the homotopy algebraic formulation based on the A_∞ structure [8]. In spite of these successes and several related developments [9–16], these two formulations are not yet fully satisfactory. Gauge-invariant actions are only constructed for the NS sector in the heterotic string field theory and for the NS–NS sector in the type II superstring field theory. The purpose of this paper is to fill in some of these missing pieces by constructing a complete gauge-invariant action for the heterotic string field theory in both the homotopy algebraic and the WZW-like formulations.

In string field theory interaction of strings is described by means of string products. The way to construct their basic part, which defines how strings connect, is well established in bosonic string field theory [17–19]. The problem in superstring field theories is to find a general prescription to insert appropriate operators for saturating the picture number and to construct the proper string products required for the gauge-invariant action.

In the homotopy algebraic formulation, such a prescription is given by taking homotopy algebraic structures, an A_∞ structure for the open superstring and an L_∞ structure for the heterotic and type II

[1] Wess–Zumino–Witten.

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superstring, as a guiding principle. Proper string products for the NS sector of the open and heterotic string and for the NS–NS sector of the type II superstring have been constructed as a solution of differential equations for its generating function [1,2]. Although this was successfully extended to the structure including the Ramond sector in the open superstring [8], similar extension in the heterotic and type II superstring is not sufficient to construct gauge-invariant actions because of the lack of cyclicity [10]. In this paper we propose a similar but slightly different prescription to construct string products realizing a *cyclic* L_∞ algebra, and construct a complete gauge-invariant action for the heterotic string field theory in the homotopy algebraic formulation. Then, after confirming that it reproduces the first-quantized four-point amplitudes, we rewrite the action in the WZW-like form and also construct a complete WZW-like action through a field redefinition.

The paper is organized as follows. In Sect. 2 we review the homotopy algebraic formulation for the heterotic string field theory. A gauge-invariant action and gauge transformation are constructed on the assumption that the proper string products realizing a cyclic L_∞ structure are given for any combination of the NS and Ramond string fields. Section 3 is devoted to a concrete construction of such proper string products realizing a cyclic L_∞ structure in two steps. First we consider an L_∞ algebra, which we call a combined L_∞ algebra in this paper, respecting (not the Ramond number but) the cyclic Ramond number. It naturally decomposes two L_∞ algebras, which can be called the dynamical and constraint L_∞ algebras [11]. Although these two are neither cyclic nor closed in the small Hilbert space, the combined L_∞ algebra can be cyclic with respect to the simple symmetric symplectic form. We give in the second step a similarity transformation which transforms two decomposed algebras to a desired cyclic L_∞ algebra and the constraint restricting the products in the small Hilbert space. Then, a concrete prescription to construct the proper string products realizing the combined L_∞ algebra is given. After decomposing the commutator of coderivations into two operations projecting onto the definite cyclic Ramond number, we propose equations for the generating function of (slightly generalized) string products generalizing the L_∞ relation and the closedness condition in the small Hilbert space. Introducing another kind of string product, gauge products, we show that the solution of these equations can be obtained by solving some differential equations of generating functions of string and gauge products iteratively. We confirm, in Sect. 4, that the heterotic string field theory we constructed reproduces the well-known first-quantized four-point amplitudes, including those with the Ramond external states. In Sect. 5 we give a field redefinition which maps the string fields to those in the WZW-like formulation. After rewriting the action in the WZW-like form, we can obtain a complete WZW-like action through this field redefinition. The gauge invariance of the WZW-like action is verified independently without referring to the L_∞ structure. After a summary and discussion in Srct. 6, we add four appendices for details which could not be included in the text. For the reader unfamiliar with the coalgebraic representation of L_∞ algebra, we briefly summarize it in Appendix A. Appendix B is devoted to discussing the ways to distinguish string products according to the number of Ramond states. In Appendix C we prove the cyclicity of the string products constructed in the text. The identity used in Sect. 5 is proved in Appendix D.

2. Heterotic string field theory in the homotopy algebraic formulation

Let us first summarize several basics of heterotic string field theory in the homotopy algebraic formulation. The heterotic string field Φ has two components:

$$\Phi = \Phi_{\mathrm{NS}} + \Phi_{\mathrm{R}} \in \mathcal{H} = \mathcal{H}_{\mathrm{NS}} + \mathcal{H}_{\mathrm{R}}. \tag{2.1}$$

The first component $\Phi_{\rm NS}$ is a Grassmann even NS string field in the small Hilbert space $\mathcal{H}_{\rm NS}$ at ghost number 2 and picture number -1. The second component $\Phi_{\rm R}$ is a Grassmann even R string field in the small Hilbert space $\mathcal{H}_{\rm R}$ at ghost number 2 and picture number $-1/2$. Since the heterotic string is a closed string, the string field Φ is restricted by the closed string constraints:

$$b_0^-\Phi \;=\; L_0^-\Phi \;=\; 0. \tag{2.2}$$

Additionally, the R string field $\Phi_{\rm R}$ satisfies the condition

$$XY\Phi_{\rm R} \;=\; \Phi_{\rm R}, \tag{2.3}$$

where X and Y are defined by

$$X \;=\; -\,\delta(\beta_0)G_0 + \delta'(\beta_0)b_0, \tag{2.4}$$

$$Y \;=\; -\,2c_0^+\delta'(\gamma_0). \tag{2.5}$$

The operator X is the picture-changing operator acting on states with picture number $-1/2$ and Y is its inverse in the following sense acting on those with picture number $-3/2$. They are Belavin–Polyakov–Zamolodchikov (BPZ) even and satisfy

$$XYX \;=\; X, \qquad YXY \;=\; Y, \qquad [Q,X] \;=\; 0, \tag{2.6}$$

from which we can show that the operator XY in the constraint in Eq. (2.3) is a projection operator. We call the space of the states restricted by Eq. (2.3) the restricted Hilbert space, or sometimes simply the restricted space, denoted by $\mathcal{H}^{\rm res}$. Note that the Becchi–Rouet–Stora–Tyutin (BRST) operator is closed in the restricted space: $XYQXY = QXY$.

We define a symplectic form of the small Hilbert space $\omega_{\rm s}$ by

$$\omega_{\rm s}(\Phi_1,\Phi_2) \;=\; (-1)^{|\Phi_1|}\langle\Phi_1,\Phi_2\rangle \;=\; (-1)^{|\Phi_1|}\langle\Phi_1|c_0^-|\Phi_2\rangle, \tag{2.7}$$

where $\langle\Phi_1|$ is the BPZ conjugate state of $|\Phi_1\rangle$. The symbol $|\Phi|$ denotes the Grassmann property of the string field Φ : $|\Phi| = 0$ (1) if the string field Φ is Grassmann even (odd). The symplectic form $\omega_{\rm s}$ is graded anti-symmetric:

$$\omega_{\rm s}(\Phi_1,\Phi_2) \;=\; -(-1)^{|\Phi_1||\Phi_2|}\omega_{\rm s}(\Phi_2,\Phi_1). \tag{2.8}$$

The BRST charge satisfies

$$\omega_{\rm s}(Q\Phi_1,\Phi_2) \;=\; -(-1)^{|\Phi_1|}\omega_{\rm s}(\Phi_1,Q\Phi_2). \tag{2.9}$$

The symplectic form Ω of the restricted Hilbert space is then defined by

$$\Omega(\Phi_1,\Phi_2) \;=\; \omega_{\rm s}(\Phi_{1\rm NS},\Phi_{2\rm NS}) + \omega_{\rm s}(\Phi_{1\rm R},Y\Phi_{2\rm R}) \tag{2.10}$$

for restricted fields Φ_1 and Φ_2. We also define here a symplectic form of the large Hilbert space $\omega_{\rm l}$ by

$$\omega_{\rm l}(\varphi_1,\varphi_2) \;=\; -(-1)^{|\varphi_1|}\langle\varphi_1,\varphi_2\rangle_{\rm l} \;=\; -(-1)^{|\varphi_1|}{}_{\rm l}\langle\varphi_1|c_0^-|\varphi_2\rangle_{\rm l} \tag{2.11}$$

for later use, where φ_1 and φ_2 are some string fields in the large Hilbert space $\mathcal{H}_l : \varphi_1, \varphi_2 \in \mathcal{H}_l$. The η satisfies

$$\omega_l(\eta\varphi_1, \varphi_2) = -(-1)^{|\varphi_1|}\omega_l(\varphi_1, \eta\varphi_2). \tag{2.12}$$

If one of the arguments, suppose $\varphi_2 \equiv \Phi_2$, is in the small Hilbert space, we can relate ω_l to ω_s as[2]

$$\omega_l(\varphi_1, \Phi_2) = \omega_s(\eta\varphi_1, \Phi_2). \tag{2.13}$$

In the large Hilbert space, X can be written as the BRST exact form $X = \{Q, \Xi\}$ with

$$\Xi = \xi + (\Theta(\beta_0)\eta\xi - \xi)P_{-3/2} + (\xi\eta\Theta(\beta_0) - \xi)P_{-1/2}, \tag{2.14}$$

where P_n is the projector onto the states with picture number n.

The kinetic term of the action is written by using these symplectic forms as

$$\begin{aligned} S_0 &= \frac{1}{2}\Omega(\Phi, Q\Phi) \\ &= \frac{1}{2}\omega_s(\Phi_{NS}, Q\Phi_{NS}) + \frac{1}{2}\omega_s(\Phi_R YQ\Phi_R). \end{aligned} \tag{2.15}$$

The other fundamental ingredients of heterotic string field theory are multi-closed-string products,

$$L_n(\Phi_1, \dots, \Phi_n), \qquad (n \geq 1), \tag{2.16}$$

which make a string field from n string fields $\Phi_1, \dots, \Phi_n$. They are graded symmetric under interchange of the n string fields. If each string product carries proper ghost and picture numbers, as explained in detail in the next section, the heterotic string interaction is described by using such string products with $n \geq 2$. In addition, since the heterotic string field in this formulation is in the restricted small Hilbert space, the string products have also to be closed in the restricted space: The R component of Eq. (2.16) has also to be in the restricted Hilbert space. The action of heterotic string field theory is written as

$$S = \sum_{n=0}^{\infty} \frac{1}{(n+2)!}\,\Omega(\Phi, L_{n+1}(\underbrace{\Phi, \dots, \Phi}_{n+1})), \tag{2.17}$$

where the one-string product is identified as the BRST charge: $L_1 = Q$. This is invariant under the gauge transformation

$$\delta\Phi = \sum_{n=0}^{\infty} \frac{1}{n!} L_{n+1}(\underbrace{\Phi, \dots, \Phi}_{n}, \Lambda) \tag{2.18}$$

if the string products L_n satisfy the L_∞ relations

$$\sum_{\sigma}\sum_{m=1}^{n}(-1)^{\epsilon(\sigma)}\frac{1}{m!(n-m)!}L_{n-m+1}(L_m(\Phi_{\sigma(1)}, \dots, \Phi_{\sigma(m)}), \Phi_{\sigma(m+1)}, \dots, \Phi_{\sigma(n)}) = 0 \tag{2.19}$$

[2] Here we assume that the BPZ inner products in the large and small Hilbert spaces are related as $\langle\xi\Phi_1, \Phi_2\rangle_l = \langle\Phi_1, \Phi_2\rangle$. We denote the zero-mode ξ_0 as ξ in this paper for notational simplicity.

and the cyclicity condition

$$\Omega(\Phi_1, L_n(\Phi_2, \ldots, \Phi_{n+1})) \;=\; -(-1)^{|\Phi_1|}\,\Omega(L_n(\Phi_1, \ldots, \Phi_n), \Phi_{n+1}). \tag{2.20}$$

Here, σ in Eq. (2.19) denotes the permutation from $\{1, \ldots, n\}$ to $\{\sigma(1), \ldots, \sigma(n)\}$ and the factor $\epsilon(\sigma)$ is the sign factor of permutation of string fields from $\{\Phi_1, \ldots, \Phi_n\}$ to $\{\Phi_{\sigma(1)}, \ldots, \Phi_{\sigma(n)}\}$. If the set of string products satisfies these conditions it is called a cyclic L_∞ algebra. The problem of constructing the heterotic string field theory reduces to the problem of constructing a set of string products realizing a cyclic L_∞ algebra.[3] However, the asymmetry of the symplectic form Ω between the NS and the R sectors complicates the construction of string products cyclic across both sectors [10]. In the next section we propose a way to construct them in two steps.

3. Constructing string products with cyclic L_∞ structure

Now let us construct a set of string products realizing a cyclic L_∞ algebra. We use a coalgebraic representation which is convenient to discuss an infinite number of multi-string products collectively. Its basic definitions and properties are summarized in Appendix A to make the paper self-contained.

3.1. *Prescription*

Denote a coderivation corresponding to an $n+2$ string product ($n \geq 0$) with picture number $p \geq 0$ as $\boldsymbol{B}^{(p)}_{n+2}$. As mentioned in the previous section, string products have to have proper ghost and picture numbers to describe interactions of heterotic strings. Here we only need to consider the picture number since the heterotic string field Φ has the same ghost number as the bosonic closed string field.

To describe the heterotic string interaction the output string state has to have the same picture number as the heterotic string field: the picture number of its NS (R) component has to be equal to -1 $(-1/2)$. In order to discuss this kind of picture number counting it is useful to introduce the Ramond number and the cyclic Ramond number [10,15] by

$$\begin{pmatrix} \text{Ramond} \\ \text{cyclic Ramond} \end{pmatrix} \text{number} \;=\; \#\,\text{of Ramond inputs} \mp \#\,\text{of Ramond outputs}. \tag{3.1}$$

Let us first suppose that $2r$ of $n+2$ inputs are the R states. Since the R (NS) states represent the space-time fermions (bosons), the output is an NS state. Such a string product is characterized by the Ramond number $2r$ and the cyclic Ramond number $2r$. From picture number conservation we have

$$\left(-\frac{1}{2}\right) \times 2r + (-1) \times (n+2-2r) + p \;=\; -1. \tag{3.2a}$$

If $2r+1$ of the inputs are the R states, the output is the R state and we have

$$\left(-\frac{1}{2}\right) \times (2r+1) + (-1) \times (n+1-2r) + p \;=\; -\frac{1}{2}. \tag{3.2b}$$

[3] We also call them string products with an L_∞ structure.

This is the case characterized by the Ramond number $2r$ and the cyclic Ramond number $2r+2$. These two Eqs. (3.2) have a common solution: $n = p + r - 1$. A candidate coderivation is therefore

$$\boldsymbol{Q} + \sum_{p,r=0}^{\infty} \left(\boldsymbol{B}^{(p)}_{p+r+1}|^{2r}_{2r} + \boldsymbol{B}^{(p)}_{p+r+1}|^{2r+2}_{2r} \right) \;=\; \boldsymbol{Q} + \sum_{p,r=0}^{\infty} \boldsymbol{B}^{(p)}_{p+r+1}|_{2r}, \tag{3.3}$$

with $\boldsymbol{B}^{(0)}_1|_0 \equiv 0$. However, the cyclicity cannot be transparent in this form since the Ramond number is not invariant under the "cyclic permutation" as in Eq. (2.20). Instead, we consider the string products[4]

$$\boldsymbol{B} \;\equiv\; \sum_{p,r=0}^{\infty} \boldsymbol{B}^{(p)}_{p+r+1}|^{2r} \;=\; \sum_{p,r=0}^{\infty} \left(\boldsymbol{B}^{(p)}_{p+r+1}|^{2r}_{2r} + \boldsymbol{B}^{(p)}_{p+r+1}|^{2r}_{2r-2} \right), \tag{3.4}$$

respecting the cyclic Ramond number which is invariant under the permutation, which makes it possible to construct the cyclic coderivation, at the cost, however, that the string products of Eq. (3.4) do not satisfy the condition in Eq. (3.2b). The picture number deficit of the string products in the second term is equal to 1: the output of the second term is the R state with picture number $-1/2 - 1 = -3/2$. This combination of string products appears naturally as the difference of two coderivations $\boldsymbol{D}$ and $\boldsymbol{C}$ with picture number deficit 0 and 1 respectively:

$$\boldsymbol{D} - \boldsymbol{C} \;=\; \boldsymbol{Q} - \boldsymbol{\eta} + \boldsymbol{B}, \tag{3.5}$$

with

$$\pi_1 \boldsymbol{D} \;=\; \pi_1 \boldsymbol{Q} + \sum_{p,r=0}^{\infty} \pi_1 \boldsymbol{B}^{(p)}_{p+r+1}|^{2r}_{2r} \;=\; \pi_1 \boldsymbol{Q} + \pi^0_1 \boldsymbol{B}, \tag{3.6}$$

$$\pi_1 \boldsymbol{C} \;=\; \pi_1 \boldsymbol{\eta} - \sum_{p,r=0}^{\infty} \pi_1 \boldsymbol{B}^{(p)}_{p+r+1}|^{2r}_{2r-2} \;=\; \pi_1 \boldsymbol{\eta} - \pi^1_1 \boldsymbol{B}. \tag{3.7}$$

Here, π^0_1 (π^1_1) is the projection operator onto $\mathcal{H}_{\rm NS}$ ($\mathcal{H}_{\rm R}$) introduced in Appendix A. This combination of string products can be cyclic with respect to the symmetric symplectic form $\omega_{\rm s}$.

Note that if we suppose that the coderivation $\boldsymbol{D} - \boldsymbol{C}$ satisfies the L_∞ relation

$$[\boldsymbol{D} - \boldsymbol{C}, \boldsymbol{D} - \boldsymbol{C}] = 0, \tag{3.8}$$

it is not closed in the small Hilbert space since $[\boldsymbol{\eta}, \boldsymbol{D} - \boldsymbol{C}] \neq 0$. As the first step, we construct an L_∞ algebra $\boldsymbol{D} - \boldsymbol{C}$ cyclic with respect to the symplectic form $\omega_{\rm l}$, which is comparatively easy due to the symmetry of $\omega_{\rm l}$ between the NS and R sectors. We denote this cyclic L_∞ algebra as $(\mathcal{H}_{\rm l}, \omega_{\rm l}, \boldsymbol{D} - \boldsymbol{C})$.

By decomposing Eq. (3.8) with the picture number deficit, we can find that it is equivalent to the set of conditions

$$[\boldsymbol{D}, \boldsymbol{D}] \;=\; [\boldsymbol{C}, \boldsymbol{C}] \;=\; [\boldsymbol{D}, \boldsymbol{C}] \;=\; 0. \tag{3.9}$$

[4] Here and hereafter we use the convention that a quantity with the Ramond or cyclic Ramond number outside the range given in Appendix B is identically equal to zero.

These two mutually commutative L_∞ algebras $\boldsymbol{D}$ and $\boldsymbol{C}$ are the heterotic string analogs of the dynamical and constraint L_∞ algebras in Ref. [11], respectively. They are neither cyclic nor closed in the small Hilbert space. We denote them as $(\mathcal{H}_l, \boldsymbol{D})$, $(\mathcal{H}_l, \boldsymbol{C})$.

Once the cyclic L_∞ algebra $(\mathcal{H}_l, \omega_l, \boldsymbol{D}-\boldsymbol{C})$ is constructed, a desired cyclic L_∞ algebra $(\mathcal{H}^{\mathrm{res}}, \Omega, \boldsymbol{L})$ can be obtained in a similar manner to Ref. [11]. Let us consider, in the second step, a simultaneous similarity transformation of $\boldsymbol{D}$ and $\boldsymbol{C}$ generated by an invertible cohomomorphism $\hat{\boldsymbol{F}}$. Since a similarity transformation preserves L_∞ structure, they are still mutually commutative L_∞ algebras after transformation. Suppose that $\hat{\boldsymbol{F}}$ transforms $\boldsymbol{C}$ to $\hat{\boldsymbol{F}}^{-1}\boldsymbol{C}\hat{\boldsymbol{F}} = \boldsymbol{\eta}$. Then $\boldsymbol{L} \equiv \hat{\boldsymbol{F}}^{-1}\boldsymbol{D}\hat{\boldsymbol{F}}$ is commutative to $\boldsymbol{\eta}$ and hence an L_∞ algebra in the small Hilbert space $(\mathcal{H}, \boldsymbol{L})$. In order that $\boldsymbol{L}$ is further an L_∞ algebra in the restricted space $\mathcal{H}^{\mathrm{res}}$, the Ramond component of the output $\pi_1^1\boldsymbol{L}$ must be in the restricted Hilbert space. So we assume here that $\boldsymbol{L}$ has the form

$$\pi_1\boldsymbol{L} \ = \ \pi_1\boldsymbol{Q} + \pi_1^0\boldsymbol{b} + X\pi_1^1\boldsymbol{b}, \tag{3.10}$$

with $\boldsymbol{b} = \sum_{n=2}^{\infty}\boldsymbol{b}_n$. If $\boldsymbol{L}$ has this form, we have

$$\Omega(\Phi_1, L_n(\Phi_2, \dots, \Phi_{n+1})) \ = \ \omega_{\mathrm{s}}(\Phi_1, b_n(\Phi_2, \dots, \Phi_{n+1})), \qquad (n \geq 2) \tag{3.11}$$

for $\Phi_1, \dots, \Phi_{n+1} \in \mathcal{H}^{\mathrm{res}}$, and hence the cyclicity of $\boldsymbol{L}$ with respect to Ω is translated to simpler cyclicity of $\boldsymbol{b}$ with respect to ω_{s}. We need two steps since $\boldsymbol{b}$ itself cannot be string products of an L_∞ algebra since it does not have a definite picture number deficit.

Next, let us show that the desired cohomomorphism $\hat{\boldsymbol{F}}$ is concretely given by

$$\pi_1\hat{\boldsymbol{F}}^{-1} \ = \ \pi_1\mathbb{I}_{\mathcal{SH}} - \Xi\pi_1^1\boldsymbol{B}. \tag{3.12}$$

Acting $\hat{\boldsymbol{F}}$ from the right of both sides, we have

$$\pi_1\hat{\boldsymbol{F}} \ = \ \pi_1\mathbb{I}_{\mathcal{SH}} + \Xi\pi_1^1\boldsymbol{B}\hat{\boldsymbol{F}}. \tag{3.13}$$

By decomposing Eq. (3.12) we have

$$\pi_1^0\hat{\boldsymbol{F}}^{-1} = \pi_1^0\mathbb{I}_{\mathcal{SH}}, \tag{3.14}$$

$$\pi_1^1\hat{\boldsymbol{F}}^{-1} = \pi_1^1(\mathbb{I}_{\mathcal{SH}} - \Xi\boldsymbol{B}) \ = \ \pi_1^1(\eta\Xi\,\mathbb{I}_{\mathcal{SH}} + \Xi\boldsymbol{C}). \tag{3.15}$$

Using $\pi_1^0\boldsymbol{C} = \pi_1^0\boldsymbol{\eta}$, $\pi_1^0\hat{\boldsymbol{F}} = \pi_1^0\hat{\boldsymbol{F}}^{-1}\hat{\boldsymbol{F}} = \pi_1^0\mathbb{I}_{\mathcal{SH}}$, and $\boldsymbol{C}^2 = 0$, we find that

$$\pi_1^0\hat{\boldsymbol{F}}^{-1}\boldsymbol{C}\hat{\boldsymbol{F}} = \pi_1^0\boldsymbol{\eta}, \tag{3.16}$$

$$\begin{aligned}\pi_1^1\hat{\boldsymbol{F}}^{-1}\boldsymbol{C}\hat{\boldsymbol{F}} &= \eta\Xi\pi_1^1(\boldsymbol{\eta} - \boldsymbol{B})\hat{\boldsymbol{F}} \\ &= \pi_1^1\boldsymbol{\eta}(\mathbb{I}_{\mathcal{SH}} - \Xi\boldsymbol{B})\hat{\boldsymbol{F}} \ = \ \pi_1^1\boldsymbol{\eta},\end{aligned} \tag{3.17}$$

and hence

$$\pi_1\hat{\boldsymbol{F}}^{-1}\boldsymbol{C}\hat{\boldsymbol{F}} \ = \ \pi_1\boldsymbol{\eta}. \tag{3.18}$$

Similarly, from $[\boldsymbol{C},\boldsymbol{D}]=0$ and $\pi_1^1\boldsymbol{D}=\pi_1^1\boldsymbol{Q}$, we find that

$$\begin{aligned}\pi_1^1\boldsymbol{B}\boldsymbol{D} &= \pi_1^1(\boldsymbol{\eta}-\boldsymbol{C})\boldsymbol{D}\\ &= \boldsymbol{\eta}\pi_1^1\boldsymbol{Q}+\pi_1^1\boldsymbol{Q}\boldsymbol{C}=-Q\pi_1^1\boldsymbol{B},\end{aligned} \tag{3.19}$$

and then we can show that

$$\pi_1^0\hat{\boldsymbol{F}}^{-1}\boldsymbol{D}\hat{\boldsymbol{F}} = \pi_1^0(\boldsymbol{Q}+\boldsymbol{B}\hat{\boldsymbol{F}}), \tag{3.20}$$

$$\pi_1^1\hat{\boldsymbol{F}}^{-1}\boldsymbol{D}\hat{\boldsymbol{F}} = \pi_1^1(\boldsymbol{Q}+X\boldsymbol{B}\hat{\boldsymbol{F}}), \tag{3.21}$$

which provides an expected form of $\boldsymbol{L}$ as

$$\pi_1\boldsymbol{L} \;=\; \pi_1\hat{\boldsymbol{F}}^{-1}\boldsymbol{D}\hat{\boldsymbol{F}} \;=\; \pi_1\boldsymbol{Q}+\pi_1^0\boldsymbol{B}\hat{\boldsymbol{F}}+X\pi_1^1\boldsymbol{B}\hat{\boldsymbol{F}}. \tag{3.22}$$

Finally, we can also show, as proved in Appendix C, that if $\boldsymbol{B}$ is cyclic with respect to ω_l then $\pi_1\boldsymbol{b}=\pi_1\boldsymbol{B}\hat{\boldsymbol{F}}$ is cyclic with respect to ω_s. In this way we can obtain the desired cyclic L_∞ algebra $(\mathcal{H}^{\mathrm{res}},\Omega,\boldsymbol{L})$ from a cyclic L_∞ algebra $(\mathcal{H}_l,\omega_l,\boldsymbol{D}-\boldsymbol{C})$.

3.2. *Explicit construction*

Now, the remaining task is to construct concretely a cyclic L_∞ algebra $(\mathcal{H}_l,\omega_l,\boldsymbol{D}-\boldsymbol{C})$. Let us start with considering the string products with picture number zero, $\boldsymbol{L}^{(0)}=\sum_{n=0}^{\infty}\boldsymbol{L}^{(0)}_{n+1}$. The cyclic L_∞ algebra $(\mathcal{H}_l,\omega_l,\boldsymbol{L}^{(0)})$ can easily be constructed in the same way as in the bosonic closed string field theory. Hereafter we call it the bosonic L_∞ algebra, which is assumed to be known. We define a generating function

$$\boldsymbol{L}^{(0)}(s) \;=\; \boldsymbol{Q}+\sum_{m,r=0}^{\infty}s^m\boldsymbol{L}^{(0)}_{m+r+1}|^{2r} \;\equiv\; \boldsymbol{Q}+\boldsymbol{L}^{(0)}_B(s); \tag{3.23}$$

counting the picture number deficit, the picture number of its NS component $\pi_1^0\boldsymbol{L}^{(0)}$ is $-1-m$, and that of its R component $\pi_1^1\boldsymbol{L}^{(0)}$ is $-3/2-m$. It reduces to $\boldsymbol{L}^{(0)}$ at $s=1$. From the L_∞ relation $[\boldsymbol{L}^{(0)},\boldsymbol{L}^{(0)}]=0$, we can show that $\boldsymbol{L}^{(0)}_B(s)$ satisfies

$$[\boldsymbol{Q},\boldsymbol{L}^{(0)}_B(s)]+\frac{1}{2}[\boldsymbol{L}^{(0)}_B(s),\boldsymbol{L}^{(0)}_B(s)]^1+\frac{s}{2}[\boldsymbol{L}^{(0)}_B(s),\boldsymbol{L}^{(0)}_B(s)]^2 \;=\; 0, \tag{3.24a}$$

and is closed in the small Hilbert space

$$[\boldsymbol{\eta},\boldsymbol{L}^{(0)}_B(s)] \;=\; 0, \tag{3.24b}$$

where $[\ ,\ ]^{1,2}$ are the operations introduced in Eq. (B.13), which are obtained by projecting the commutator onto the components with the definite cyclic Ramond number.

On the other hand, the L_∞ relation $[\boldsymbol{D}-\boldsymbol{C},\boldsymbol{D}-\boldsymbol{C}]=0$ is equivalent to the equations

$$[\boldsymbol{Q},\boldsymbol{B}(t)]+\frac{1}{2}[\boldsymbol{B}(t),\boldsymbol{B}(t)]^1 = 0, \tag{3.25a}$$

$$[\boldsymbol{\eta},\boldsymbol{B}(t)]-\frac{t}{2}[\boldsymbol{B}(t),\boldsymbol{B}(t)]^2 = 0 \tag{3.25b}$$

for the generating function counting the picture number,

$$\boldsymbol{B}(t) \;=\; \sum_{n,r=0}^{\infty} t^n \boldsymbol{B}^{(n)}_{n+r+1}\big|^{2r}. \tag{3.26}$$

We extend Eqs. (3.25) to the equations

$$\boldsymbol{I}(s,t) \;\equiv [\boldsymbol{Q},\boldsymbol{B}(s,t)] + \frac{1}{2}[\boldsymbol{B}(s,t),\boldsymbol{B}(s,t)]^1 + \frac{s}{2}[\boldsymbol{B}(s,t),\boldsymbol{B}(s,t)]^2 \;=\; 0, \tag{3.27a}$$

$$\boldsymbol{J}(s,t) \;\equiv [\boldsymbol{\eta},\boldsymbol{B}(s,t)] - \frac{t}{2}[\boldsymbol{B}(s,t),\boldsymbol{B}(s,t)]^2 \;=\; 0 \tag{3.27b}$$

by introducing the string products with non-zero picture number deficit counted by s in

$$\boldsymbol{B}(s,t) \;=\; \sum_{m,n,r=0}^{\infty} s^m t^n \boldsymbol{B}^{(n)}_{m+n+r+1}\big|^{2r} \;=\; \sum_{n=0}^{\infty} t^n\, \boldsymbol{B}^{(n)}(s). \tag{3.28}$$

The desired string products are included as $\boldsymbol{B}(t) = \boldsymbol{B}(0,t)$ since Eqs. (3.27) reduce to Eqs. (3.25) at $s = 0$. In the following we construct the string products satisfying Eqs. (3.27).

Let us first show that such string products can be obtained by postulating the differential equations

$$\partial_t \boldsymbol{B}(s,t) = [\boldsymbol{Q},\boldsymbol{\lambda}(s,t)] + [\boldsymbol{B}(s,t),\boldsymbol{\lambda}(s,t)]^1 + s[\boldsymbol{B}(s,t),\boldsymbol{\lambda}(s,t)]^2, \tag{3.29a}$$

$$[\boldsymbol{\eta},\boldsymbol{\lambda}(s,t)] = \partial_s \boldsymbol{B}(s,t) + t\,[\boldsymbol{B}(s,t),\boldsymbol{\lambda}(s,t)]^2 \tag{3.29b}$$

by introducing a degree-even coderivation

$$\boldsymbol{\lambda}(s,t) \;=\; \sum_{m,n,r=0}^{\infty} s^m t^n \boldsymbol{\lambda}^{(n+1)}_{m+n+r+2}\big|^{2r} \;=\; \sum_{n=0}^{\infty} t^n\, \boldsymbol{\lambda}^{(n+1)}(s), \tag{3.30}$$

which is an analog of the gauge products in Ref. [2]. Equations (3.29) imply

$$\partial_t \boldsymbol{I}(s,t) \;= [\boldsymbol{I}(s,t),\boldsymbol{\lambda}(s,t)]^1 + s\,[\boldsymbol{I}(s,t),\boldsymbol{\lambda}(s,t)]^2, \tag{3.31}$$

$$\partial_t \boldsymbol{J}(s,t) \;=\; -\,\partial_s \boldsymbol{I}(s,t) - t[\boldsymbol{I}(s,t),\boldsymbol{\lambda}(s,t)]^2 + [\boldsymbol{J}(s,t),\boldsymbol{\lambda}(s,t)]^1 + s[\boldsymbol{J}(s,t),\boldsymbol{\lambda}(s,t)]^2. \tag{3.32}$$

Since these equations are homogeneous in $\boldsymbol{I}(s,t)$ and $\boldsymbol{J}(s,t)$, Eqs. (3.27) follow from the equations at $t = 0$ which agree with Eqs. (3.24) satisfied by the bosonic products $\boldsymbol{L}^{(0)}_B(s)$ assumed to be known. All the products $\boldsymbol{B}(s,t)$ satisfying Eqs. (3.27) can be determined by solving the differential equations in Eqs. (3.29) under the condition

$$\boldsymbol{B}(s,0) = \boldsymbol{B}^{(0)}(s) = \boldsymbol{L}^{(0)}_B(s). \tag{3.33}$$

Finally, let us find a concrete expression of the string products. By expanding Eqs. (3.29) in t, we obtain

$$(n+1)\boldsymbol{B}^{(n+1)}(s) \;= [\boldsymbol{Q},\boldsymbol{\lambda}^{(n+1)}(s)] + \sum_{n'=0}^{n}[\boldsymbol{B}^{(n-n')}(s),\boldsymbol{\lambda}^{(n'+1)}(s)]^1 + \sum_{n'=0}^{n} s\,[\boldsymbol{B}^{(n-n')}(s),\boldsymbol{\lambda}^{(n'+1)}(s)]^2, \tag{3.34a}$$

$$[\boldsymbol{\eta},\boldsymbol{\lambda}^{(n+1)}(s)] \;= \partial_s \boldsymbol{B}^{(n)}(s) \;+ \sum_{n'=0}^{n-1}[\boldsymbol{B}^{(n-n'-1)}(s),\boldsymbol{\lambda}^{(n'+1)}(s)]^2, \tag{3.34b}$$

which can be solved iteratively. For $n = 0$ we have

$$\begin{aligned} \boldsymbol{B}^{(1)}(s) &= [\boldsymbol{Q}, \boldsymbol{\lambda}^{(1)}(s)] + [\boldsymbol{L}_B^{(0)}(s), \boldsymbol{\lambda}^{(1)}(s)]^1 + s\,[\boldsymbol{L}_B^{(0)}(s), \boldsymbol{\lambda}^{(1)}(s)]^2, && (3.35a)\\ [\boldsymbol{\eta}, \boldsymbol{\lambda}^{(1)}(s)] &= \partial_s \boldsymbol{L}_B^{(0)}(s). && (3.35b)\end{aligned}$$

Since $[\boldsymbol{\eta}, \boldsymbol{L}_B^{(0)}(s)] = 0$, we can consistently determine the gauge products $\boldsymbol{\lambda}^{(1)}(s)$ from Eq. (3.35b) as

$$\begin{aligned} \boldsymbol{\lambda}^{(1)}(s) &= \sum_{m,r=0}^{\infty} (m+1)s^m \frac{1}{m+r+3} \Big(\xi\, L_{m+r+2}^{(0)}|^{2r} - L_{m+r+2}^{(0)}|^{2r}(\xi \wedge \mathbb{I}_{m+r+1}) \Big) \\ &\equiv \xi \circ \partial_s \boldsymbol{L}_B^{(0)}(s), \end{aligned} \tag{3.36}$$

respecting the cyclicity. Then, all the terms on the right-hand side of Eq. (3.35a) are given, which determines $\boldsymbol{B}^{(1)}(s)$. In order to go to the next step and beyond, we note here that the right-hand side of Eq. (3.34b),

$$\boldsymbol{K}^{(n)}(s) \equiv \partial_s \boldsymbol{B}^{(n)}(s) + \sum_{m=0}^{n-1} [\boldsymbol{B}^{(n-m-1)}(s), \boldsymbol{\lambda}^{(m+1)}(s)]^2, \tag{3.37}$$

satisfies

$$\begin{aligned} [\boldsymbol{\eta}, \boldsymbol{K}^{(n)}(s)] &= \partial_s \boldsymbol{J}^{(n)}(s) + \sum_{n'=0}^{n-1} [\boldsymbol{J}^{(n-n'-1)}(s), \boldsymbol{\lambda}^{(n'+1)}(s)]^2 \\ &\quad - \sum_{n'=0}^{n-1} [\boldsymbol{B}^{(n-n'-1)}(s), \Big([\boldsymbol{\eta}, \boldsymbol{\lambda}^{(n'+1)}(s)] - \boldsymbol{K}^{(n')}(s) \Big)]^2, \end{aligned} \tag{3.38}$$

where

$$\boldsymbol{J}^{(n)}(s) \equiv \frac{1}{n!} \partial_t^n \boldsymbol{J}(s,t)|_{t=0} = [\boldsymbol{\eta}, \boldsymbol{B}^{(n)}(s)] - \frac{1}{2} \sum_{n'=0}^{n-1} [\boldsymbol{B}^{(n-n'-1)}(s), \boldsymbol{B}^{(n')}(s)]^2. \tag{3.39}$$

In the next step, $n = 1$, this becomes

$$\begin{aligned} [\boldsymbol{\eta}, \boldsymbol{K}^{(1)}(s)] &= \partial_s \boldsymbol{J}^{(1)}(s) + [\boldsymbol{J}^{(0)}(s), \boldsymbol{\lambda}^{(1)}(s)]^2 - [\boldsymbol{B}^{(0)}(s), \Big([\boldsymbol{\eta}, \boldsymbol{\lambda}^{(1)}(s)] - \boldsymbol{K}^{(0)}(s) \Big)] \\ &= 0, \end{aligned} \tag{3.40}$$

since $\boldsymbol{B}^{(0)}(s)$, $\boldsymbol{B}^{(1)}(s)$, and $\boldsymbol{\lambda}^{(1)}(s)$ are already determined so as to satisfy

$$\boldsymbol{J}^{(0)}(s) = \boldsymbol{J}^{(1)}(s) = [\boldsymbol{\eta}, \boldsymbol{\lambda}^{(1)}(s)] - \boldsymbol{K}^{(0)}(s) = 0. \tag{3.41}$$

This enables us to determine $\boldsymbol{\lambda}^{(2)}(s)$ consistently from Eq. (3.34b) with $n = 1$,

$$[\boldsymbol{\eta}, \boldsymbol{\lambda}^{(2)}] - \boldsymbol{K}^{(1)} = 0, \tag{3.42}$$

as

$$\boldsymbol{\lambda}^{(2)}(s) = \xi \circ \Big(\partial_s \boldsymbol{B}^{(1)}(s) + [\boldsymbol{B}^{(0)}(s), \boldsymbol{\lambda}^{(1)}(s)] \Big), \tag{3.43}$$

$$
\begin{array}{cccccccc}
\boldsymbol{L}_B^{(0)}(s) & \boldsymbol{B}^{(1)}(s) & \boldsymbol{B}^{(2)}(s) & \cdots & \boldsymbol{B}^{(n)}(s) & \cdots & \cdots \\
\searrow\uparrow & \searrow\uparrow & \searrow & \cdots & \searrow\uparrow & \searrow & \cdots & \cdots \\
& \boldsymbol{\lambda}^{(1)}(s) & \boldsymbol{\lambda}^{(2)}(s) & \cdots & \boldsymbol{\lambda}^{(n)}(s) & \boldsymbol{\lambda}^{(n+1)}(s) & \cdots
\end{array}
$$

Fig. 1. The flow along which all the $\boldsymbol{B}^{(n)}(s)$ and $\boldsymbol{\lambda}^{(n)}(s)$ are determined recursively from $\boldsymbol{B}^{(0)}(s) = \boldsymbol{L}_B^{(0)}(s)$.

and to determine $\boldsymbol{B}^{(2)}$ from Eq. (3.34a) with $n = 1$. Similarly, when we solved Eqs. (3.34) iteratively and determined $\boldsymbol{\lambda}^{(n)}(s)$ and $\boldsymbol{B}^{(n)}(s)$ for $n \leq n_0$, the equations

$$\boldsymbol{J}^{(n)}(s) \;=\; [\boldsymbol{\eta}, \boldsymbol{\lambda}^{(n)}(s)] - \boldsymbol{K}^{(n-1)}(s) \;=\; 0 \tag{3.44}$$

hold for $n \leq n_0$ by construction, which provides $[\boldsymbol{\eta}, \boldsymbol{K}^{(n_0)}] = 0$ from Eq. (3.38) and guarantees to go one step forward consistently. Thus we can recursively determine all the $\boldsymbol{B}^{(n)}(s)$ and $\boldsymbol{\lambda}^{(n)}(s)$, hence $\boldsymbol{B}(s,t)$ and $\boldsymbol{\lambda}(s,t)$, along the flow depicted in Fig. 1. The concrete expression can be obtained through the recursive relations

$$\boldsymbol{\lambda}^{(n+1)}(s) \;= \xi \circ \Big(\partial_s \boldsymbol{B}^{(n)}(s) \;+ \sum_{n'=0}^{n-1} [\boldsymbol{B}^{(n-n'-1)}(s), \boldsymbol{\lambda}^{(n'+1)}(s)]^2\Big), \tag{3.45a}$$

$$\begin{aligned}(n+1)\boldsymbol{B}^{(n+1)}(s) \;&= [\boldsymbol{Q}, \boldsymbol{\lambda}^{(n+1)}(s)] \\ &\quad + \sum_{n'=0}^{n} [\boldsymbol{B}^{(n-n')}(s), \boldsymbol{\lambda}^{(n'+1)}(s)]^1 + \sum_{n'=0}^{n} s\,[\boldsymbol{B}^{(n-n')}(s), \boldsymbol{\lambda}^{(n'+1)}(s)]^2,\end{aligned} \tag{3.45b}$$

starting from the initial condition $\boldsymbol{B}^{(0)}(s) = \boldsymbol{L}_B^{(0)}(s)$. All the $\boldsymbol{B}(s,t)$ and $\boldsymbol{\lambda}(s,t)$ are cyclic with respect to ω_l by construction. Thus a cyclic L_∞ $(\mathcal{H}_l, \omega_l, \boldsymbol{D} - \boldsymbol{C})$ is obtained.

4. Four-point amplitudes

It is interesting how the heterotic string field theory we have constructed reproduces the first-quantized amplitudes. In this section we demonstrate it by focusing on the four-point amplitudes as the simplest example. We concretely calculate three types of four-point amplitudes, four-NS, two-NS-two-R, and four-R, following a similar process to the one given in Ref. [1]. In this section we denote the string fields $\Phi_{\rm NS}$ and $\Phi_{\rm R}$ as Φ and Ψ, respectively, for notational simplicity. We take the Siegel gauge satisfying the conditions

$$b_0^+ \Phi \;=\; \beta_0 \Psi \;=\; 0. \tag{4.1}$$

The propagators in this gauge are obtained in the usual way as

$$\Pi_{\rm NS} \;=\; -\frac{b_0^+ b_0^-}{L_0^+}\mathcal{P}, \qquad \Pi_{\rm R} \;=\; -\frac{b_0^+ b_0^- X}{L_0^+}\mathcal{P}, \tag{4.2}$$

where

$$\mathcal{P} \;\equiv\; \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta L_0^-}. \tag{4.3}$$

4.1. Four-NS amplitude

To warm up let us first consider the four-NS amplitude, although it can be calculated without using the new results obtained in this paper. The first-quantized four-NS amplitude is expressed, for example, in the form

$$\mathcal{A}_4^{\rm 1st}(\Phi_1,\Phi_2,\Phi_3,\Phi_4) \;=\; \int d^2z\,\langle\big((X_0)^2\Phi_1(0)\big)\,(b_{-1}^+b_{-1}^-\Phi_2(z))\Phi_3(1)\Phi_4(\infty)\rangle\rangle, \tag{4.4}$$

where $X_0=\{Q,\xi\}$ and $\Phi_1,\dots,\Phi_4$ are on-shell physical NS vertex operators in the -1 picture: it satisfies $Q\Phi=0$. The correlator is evaluated in the small Hilbert space on the complex z-plane. We put two X_0s in front of Φ_1, which is possible since the physical amplitudes are independent of the position of the picture-changing operators. Since the structure of bosonic moduli is common to the bosonic closed string, we can express this using the bosonic closed string products $L_n^{(0)}$ as

$$\begin{aligned}\mathcal{A}_4^{\rm 1st}(\Phi_1,\Phi_2,\Phi_3,\Phi_4) \;=\omega_{\rm s}\Big((X_0)^2\Phi_1,\Big(&L_3^{(0)}(\Phi_2,\Phi_3,\Phi_4)-L_2^{(0)}(\Phi_2,\frac{b_0^+}{L_0^+}L_2^{(0)}(\Phi_3,\Phi_4))\\&-L_2^{(0)}(\Phi_3,\frac{b_0^+}{L_0^+}L_2^{(0)}(\Phi_4,\Phi_2))-L_2^{(0)}(\Phi_4,\frac{b_0^+}{L_0^+}L_2^{(0)}(\Phi_2,\Phi_3))\Big)\Big).\end{aligned} \tag{4.5}$$

The moduli integral $b_0^-\mathcal{P}$ is hidden behind the definition of the linear map $L_n^{(0)}$. This can be regarded as a multi-linear map:

$$\langle\mathcal{A}_4|:\mathcal{H}_Q^{\rm NS}\otimes(\mathcal{H}_Q^{\rm NS})^{\wedge3}\;\longrightarrow\;\mathbb{C}, \tag{4.6}$$

where $\mathcal{H}_Q^{\rm NS}\subset\mathcal{H}_{\rm NS}$ is the subspace of states annihilated by Q. Putting $\Phi_1,\dots,\Phi_4$ out, we can express Eq. (4.5) as

$$\langle\mathcal{A}_4^{1st}|\;=\;\langle\omega_{\rm s}|(X_0)^2\otimes\Big(L_3^{(0)}-L_2^{(0)}\big(\mathbb{I}\wedge\frac{b_0^+}{L_0^+}L_2^{(0)}\big)\Big). \tag{4.7}$$

Here, $\langle\omega_{\rm s}|$ is a bilinear map representation of the symplectic form $\omega_{\rm s}$. We can also write Eq. (4.7) using the coderivations as

$$\langle\mathcal{A}_4^{1st}|\;=\;\langle\omega_{\rm s}|(X_0)^2\otimes\pi_1^0\Big(\boldsymbol{L}_3^{(0)}|_0^0-\boldsymbol{L}_2^{(0)}|_0^0\frac{b_0^+}{L_0^+}\boldsymbol{L}_2^{(0)}|_0^0\Big), \tag{4.8}$$

where $\frac{b_0^+}{L_0^+}\boldsymbol{L}_2^{(0)}$ is the coderivation derived from $\frac{b_0^+}{L_0^+}L_2^{(0)}$.

On the other hand, the four-NS amplitude obtained from the heterotic string field theory is given by

$$\begin{aligned}&\mathcal{A}_4(\Phi_1,\Phi_2,\Phi_3,\Phi_4)\\&=\omega_{\rm s}\Big(\Phi_1,\Big(L_3^{(2)}(\Phi_2,\Phi_3,\Phi_4)-L_2^{(1)}(\Phi_2,\frac{b_0^+b_0^-}{L_0^+}\mathcal{P}c_0^-L_2^{(1)}(\Phi_3,\Phi_4))\\&\quad-L_2^{(1)}(\Phi_3,\frac{b_0^+b_0^-}{L_0^+}\mathcal{P}c_0^-L_2^{(1)}(\Phi_4,\Phi_2))-L_2^{(1)}(\Phi_4,\frac{b_0^+b_0^-}{L_0^+}\mathcal{P}c_0^-L_2^{(1)}(\Phi_2,\Phi_3))\Big)\Big)\end{aligned}$$

$$= \omega_{\rm s}\bigg(\Phi_1, \Big(L_3^{(2)}(\Phi_2, \Phi_3, \Phi_4) - L_2^{(1)}(\Phi_2, \frac{b_0^+}{L_0^+} L_2^{(1)}(\Phi_3, \Phi_4))$$
$$- L_2^{(1)}(\Phi_3 \frac{b_0^+}{L_0^+} L_2^{(1)}(\Phi_4 \Phi_2)) - L_2^{(1)}(\Phi_4, \frac{b_0^+}{L_0^+} L_2^{(1)}(\Phi_2, \Phi_3)) \Big) \bigg). \tag{4.9}$$

The second equality follows from the fact that the string field $L_2^{(1)}(\Phi_1, \Phi_2)$ satisfies the closed string constraints in Eq. (2.2). Using the coderivations this can also be written as

$$\langle \mathcal{A}_4 | \ = \ \langle \omega_{\rm s} | \mathbb{I} \otimes \pi_1^0 \Big(\boldsymbol{L}_3^{(2)}|_0^0 - \boldsymbol{L}_2^{(1)}|_0^0 \frac{b_0^+}{L_0^+} \boldsymbol{L}_2^{(1)}|_0^0 \Big). \tag{4.10}$$

We show that Eq. (4.10) actually agrees with the first-quantized expression in Eq. (4.8).

First, we note that $\boldsymbol{L}|^0 = \boldsymbol{B}|^0$ since $\hat{\boldsymbol{F}} = \mathbb{I}$ in the pure NS sector. Substituting the relations

$$\boldsymbol{B}_2^{(1)}|_0^0 \ = \ [\boldsymbol{Q}, \boldsymbol{\lambda}_2^{(1)}|_0^0], \qquad 2\boldsymbol{B}_3^{(2)}|_0^0 \ = \ [\boldsymbol{Q}, \boldsymbol{\lambda}_3^{(2)}|_0^0] + [\boldsymbol{B}_2^{(1)}|_0^0, \boldsymbol{\lambda}_2^{(1)}|_0^0\,] \tag{4.11}$$

following from Eq. (3.34a) into Eq. (4.10), we have

$$\langle \mathcal{A}_4 | \ = \ \frac{1}{2} \langle \omega_{\rm s} | \mathbb{I} \otimes \pi_1^0 \Big([\boldsymbol{Q}, \boldsymbol{\lambda}_3^{(2)}|_0^0] + [\boldsymbol{B}_2^{(1)}|_0^0, \boldsymbol{\lambda}_2^{(1)}|_0^0]$$
$$- [\boldsymbol{Q}, \boldsymbol{\lambda}_2^{(1)}|_0^0] \frac{b_0^+}{L_0^+} \boldsymbol{B}_2^{(1)}|_0^0 - \boldsymbol{B}_2^{(1)}|_0^0 \frac{b_0^+}{L_0^+} [\boldsymbol{Q}, \boldsymbol{\lambda}_2^{(1)}|_0^0] \Big). \tag{4.12}$$

Moving to the large Hilbert space using $\langle \omega_{\rm s} | = \langle \omega_{\rm l} | \xi \otimes \mathbb{I}$ following from Eq. (2.13), one can pull $\boldsymbol{Q}$ out from Eq. (4.12) and obtain

$$\langle \mathcal{A}_4 | \ = \ -\frac{1}{2} \langle \omega_{\rm l} | X_0 \otimes \pi_1^0 \Big(\boldsymbol{\lambda}_3^{(2)}|_0^0 - \boldsymbol{\lambda}_2^{(1)}|_0^0 \frac{b_0^+}{L_0^+} \boldsymbol{B}_2^{(1)}|_0^0 - \boldsymbol{B}_2^{(1)}|_0^0 \frac{b_0^+}{L_0^+} \boldsymbol{\lambda}_2^{(1)}|_0^0 \Big), \tag{4.13}$$

except for the terms vanishing when they hit the states in $\mathcal{H}_Q^{\rm NS}$. Similarly, we can further rewrite Eq. (4.13) as

$$\langle \mathcal{A}_4 | \ = \frac{1}{2} \langle \omega_{\rm l} | \xi X_0 \otimes \pi_1^0 \Big([\boldsymbol{\eta}, \boldsymbol{\lambda}_3^{(2)}|_0^0] - [\boldsymbol{\eta}, \boldsymbol{\lambda}_2^{(1)}|_0^0] \frac{b_0^+}{L_0^+} \boldsymbol{B}_2^{(1)}|_0^0 - \boldsymbol{B}_2^{(1)}|_0^0 \frac{b_0^+}{L_0^+} [\boldsymbol{\eta}, \boldsymbol{\lambda}_2^{(1)}|_0^0] \Big)$$
$$= \frac{1}{2} \langle \omega_{\rm l} | \xi X_0 \otimes \pi_1^0 \Big(\boldsymbol{B}_3^{(1)}|_0^0 - \boldsymbol{L}_2^{(0)}|_0^0 \frac{b_0^+}{L_0^+} \boldsymbol{B}_2^{(1)}|_0^0 - \boldsymbol{B}_2^{(1)}|_0^0 \frac{b_0^+}{L_0^+} \boldsymbol{L}_2^{(0)}|_0^0 \Big) \tag{4.14}$$

using the relations

$$[\boldsymbol{\eta}, \boldsymbol{\lambda}_3^{(2)}|_0^0] = \boldsymbol{B}_3^{(1)}|_0^0, \qquad [\boldsymbol{\eta}, \boldsymbol{\lambda}_2^{(1)}|_0^0] = \boldsymbol{L}_2^{(0)}|_0^0 \tag{4.15}$$

following from Eq (3.34b), except for the terms vanishing when they hit the states in the small Hilbert space. We can repeat similar steps once more. Again using

$$\boldsymbol{B}_2^{(1)}|_0^0 \ = \ [\boldsymbol{Q}, \boldsymbol{\lambda}_2^{(1)}|_0^0], \qquad \boldsymbol{B}_3^{(1)}|_0^0 \ = \ [\boldsymbol{Q}, \boldsymbol{\lambda}_3^{(1)}|_0^0] + [\boldsymbol{L}_2^{(0)}|_0^0, \boldsymbol{\lambda}_2^{(1)}|_0^0], \tag{4.16a}$$
$$[\boldsymbol{\eta}, \boldsymbol{\lambda}_3^{(1)}|_0^0] \ = \ 2\boldsymbol{L}_3^{(0)}|_0^0, \qquad [\boldsymbol{\eta}, \boldsymbol{\lambda}_2^{(1)}|_0^0] \ = \ \boldsymbol{L}_2^{(0)}|_0^0, \tag{4.16b}$$

we find that it agrees with the first-quantized amplitude in Eq. (4.8):

$$\begin{aligned}\langle\mathcal{A}_4| &= -\frac{1}{2}\langle\omega_{\rm l}|(X_0)^2\otimes\pi_1^0\Big(\boldsymbol{\lambda}_3^{(1)}|_0^0-\boldsymbol{\lambda}_2^{(1)}|_0^0\,\frac{b_0^+}{L_0^+}\,\boldsymbol{L}_2^{(0)}|_0^0-\boldsymbol{L}_2^{(0)}\,\frac{b_0^+}{L_0^+}\,\boldsymbol{\lambda}_2^{(1)}|_0^0\Big)\\ &=\langle\omega_{\rm s}|(X_0)^2\otimes\pi_1^0\Big(\boldsymbol{L}_3^{(0)}|_0^0-\boldsymbol{L}_2^{(0)}|_0^0\,\frac{b_0^+}{L_0^+}\,\boldsymbol{L}_2^{(0)}|_0^0\Big).\end{aligned}\tag{4.17}$$

4.2. *Two-NS-two-R amplitude*

Similarly, let us show the agreement of the two-NS-two-R amplitude. The first quantized amplitude is now expressed as the multi-linear map

$$\langle\mathcal{A}_4|\,:\,\mathcal{H}_Q^{\rm R}\otimes\mathcal{H}_Q^{\rm R}\wedge(\mathcal{H}_Q^{\rm NS})^{\wedge 2}\to\mathbb{C},\tag{4.18}$$

which can be rewritten using the coderivations as

$$\langle\mathcal{A}_4^{1st}|\;=\;\langle\omega_{\rm s}|X_0\otimes\pi_1^1\Big(\boldsymbol{L}_3^{(0)}|_0^2-\boldsymbol{L}_2^{(0)}|_0^2\,\frac{b_0^+}{L_0^+}\,\boldsymbol{L}_2^{(0)}|_0^0-\boldsymbol{L}_2^{(0)}|_0^2\,\frac{b_0^+}{L_0^+}\,\boldsymbol{L}_2^{(0)}|_0^2\Big).\tag{4.19}$$

In the heterotic string field theory, on the other hand, the two-NS-two-R amplitude can be calculated as

$$\begin{aligned}\langle\mathcal{A}_4| &=\langle\omega_{\rm s}|\mathbb{I}\otimes\pi_1^1\Big(\boldsymbol{b}_3|_0^2-\boldsymbol{b}_2|_0^2\,\frac{b_0^+}{L_0^+}\,\boldsymbol{L}_2|_0^0-\boldsymbol{b}_2|_0^2\,\frac{b_0^+X}{L_0^+}\,\boldsymbol{L}_2|_0^2\Big)\\ &=\langle\omega_{\rm s}|\mathbb{I}\otimes\pi_1^1\Big(\boldsymbol{B}_3^{(1)}|_0^2+\boldsymbol{L}_2^{(0)}|_0^2\,\Xi\boldsymbol{L}_2^{(0)}|_0^2-\boldsymbol{L}_2^{(0)}|_0^2\,\frac{b_0^+}{L_0^+}\,\boldsymbol{B}_2^{(1)}|_0^0-\boldsymbol{L}_2^{(0)}|_0^2\,\frac{b_0^+X}{L_0^+}\,\boldsymbol{L}_2^{(0)}|_0^2\Big),\end{aligned}\tag{4.20}$$

where we used $\boldsymbol{B}_n^{(0)}=\boldsymbol{L}_n^{(0)}$ and

$$\pi_1^1\boldsymbol{b}_2\;=\;\boldsymbol{L}_2^{(0)}|_0^2,\qquad\pi_1^1\boldsymbol{b}_3\;=\;\boldsymbol{L}_3^{(0)}|_2^4+\boldsymbol{B}_3^{(1)}|_0^2+\boldsymbol{L}_2^{(0)}|_0^2\,\Xi\boldsymbol{L}_2^{(0)}|_0^2\tag{4.21}$$

following from the definition $\pi_1\boldsymbol{b}=\pi_1\boldsymbol{B}\hat{\boldsymbol{F}}$. Again, $\Xi\boldsymbol{L}_2^{(0)}$ and $\frac{b_0^+X}{L_0^+}\boldsymbol{L}_2^{(0)}$ are coderivations derived from $\Xi L_2^{(0)}$ and $\frac{b_0^+X}{L_0^+}L_2^{(0)}$, respectively. Using the relations

$$\boldsymbol{B}_2^{(1)}|_0^0\;=\;[\boldsymbol{Q},\boldsymbol{\lambda}_2^{(1)}|_0^0]\qquad\boldsymbol{B}_3^{(1)}|_0^2\;=\;[\boldsymbol{Q},\boldsymbol{\lambda}_3^{(1)}|_0^2]+[\boldsymbol{L}_2^{(0)}|_0^2,\boldsymbol{\lambda}_2^{(1)}|_0^0]|_0^2,\tag{4.22a}$$

$$[\boldsymbol{\eta},\boldsymbol{\lambda}_2^{(1)}|_0^0]\;=\;\boldsymbol{L}_2^{(0)}|_0^0,\qquad[\boldsymbol{\eta},\boldsymbol{\lambda}_3^{(1)}|_0^2]\;=\;\boldsymbol{L}_3^{(0)}|_0^2,\tag{4.22b}$$

and $X=[\boldsymbol{Q},\Xi]$, we find the amplitude obtained from the heterotic string field theory agrees with the first-quantized amplitude:

$$\begin{aligned}\langle\mathcal{A}_4| &= -\langle\omega_{\rm l}|X_0\otimes\pi_1^1\Big(\boldsymbol{\lambda}_3^{(1)}|_0^2-\boldsymbol{L}_2^{(0)}|_0^2\,\frac{b_0^+}{L_0^+}\,\boldsymbol{\lambda}_2^{(1)}|_0^0-\boldsymbol{L}_2^{(0)}|_0^2\,\frac{b_0^+\,\Xi}{L_0^+}\,\boldsymbol{L}_2^{(0)}|_0^2\Big)\\ &=\langle\omega_{\rm s}|X_0\otimes\pi_1^1\Big(\boldsymbol{L}_3^{(0)}|_0^2-\boldsymbol{L}_2^{(0)}|_0^2\,\frac{b_0^+}{L_0^+}\,\boldsymbol{L}_2^{(0)}|_0^0-\boldsymbol{L}_2^{(0)}|_0^2\,\frac{b_0^+}{L_0^+}\,\boldsymbol{L}_2^{(0)}|_0^2\Big)\Big)\;=\;\langle\mathcal{A}_4^{1st}|.\end{aligned}\tag{4.23}$$

4.3. *Four-R amplitude*

Finally, the four-R amplitude obtained from the heterotic string field theory expressed as the map

$$\langle \mathcal{A}_4| \ : \ \mathcal{H}_Q^{\mathrm{R}} \otimes (\mathcal{H}_Q^{\mathrm{R}})^{\wedge 3} \ \to \ \mathbb{C} \tag{4.24}$$

can be calculated as

$$\begin{aligned}\langle \mathcal{A}_4| \ &= \langle \omega_{\mathrm{s}}| \mathbb{I} \otimes \pi_1^1 \Big(\boldsymbol{b}_3|_2^4 - \boldsymbol{b}_2|_0^2 \frac{b_0^+}{L_0^+} \boldsymbol{L}_2|_2^2 \Big) \\ &= \langle \omega_{\mathrm{s}}| \mathbb{I} \otimes \pi_1^1 \Big(\boldsymbol{L}_3^{(0)}|_2^4 - \boldsymbol{L}_2^{(0)}|_0^2 \frac{b_0^+}{L_0^+} \boldsymbol{L}_2^{(0)}|_2^2 \Big)\end{aligned} \tag{4.25}$$

by using Eq. (4.21). This is nothing but the four-R amplitude in the first-quantized formulation.

5. Gauge-invariant action in WZW-like formulation

So far we have constructed a complete gauge-invariant action for the heterotic string field theory based on the cyclic L_∞ algebra in the small Hilbert space. In this section we also construct a WZW-like action by using a field redefinition, and show its gauge invariance independently.

5.1. *Complete action and gauge transformation*

First, we note that if we project $\boldsymbol{B}(s,t)$ and $\boldsymbol{\lambda}(s,t)$ onto the pure NS sector,

$$\boldsymbol{B}(s,t)|^0 \ \equiv \ \sum_{m,n=0}^{\infty} s^m t^n \boldsymbol{B}_{m+n+1}^{(n)}|^0 \ = \ \sum_{m,n=0}^{\infty} s^m \boldsymbol{B}^{[m]}(t)|^0, \tag{5.1}$$

$$\boldsymbol{\lambda}(s,t)|^0 \ \equiv \ \sum_{m,n=0}^{\infty} s^m t^n \boldsymbol{\lambda}_{m+n+2}^{(n+1)}|^0 \ = \ \sum_{m,n=0}^{\infty} s^m \boldsymbol{\lambda}^{[m]}(t)|^0, \tag{5.2}$$

the differential equation in Eqs. (3.29) reduces to

$$\partial_t \boldsymbol{L}(s,t)|^0 \ = [\boldsymbol{L}(s,t)|^0, \boldsymbol{\lambda}(s,t)|^0], \tag{5.3a}$$

$$[\boldsymbol{\eta}, \boldsymbol{\lambda}(s,t)|^0] \ = \partial_s \boldsymbol{L}(s,t)|^0, \tag{5.3b}$$

where $\boldsymbol{L}(s,t)|^0 \equiv \boldsymbol{Q}|^0 + \boldsymbol{B}(s,t)|^0$, and then, from Eqs. (3.27), $\boldsymbol{L}(s,t)|^0$ satisfies

$$[\boldsymbol{L}(s,t)|^0, \boldsymbol{L}(s,t)|^0] \ = \ 0, \qquad [\boldsymbol{\eta}, \boldsymbol{L}(s,t)|^0] \ = \ 0. \tag{5.4}$$

The differential equations in Eqs. (5.3) and L_∞ relations in Eq. (5.4) are nothing but those introduced in Ref. [2]. Thus, by construction, the string and gauge products restricted in the pure NS sector, $\boldsymbol{B}(s,t)|^0$ and $\boldsymbol{\lambda}(s,t)|^0$, reduce to those in Ref. [2]. This implies that the L_∞ algebra restricting Eq. (3.22) in the pure NS sector $\boldsymbol{Q} + \boldsymbol{B}^{[0]}|^0$ can be written in the form of the similarity transformation,

$$\boldsymbol{Q} + \boldsymbol{B}^{[0]}|^0 = \ \hat{\boldsymbol{g}}^{-1} \boldsymbol{Q} \hat{\boldsymbol{g}}, \tag{5.5}$$

generated by the cohomomorphism [13]

$$\hat{\boldsymbol{g}} \ = \ \vec{\mathcal{P}} \exp \Big(\int_0^1 dt \, \boldsymbol{\lambda}^{[0]}(t)|^0 \Big). \tag{5.6}$$

Here, $\vec{\mathcal{P}}$ denotes the path-ordered product from left to right. Using this fact we find that the string products $\boldsymbol{L}$ are transformed by (the inverse of) this similarity transformation as

$$\pi_1\tilde{\boldsymbol{L}} \equiv \pi_1\hat{\boldsymbol{g}}\,\boldsymbol{L}\,\hat{\boldsymbol{g}}^{-1} = \pi_1\boldsymbol{Q} + \pi_1^0\tilde{\boldsymbol{b}} + X\pi_1^1\tilde{\boldsymbol{b}}, \tag{5.7}$$

where

$$\tilde{\boldsymbol{b}} = \hat{\boldsymbol{g}}\,(\boldsymbol{b} - \boldsymbol{B}^{[0]}|^0)\,\hat{\boldsymbol{g}}^{-1}. \tag{5.8}$$

By construction, $\boldsymbol{\lambda}^{[0]}(t)|^0$ is cyclic with respect to ω_{l}. This implies that the similarity transformation generated by $\hat{\boldsymbol{g}}$ preserves the cyclicity, and thus $\tilde{\boldsymbol{b}}$ is also cyclic with respect to ω_{l}.

Next, we rewrite the action in Eq. (2.17) in the WZW-like form by extending the NS string field Φ_{NS} to $\Phi_{\mathrm{NS}}(t)$ with $t \in [0,1]$ satisfying $\Phi_{\mathrm{NS}}(1) = \Phi_{\mathrm{NS}}$ and $\Phi_{\mathrm{NS}}(0) = 0$. Using the cyclicity we find

$$\begin{aligned}
S =& \frac{1}{2}\,\omega_{\mathrm{s}}\big(\Phi_{\mathrm{R}}, YQ\Phi_{\mathrm{R}}\big) + \sum_{n=0}^{\infty}\frac{1}{(n+2)!}\,\omega_{\mathrm{s}}\big(\Phi_{\mathrm{NS}}, L_{n+1}({\Phi_{\mathrm{NS}}}^{n+1})\big)\\
&+ \sum_{n=1}^{\infty}\sum_{r=0}^{\infty}\frac{1}{(n+2r+2)(n-1)!(2r+2)!}\,\omega_{\mathrm{s}}\big(\Phi_{\mathrm{NS}}, b_{n+2r+1}({\Phi_{\mathrm{NS}}}^{n-1}, {\Phi_{\mathrm{R}}}^{2r+2})\big)\\
&+ \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{1}{(n+2r+2)n!(2r+1)!}\,\omega_{\mathrm{s}}\big(\Phi_{\mathrm{R}}, b_{n+2r+1}({\Phi_{\mathrm{NS}}}^{n}, {\Phi_{\mathrm{R}}}^{2r+1})\big)\\
=& \sum_{n=0}^{\infty}\frac{1}{(n+1)!}\int_0^1 dt\;\omega_{\mathrm{s}}\big(\partial_t\Phi_{\mathrm{NS}}(t), L_{n+1}(\Phi_{\mathrm{NS}}(t)^{n+1})\big)\\
&+ \frac{1}{2}\,\omega_{\mathrm{s}}\big(\Phi_{\mathrm{R}}, YQ\Phi_{\mathrm{R}}\big) + \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{1}{n!(2r+2)!}\,\omega_{\mathrm{s}}\big(\Phi_{\mathrm{R}}, b_{n+2r+1}({\Phi_{\mathrm{NS}}}^{n}, {\Phi_{\mathrm{R}}}^{2r+1})\big)\\
=& \int_0^1 dt\;\omega_{\mathrm{l}}\big(\xi\,\partial_t\Phi_{\mathrm{NS}}(t), \pi_1^0\boldsymbol{L}(e^{\wedge\Phi_{\mathrm{NS}}(t)})\big)\\
&+ \frac{1}{2}\,\omega_{\mathrm{s}}(\Phi_{\mathrm{R}}, YQ\Phi_{\mathrm{R}}) + \sum_{r=0}^{\infty}\frac{1}{(2r+2)!}\,\omega_{\mathrm{s}}\big(\Phi_{\mathrm{R}}, \pi_1^1\boldsymbol{b}(e^{\wedge\Phi_{\mathrm{NS}}}\wedge {\Phi_{\mathrm{R}}}^{\wedge 2r+1}\big).
\end{aligned} \tag{5.9}$$

Here, the first term can be mapped to the WZW-like action as follows [13]. Using the similarity transformation in Eq. (5.7) and the identity

$$\omega_{\mathrm{l}}(\pi_1\hat{\boldsymbol{g}}\boldsymbol{l}_1(e^{\wedge\Phi}), \pi_1\hat{\boldsymbol{g}}\boldsymbol{l}_2(e^{\wedge\Phi})) = \omega_{\mathrm{l}}(\pi_1\boldsymbol{l}_1(e^{\wedge\Phi}), \pi_1\boldsymbol{l}_2(e^{\wedge\Phi})), \tag{5.10}$$

for odd coderivations $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$, which we prove in Appendix D, the first term can be written as

$$\int_0^1 dt\;\omega_{\mathrm{l}}\big(\xi\,\partial_t\Phi_{\mathrm{NS}}(t), \pi_1^0\boldsymbol{L}(e^{\wedge\Phi_{\mathrm{NS}}(t)})\big) = \int_0^1 dt\;\omega_{\mathrm{l}}\big(\pi_1\hat{\boldsymbol{g}}\boldsymbol{\xi}_t(e^{\wedge\Phi_{\mathrm{NS}}(t)}), \pi_1^0\boldsymbol{Q}\hat{\boldsymbol{g}}(e^{\wedge\Phi_{\mathrm{NS}}(t)})\big), \tag{5.11}$$

where $\boldsymbol{\xi}_t$ is the one coderivation derived from $\xi\,\partial_t$. By this transformation, the constraint $\eta\Phi = 0$ restricting Φ in the small Hilbert space is mapped to the constraint

$$0 = \pi_1\hat{\boldsymbol{g}}\boldsymbol{\eta}(e^{\wedge(\Phi_{\mathrm{NS}}+\Phi_{\mathrm{R}})}) = \pi_1^0\boldsymbol{L}^{\eta}(e^{\wedge\pi_1^0\hat{\boldsymbol{g}}(e^{\wedge\Phi_{\mathrm{NS}}})}) + \pi_1^1\eta\Phi_{\mathrm{R}}, \tag{5.12}$$

where $\boldsymbol{L}^{\eta} \equiv \hat{\boldsymbol{g}}\, \boldsymbol{\eta}\, \hat{\boldsymbol{g}}^{-1}$. Since this $\boldsymbol{L}^{\eta}$ is nothing but the dual L_∞ products in Ref. [14], the NS component of Eq. (5.12) is the Maurer–Cartan equation for the pure-gauge string field $G_\eta(V)$ in the WZW-like formulation:

$$\boldsymbol{L}^{\eta}(e^{\wedge G_\eta(V)}) \;=\; 0. \tag{5.13}$$

This suggests that we can identify the string fields $(\Phi_{\rm NS}, \Phi_{\rm R})$ with the string fields (V, Ψ) in the WZW-like formulation through the relations

$$\pi_1^0 \hat{\boldsymbol{g}}(e^{\wedge \Phi_{\rm NS}}) \;=\; G_\eta(V), \qquad \Phi_{\rm R} \;=\; \Psi. \tag{5.14}$$

Then the associated fields $B_d(V(t))$ $(d = t, \delta)$ are written as

$$B_d(V(t)) \;=\; \pi_1^0 \hat{\boldsymbol{g}} \boldsymbol{\xi_d}(e^{\wedge \Phi_{\rm NS}(t)}) \tag{5.15}$$

under this identification, where $\boldsymbol{\xi_\delta}$ is the one coderivation $\boldsymbol{\xi_\delta}$ derived from $\xi\delta$. The identities characterizing the associated field,

$$dG_\eta(V(t)) \;=\; \pi_1^0 \boldsymbol{L}^{\eta}(e^{\wedge G_\eta(t)} \wedge B_d(V(t))), \tag{5.16a}$$

$$\partial_t B_\delta(V(t)) - \delta B_t(V(t)) + \pi_1^0 \boldsymbol{L}^{\eta}(e^{\wedge G_\eta(t)} \wedge B_t(V(t)) \wedge B_\delta(V(t))) \;=\; 0, \tag{5.16b}$$

follow from the identifications in Eqs. (5.14) and (5.15). Eventually the first term of the action in Eq. (5.9) is mapped to (the dual form of) the WZW-like action in the pure NS sector:

$$\int_0^1 dt\, \omega_{\rm l}\big(\xi\,\partial_t \Phi_{\rm NS}(t), \pi_1^0 \boldsymbol{L}(e^{\wedge \Phi_{\rm NS}(t)})\big) = \int_0^1 dt\, \omega_{\rm l}\big(B_t(V(t)), QG_\eta(V(t))\big). \tag{5.17}$$

If we note that $\pi_1^1 \boldsymbol{b} = \pi_1^1 \tilde{\boldsymbol{b}} \hat{\boldsymbol{g}}$, the whole action is finally mapped to the complete WZW-like action[5]

$$\begin{aligned} S \;=&\; \int_0^1 dt\, \omega_{\rm l}\big(B_t(V(t)), QG_\eta(V(t))\big) \\ &+ \frac{1}{2}\,\omega_{\rm s}(\Psi, YQ\Psi) + \sum_{r=0}^{\infty} \frac{1}{(2r+2)!}\,\omega_{\rm s}\big(\Psi, \pi_1^1 \tilde{\boldsymbol{b}}(e^{\wedge G_\eta(V)} \wedge \Psi^{\wedge 2r+1})\big). \end{aligned} \tag{5.18}$$

Since the field redefinition in Eq. (5.14) does not uniquely determine the NS string field V, we have an extra gauge invariance in the WZW-like formulation,

$$B_\delta(V) = D_\eta \Omega, \qquad \delta\Psi \;=\; 0, \tag{5.19}$$

which keeps $G_\eta(V)$ invariant. In addition, the gauge transformation in the small Hilbert space formulation,

$$\pi_1 \delta(e^{\wedge(\Phi_{\rm NS}+\Phi_{\rm R})}) \;=\; \pi_1 \boldsymbol{L}(e^{\wedge(\Phi_{\rm NS}+\Phi_{\rm R})} \wedge (\Lambda_{\rm NS} + \Lambda_{\rm R})), \tag{5.20}$$

[5] The last term can also be written as $\sum_{r=0}^{\infty} \frac{1}{(2r+2)!}\,\omega_{\rm l}\big(\Psi, \pi_1^1 \hat{\boldsymbol{F}}_g(e^{\wedge G_\eta(V)} \wedge \Psi^{\wedge 2r+1})\big)$ with $\hat{\boldsymbol{F}}_g = \hat{\boldsymbol{g}}\hat{\boldsymbol{F}}\hat{\boldsymbol{g}}^{-1}$ by using the relation $\pi_1 \hat{\boldsymbol{F}}_g = \pi_1(\mathbb{I}_{\mathcal{SH}} + \Xi \pi_1^1 \tilde{\boldsymbol{b}})$. This form can be seen as a natural extension of the result in Refs. [14,20,21].

is mapped to the gauge transformation in the WZW-like formulation, except for the terms which can be absorbed into the transformation in Eq. (5.19), as

$$B_\delta(V) \;= \pi_1^0 \tilde{\boldsymbol{L}}(e^{\wedge(G_\eta+\Psi)} \wedge (\Lambda - \xi\lambda)) \;=\; Q\Lambda + \pi_1^0 \tilde{\boldsymbol{b}}(e^{\wedge(G_\eta+\Psi)} \wedge (\Lambda - \xi\lambda)), \tag{5.21a}$$

$$\delta\Psi \;= \eta\pi_1^1 \tilde{\boldsymbol{L}}(e^{\wedge(G_\eta+\Psi)} \wedge (\Lambda - \xi\lambda)) \;=\; Q\lambda + X\eta\pi_1^1 \tilde{\boldsymbol{b}}(e^{(G_\eta+\Psi)} \wedge (\Lambda - \xi\lambda)), \tag{5.21b}$$

with the identification of gauge parameters

$$\Lambda \;=\; -\pi_1^0 \hat{\boldsymbol{g}}(e^{\wedge\Phi_{\mathrm{NS}}} \wedge \xi\Lambda_{\mathrm{NS}}), \qquad \lambda \;=\; \Lambda_{\mathrm{R}}. \tag{5.22}$$

5.2. Gauge invariance

Although it should be guaranteed by construction, we prove here that the WZW-like action in Eq. (5.18),

$$S \;=\; \int_0^1 dt\, \omega_l\big(B_t(V(t)), QG_\eta(V(t))\big) + \frac{1}{2}\omega_{\mathrm{s}}(\Psi, YQ\Psi) + \sum_{r=0}^{\infty} \frac{1}{(2r+2)!}\omega_{\mathrm{s}}\big(\Psi, \pi_1^1 \tilde{\boldsymbol{b}}(e^{\wedge G_\eta(V)} \wedge \Psi^{\wedge 2r+1})\big),$$

is invariant under the gauge transformation in Eq. (5.21),

$$B_\delta(V) \;= \pi_1^0 \tilde{\boldsymbol{L}}\big(e^{\wedge(G_\eta+\Psi)} \wedge (\Lambda - \xi\lambda)\big),$$
$$\delta\Psi \;= \eta\pi_1^1 \tilde{\boldsymbol{L}}(e^{\wedge(G_\eta+\Psi)} \wedge (\Lambda - \xi\lambda)),$$

in the WZW-like formulation independently.

Let us first consider an arbitrary variation of the action. In particular, the variation of the last term becomes

$$\begin{aligned}
&\sum_{r=0}^{\infty} \frac{1}{(2r+2)!}\delta\omega_l\big(\xi\Psi, \pi_1^1 \tilde{\boldsymbol{b}}(e^{\wedge G_\eta(V)} \wedge \Psi^{\wedge 2r+1})\big)\\
&= \sum_{r=0}^{\infty} \frac{1}{(2r+2)!}\omega_l\big(\xi\delta\Psi, \pi_1^1 \tilde{\boldsymbol{b}}(e^{\wedge G_\eta(V)} \wedge \Psi^{\wedge 2r+1})\big)\\
&\quad - \sum_{r=0}^{\infty} \frac{2r+1}{(2r+2)!}\omega_l\big(\delta\Psi, \pi_1^1 \tilde{\boldsymbol{b}}(e^{\wedge G_\eta(V)} \wedge \Psi^{\wedge 2r} \wedge \xi\Psi)\big)\\
&\quad - \sum_{r=0}^{\infty} \frac{1}{(2r+2)!}\omega_l\big(\delta G_\eta(V), \pi_1^0 \tilde{\boldsymbol{b}}(e^{\wedge G_\eta(V)} \wedge \Psi^{\wedge 2r+1} \wedge \xi\Psi)\big),
\end{aligned} \tag{5.23}$$

from the cyclicity of $\tilde{\boldsymbol{b}}$. The second term can further be calculated as

$$\begin{aligned}
&-\sum_{r=0}^{\infty} \frac{2r+1}{(2r+2)!}\omega_l\big(\delta\Psi, \pi_1^1 \tilde{\boldsymbol{b}}(e^{\wedge G_\eta(V)} \wedge \Psi^{\wedge 2r} \wedge \xi\Psi)\big)\\
&\qquad = -\sum_{r=0}^{\infty} \frac{2r+1}{(2r+2)!}\omega_l\big(\xi\delta\Psi, \pi_1^1 \boldsymbol{L}^\eta \tilde{\boldsymbol{b}}(e^{\wedge G_\eta(V)} \wedge \Psi^{\wedge 2r} \wedge \xi\Psi)\big)
\end{aligned}$$

$$= \sum_{r=0}^{\infty} \frac{2r+1}{(2r+2)!}\, \omega_{\mathrm{l}}\big(\xi\delta\Psi, \pi_1^1 \tilde{\boldsymbol{b}}(e^{\wedge G_\eta(V)} \wedge \Psi^{\wedge 2r+1})\big), \tag{5.24}$$

where we used $\pi_1^1 \boldsymbol{\eta} = \pi_1^1 \boldsymbol{L}^\eta$ and $[\boldsymbol{L}^\eta, \tilde{\boldsymbol{b}}] = 0$. Adding up the first and second terms, the result is

$$\sum_{r=0}^{\infty} \frac{1}{(2r+1)!}\, \omega_{\mathrm{l}}\big(\xi\delta\Psi\pi_1^1 \tilde{\boldsymbol{b}}(e^{(G_\eta)} \wedge \Psi^{\wedge 2r+1}\big) \;=\; \omega_{\mathrm{l}}\big(\xi\delta\Psi, \pi_1^1 \tilde{\boldsymbol{b}}(e^{(G_\eta(V)+\Psi)})\big). \tag{5.25}$$

From the relation Eq. (5.16a), the third term of Eq. (5.23) becomes

$$\begin{aligned} -\sum_{r=0}^{\infty} \frac{1}{(2r+2)!}\, \omega_{\mathrm{l}}\big(\pi_1^0 \boldsymbol{L}^\eta (e^{\wedge G_\eta(V)} \wedge B_\delta(V)), \pi_1 \tilde{\boldsymbol{b}}(e^{\wedge G_\eta(V)} \wedge \Psi^{\wedge 2r+1} \wedge \xi\Psi)\big) \\ = \sum_{r=0}^{\infty} \frac{1}{(2r+2)!}\, \omega_{\mathrm{l}}\big(B_\delta(V), \pi_1^0 \tilde{\boldsymbol{b}}(e^{\wedge G_\eta(V)} \wedge \Psi^{\wedge 2r+2})\big) \\ = \omega_{\mathrm{l}}\big(B_\delta(V), \pi_1^0 \tilde{\boldsymbol{b}}(e^{\wedge(G_\eta(V)+\Psi)}\big) \end{aligned} \tag{5.26}$$

by using the fact that $\boldsymbol{L}^\eta$ is cyclic with respect to ω_{l} and it becomes $\boldsymbol{\eta}$ outside of the pure NS sector. In total, an arbitrary variation of the action becomes

$$\begin{aligned} \delta S &= \omega_{\mathrm{l}}\Big(B_\delta(V), \big(QG_\eta + \pi_1^0 \tilde{\boldsymbol{b}}(e^{(G_\eta(V)+\Psi)})\big)\Big) + \omega_{\mathrm{s}}\Big(\delta\Psi, Y\big(Q\Psi + X\pi_1^1 \tilde{\boldsymbol{b}}(e^{(G_\eta(V)+\Psi)})\big)\Big) \\ &= \omega_{\mathrm{l}}\Big(B_\delta(V), \pi_1^0 \tilde{\boldsymbol{L}}(e^{(G_\eta(V)+\Psi)})\Big) + \omega_{\mathrm{s}}\Big(\delta\Psi, Y\pi_1^1 \tilde{\boldsymbol{L}}(e^{(G_\eta(V)+\Psi)})\Big). \end{aligned} \tag{5.27}$$

The equations of motion are therefore given by

$$\pi_1 \tilde{\boldsymbol{L}}(e^{\wedge(G_\eta(V)+\Psi)}) \;=\; 0, \tag{5.28}$$

which agrees with the equation obtained by the similarity transformations of the equation of motion in the homotopy algebraic formulation:

$$\pi_1 \boldsymbol{L}(e^{\wedge(\Phi_{\mathrm{NS}}+\Phi_{\mathrm{R}})}) \;=\; 0. \tag{5.29}$$

Now we can show that the gauge transformation of the action becomes

$$\begin{aligned} \delta S \;=\; &-\omega_{\mathrm{l}}\Big(\pi_1^0\Lambda, \pi_1^0 \boldsymbol{Q}\tilde{\boldsymbol{L}}(e^{\wedge(G_\eta+\Psi)})\Big) + \omega_{\mathrm{l}}\Big(\pi_1^0 \tilde{\boldsymbol{b}}\big((\Lambda - \xi\lambda) \wedge e^{\wedge(G_\eta+\Psi)}\big), \pi_1^0 \tilde{\boldsymbol{L}}(e^{\wedge(G_\eta+\Psi)})\Big) \\ &+ \omega_{\mathrm{l}}\Big(\pi_1^1\xi\lambda, Y\pi_1^1 \boldsymbol{Q}\tilde{\boldsymbol{L}}(e^{\wedge(G_\eta+\Psi)})\Big) + \omega_{\mathrm{l}}\Big(\pi_1^1 \tilde{\boldsymbol{b}}\big((\Lambda - \xi\lambda) \wedge e^{\wedge(G_\eta+\Psi)}\big), \pi_1^1 \tilde{\boldsymbol{L}}(e^{\wedge(G_\eta+\Psi)})\Big) \\ =\; &-\omega_{\mathrm{l}}\Big(\pi_1^0\Lambda, \pi_1^0 \boldsymbol{Q}\tilde{\boldsymbol{L}}(e^{\wedge(G_\eta+\Psi)})\Big) + \omega_{\mathrm{l}}\Big(\pi_1\xi\lambda, Y\pi_1^1 \boldsymbol{Q}\tilde{\boldsymbol{L}}(e^{\wedge(G_\eta+\Psi)})\Big) \\ &-\omega_{\mathrm{l}}\Big(\pi_1(\Lambda - \xi\lambda), \pi_1 \tilde{\boldsymbol{b}}\tilde{\boldsymbol{L}}(e^{\wedge(G_\eta+\Psi)})\Big) \\ =\; &-\omega_{\mathrm{l}}\Big(\pi_1^0\Lambda, \pi_1^0 \tilde{\boldsymbol{L}}\tilde{\boldsymbol{L}}(e^{\wedge(G_\eta+\Psi)})\Big) + \omega_{\mathrm{l}}\Big(\pi_1^1\xi\lambda, Y\pi_1^1 \tilde{\boldsymbol{L}}\tilde{\boldsymbol{L}}(e^{\wedge(G_\eta+\Psi)})\Big) \\ =\; &0. \end{aligned} \tag{5.30}$$

Hence the gauge invariance of the WZW-like action in Eq. (5.18) is shown in the WZW-like formulation independently.

6. Summary and discussion

In this paper we have constructed a complete heterotic string field theory in both the homotopy algebraic formulation and the WZW-like formulation. The complete action and gauge transformation in the homotopy algebraic formulation are given by means of string products realizing a cyclic L_∞ algebra. We have found that for constructing such string products it is useful to consider an L_∞ algebra combining two L_∞ algebras, which can easily be cyclic. Although these two, the dynamical and constraint L_∞ algebras, are neither cyclic nor closed in the small Hilbert space, we can transform them by a similarity transformation to the desired cyclic L_∞ algebra, and the constraint L_∞ algebra restricting it in the small Hilbert space. A concrete expression of the string products has been given by recursively solving differential equations for their generating functions. It has been confirmed that this heterotic string field theory reproduces all the physical four-point amplitudes at the tree level. The WZW-like action and gauge transformation have also been constructed from those in the homotopy algebraic formulation through a field redefinition.

The remaining missing pieces of the homotopy algebraic and the WZW-like formulations are complete type II superstring field theories. The prescription proposed in this paper can straightforwardly be extended to this case. We can construct string products realizing a cyclic L_∞ algebra and complete gauge-invariant actions in a similar manner. This will be reported in a separate paper (H. Kunitomo and T. Sugimoto, in preparation).

Finally, needless to say that the construction of a complete action is not the end of the story but just the beginning. We hope that the string field theory constructed in this paper provides a useful basic approach for studying various interesting nonperturbative properties of heterotic strings.

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Appendix A. Coalgebraic representation of L_∞ algebra

In this appendix we summarize basic definitions and properties of the coalgebraic representation of the L_∞ algebra.

Since the multi-closed-string products are graded symmetric under interchange of their arguments, it is useful to introduce the symmetrized tensor product by

$$\Phi_1 \wedge \Phi_2 \wedge \cdots \wedge \Phi_n \;=\; \sum_\sigma (-1)^{\epsilon(\sigma)} \Phi_{\sigma(1)} \otimes \Phi_{\sigma(2)} \otimes \cdots \otimes \Phi_{\sigma(n)}, \qquad \Phi_i \in \mathcal{H}, \tag{A.1}$$

where σ and $\epsilon(\sigma)$ denote all the permutations of $\{1, \ldots, n\}$ and the sign factor coming from the exchange $\{\Phi_1, \ldots, \Phi_n\}$ to $\{\Phi_{\sigma(1)}, \ldots, \Phi_{\sigma(n)}\}$, respectively. The string product $L_n(\Phi_1, \ldots, \Phi_n)$ can be represented as a linear operator L_n which maps the symmetrized tensor product of n copies of $\mathcal{H}$ into $\mathcal{H}$,

$$L_n \,:\, \mathcal{H}^{\wedge n} \;\to\; \mathcal{H}, \tag{A.2}$$

defined by

$$L_n(\Phi_1 \wedge \cdots \wedge \Phi_n) \;=\; L_n(\Phi_1, \ldots, \Phi_n). \tag{A.3}$$

In order to consider an infinite sequence of these multi-string products acting on different numbers of string fields, it is further useful to introduce the symmetrized tensor algebra generated by $\mathcal{H}$ as

$$\mathcal{SH} \;=\; \mathcal{H}^{\wedge 0} \oplus \mathcal{H} \oplus \mathcal{H}^{\wedge 2} \oplus \mathcal{H}^{\wedge 3} \oplus \cdots \tag{A.4}$$

Here, $\mathcal{H}^{\wedge 0}$ is a one-dimensional vector space spanned by the identity $\mathbb{I}_{\mathcal{SH}}$ of the symmetrized tensor product satisfying

$$\mathbb{I}_{\mathcal{SH}} \wedge V \;=\; V \tag{A.5}$$

for any element $V \in \mathcal{SH}$. The coderivation $\boldsymbol{L}_n$ is defined as an operator acting on $\mathcal{SH}$ by

$$\boldsymbol{L_n}\, \Phi_1 \wedge \cdots \wedge \Phi_m \;= 0, \qquad \text{for } m < n, \tag{A.6a}$$

$$\boldsymbol{L_n}\, \Phi_1 \wedge \cdots \wedge \Phi_m \;= L_n(\Phi_1 \wedge \cdots \wedge \Phi_m), \qquad \text{for } m = n, \tag{A.6b}$$

$$\boldsymbol{L_n}\, \Phi_1 \wedge \cdots \wedge \Phi_m \;= (L_n \wedge \mathbb{I}_{m-n})\ \Phi_1 \wedge \cdots \wedge \Phi_m, \qquad \text{for } m < n, \tag{A.6c}$$

with

$$\mathbb{I}_n \;=\; \frac{1}{n!} \underbrace{\mathbb{I} \wedge \cdots \wedge \mathbb{I}}_{n} \;=\; \underbrace{\mathbb{I} \otimes \cdots \otimes \mathbb{I}}_{n}. \tag{A.7}$$

Then we can consider an infinite sequence of multi-string products collectively as a general coderivation by adding them as

$$\boldsymbol{L} \;=\; \boldsymbol{L}_1 + \boldsymbol{L}_2 + \boldsymbol{L}_3 + \cdots \;=\; \sum_{n=0}^{\infty} \boldsymbol{L}_{n+1}. \tag{A.8}$$

The L_∞ relations in Eq. (2.19) are represented as nilpotency of the coderivation $\boldsymbol{L}$:

$$[\boldsymbol{L}, \boldsymbol{L}] \;=\; 0, \tag{A.9}$$

where the square bracket denotes the *graded* commutator, and the coderivation $\boldsymbol{L}$ is assumed to be degree odd.

The L_∞ structure is preserved under the L_∞ isomorphism represented by invertible cohomomorphisms. A cohomomorphism is characterized by a sequence of degree-even multi-string products[6]

$$H_1(\Phi), H_2(\Phi_1, \Phi_2), H_3(\Phi_1, \Phi_2, \Phi_3), \ldots, \tag{A.10}$$

[6] The zero-string product H_0 is also allowed in general, but we only consider the case with $H_0 = 0$ in this paper for simplicity.

and defined by a linear operator on $\mathcal{SH}$,

$$\hat{\boldsymbol{H}} = \pi_0 + \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{k_1,\ldots,k_l=1}^{\infty} (H_{k_1} \wedge \cdots \wedge H_{k_l}) \pi_{k_1+\cdots+k_l}, \tag{A.11}$$

where π_n is the projection operator onto $\mathcal{H}^{\wedge n}$:

$$\pi_n \, : \, \mathcal{SH} \, \to \, \mathcal{H}^{\wedge n}. \tag{A.12}$$

A useful relation obtained from the definition in Eq. (A.11) is

$$\begin{aligned} &\hat{\boldsymbol{H}}(\Phi_1 \wedge \cdots \wedge \Phi_k \wedge \Phi_a) \\ &\quad = \sum_{p=0}^{k} \sum_{\sigma} (-1)^{\epsilon(\sigma)} \hat{\boldsymbol{H}}(\Phi_{\sigma(1)} \wedge \cdots \wedge \Phi_{\sigma(p)}) \wedge \pi_1 \hat{\boldsymbol{H}}(\Phi_{\sigma(p+1)} \wedge \cdots \wedge \Phi_{\sigma(k)} \wedge \Phi_a), \end{aligned} \tag{A.13}$$

or equivalently,

$$\begin{aligned} &\hat{\boldsymbol{H}}(\Phi_a \wedge \Phi_1 \wedge \cdots \wedge \Phi_k) \\ &\quad = \sum_{p=0}^{k} \sum_{\sigma} (-1)^{\epsilon(\sigma)} \pi_1 \hat{\boldsymbol{H}}(\Phi_a \wedge \Phi_{\sigma(1)} \wedge \cdots \wedge \Phi_{\sigma(p)}) \wedge \hat{\boldsymbol{H}}(\Phi_{\sigma(p+1)} \wedge \cdots \wedge \Phi_{\sigma(k)}), \end{aligned} \tag{A.14}$$

where σ denotes all the decompositions of $\{1, \ldots, k\}$ to $\{\sigma(1), \ldots, \sigma(p)\}$ and $\{\sigma(p+1), \ldots, \sigma(k)\}$. The symbol $\epsilon(\sigma)$ denotes the sign factor coming from the exchanging of the string fields due to this decomposition. For an invertible cohomomorphism we have

$$\begin{aligned} &\hat{\boldsymbol{H}}^{-1}(\Phi_a \, \wedge \, \hat{\boldsymbol{H}}(\Phi_1 \, \wedge \, \cdots \, \wedge \, \Phi_k)) \\ &= \sum_{p=0}^{k} \sum_{\sigma} (-1)^{\epsilon(\sigma)} \pi_1 \hat{\boldsymbol{H}}^{-1}(\Phi_a \, \wedge \, \hat{\boldsymbol{H}}(\Phi_{\sigma(1)} \, \wedge \, \cdots \, \wedge \, \Phi_{\sigma(p)})) \, \wedge \, \Phi_{\sigma(p+1)} \, \wedge \, \cdots \, \wedge \, \Phi_{\sigma(k)}. \end{aligned} \tag{A.15}$$

A group-like element is a useful object in the tensor algebra $\mathcal{SH}$, and is defined by

$$e^{\wedge\Phi} \equiv \mathbb{I}_{\mathcal{SH}} + \Phi + \frac{1}{2!}\Phi \wedge \Phi + \frac{1}{3!}\Phi \wedge \Phi \wedge \Phi + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} \Phi^{\wedge n} \tag{A.16}$$

for a given Grassmann-even string field Φ. A coderivation $\boldsymbol{l}$ and a cohomomorphism $\hat{\boldsymbol{H}}$ act on the group-like element as

$$\boldsymbol{l}(e^{\wedge\Phi}) = e^{\wedge\Phi} \wedge \pi_1 \boldsymbol{l}(e^{\wedge\Phi}), \qquad \hat{\boldsymbol{H}}(e^{\wedge\Phi}) = e^{H[\Phi]}, \tag{A.17}$$

where $H[\Phi] = \pi_1 \hat{\boldsymbol{H}}(e^{\wedge\Phi})$. The relation

$$\hat{\boldsymbol{H}}(e^{\wedge\Phi} \wedge \Phi_a) = \hat{\boldsymbol{H}}(e^{\wedge\Phi}) \wedge \pi_1 \hat{\boldsymbol{H}}(e^{\wedge\Phi} \wedge \Phi_a) \tag{A.18}$$

that follows from the definition is useful.

Appendix B. Counting Ramond states

Let us summarize the ways to count the Ramond states in this appendix.

We introduce a projection operator π^r in the symmetrized tensor algebra, which projects onto the states containing r Ramond states:

$$\pi^r : \mathcal{SH} \to \{\Phi_{\mathrm{R}\,1} \wedge \cdots \wedge \Phi_{\mathrm{R}\,r} \wedge \Phi_{\mathrm{NS}\,1} \wedge \Phi_{\mathrm{NS}\,2} \wedge \cdots\}. \tag{B.1}$$

By multiplying this by the projection operator in Eq. (A.12), we can define a projection operator π_n^r selecting states containing r Ramond states in $\mathcal{H}^{\wedge n}$:

$$\pi_n^r = \pi_n \pi^r : \mathcal{SH} \to \{\Phi_{\mathrm{R}\,1} \wedge \cdots \wedge \Phi_{\mathrm{R}\,r} \wedge \Phi_{\mathrm{NS}\,1} \wedge \cdots \wedge \Phi_{\mathrm{NS}\,n-r}\}. \tag{B.2}$$

Note that a coderivation corresponding to the n-string product $\boldsymbol{l}_n$ satisfies

$$\pi_{m+1}\boldsymbol{l}_n = \boldsymbol{l}_n \pi_{m+n}. \tag{B.3}$$

We can decompose the coderivation $\boldsymbol{l}_n$ by the Ramond number defined by the number of Ramond inputs minus the number of Ramond outputs, which is denoted as the subscript after a vertical line, like $\boldsymbol{l}_n|_r$:

$$\pi^s \boldsymbol{l}_n|_r = \boldsymbol{l}_n|_r \pi^{r+s}. \tag{B.4}$$

If we note that the NS and Ramond states represent the space-time bosons and fermions, respectively, the Ramond number must be even. Therefore the range of the Ramond number is

$$0 \le r \le 2\left[\frac{n}{2}\right], \tag{B.5}$$

where $[x]$ is Gauss' symbol representing the greatest integer that is less than or equal to x. Combining Eqs. (B.3) and (B.4), it is found that a coderivation $\boldsymbol{l}_n|_r$ satisfies

$$\pi_{m+1}^s \boldsymbol{l}_n|_r = \boldsymbol{l}_n|_r \pi_{m+n}^{r+s}. \tag{B.6}$$

It is important to note that the Ramond number is additive when taking the (graded) commutator:

$$[\boldsymbol{l}_n|_r, \boldsymbol{l}'_m|_s]|_{r+s} = [\boldsymbol{l}_n|_r, \boldsymbol{l}'_m|_s]. \tag{B.7}$$

In order to consider the cyclicity, the cyclic Ramond number is more useful than the Ramond number because it is preserved under the cyclic permutation. The cyclic Ramond number is defined by the sum of the numbers of Ramond inputs and Ramond outputs, which must also be even. The cyclic Ramond number is denoted by a superscript after a vertical line, and a coderivation $\boldsymbol{l}_n|^r$ satisfies

$$\pi_{m+1}^s \boldsymbol{l}_n|^r = \boldsymbol{l}_n|^r \pi_{m+n}^{r-s}. \tag{B.8}$$

The range of the cyclic Ramond number is

$$0 \le r \le 2\left[\frac{n+1}{2}\right]. \tag{B.9}$$

A coderivation with a definite Ramond number or with a definite cyclic Ramond number is decomposed as

$$l_n|_{2r} = \pi_1^0\, \boldsymbol{l}_n|_{2r}^{2r} + \pi_1^1\, \boldsymbol{l}_n|_{2r}^{2r+2}, \tag{B.10}$$

$$l_n|^{2r} = \pi_1^0\, \boldsymbol{l}_n|_{2r}^{2r} + \pi_1^1\, \boldsymbol{l}_n|_{2r-2}^{2r}. \tag{B.11}$$

The cyclic Ramond number, however, is not additive when taking a commutator: the commutator of coderivations with definite cyclic Ramond numbers does not have a definite cyclic number. From the decomposition in Eq. (B.11) we have

$$[\boldsymbol{l}_n|^{2r}, \boldsymbol{l}'_m|^{2s}] = [\boldsymbol{l}_n|^{2r}, \boldsymbol{l}'_m|^{2s}]\,|^{2r+2s} + [\boldsymbol{l}_n|^{2r}, \boldsymbol{l}'_m|^{2s}]\,|^{2r+2s-2}. \tag{B.12}$$

So, it is useful to introduce operations picking up a part with definite cyclic Ramond numbers as follows. Suppose that there are two coderivations $\boldsymbol{l} = \sum_r \boldsymbol{l}|^{2r}$ and $\boldsymbol{l}' = \sum_s \boldsymbol{l}'|^{2s}$. We define two operations

$$[\boldsymbol{l}, \boldsymbol{l}']^1 = \sum_{r,s} [\boldsymbol{l}\,|^{2r}, \boldsymbol{l}'\,|^{2s}]\,|^{2r+2s}, \tag{B.13a}$$

$$[\boldsymbol{l}, \boldsymbol{l}']^2 = \sum_{r,s} [\boldsymbol{l}\,|^{2r}, \boldsymbol{l}'\,|^{2s}]\,|^{2r+2s-2}, \tag{B.13b}$$

which decompose a commutator as

$$[\boldsymbol{l}, \boldsymbol{l}'] = [\boldsymbol{l}, \boldsymbol{l}']^1 + [\boldsymbol{l}, \boldsymbol{l}']^2. \tag{B.14}$$

We can show from Eq. (B.12) that

$$[[\boldsymbol{l}\,|^{2r}, \boldsymbol{l}'\,|^{2s}]\,|^{2r+2s}, \boldsymbol{l}''\,|^{2t}]\,|^{2r+2s+2t} = [[\boldsymbol{l}\,|^{2r}, \boldsymbol{l}'\,|^{2s}], \boldsymbol{l}''\,|^{2t}]\,|^{2r+2s+2t}, \tag{B.15}$$

$$[[\boldsymbol{l}\,|^{2r}, \boldsymbol{l}'\,|^{2s}]\,|^{2r+2s-2}, \boldsymbol{l}''\,|^{2t}]\,|^{2r+2s+2t-4} = [[\boldsymbol{l}\,|^{2r}, \boldsymbol{l}'\,|^{2s}], \boldsymbol{l}''\,|^{2t}]\,|^{2r+2s+2t-4}. \tag{B.16}$$

These imply that the operations $[\ ,\ ]^1$ and $[\ ,\ ]^2$ satisfy the Jacobi identities

$$[[\boldsymbol{l}, \boldsymbol{l}']^1, \boldsymbol{l}'']^1 + (-1)^{l(l'+l'')}[[\boldsymbol{l}', \boldsymbol{l}'']^1, \boldsymbol{l}]^1 + (-1)^{l'(l''+l)}[[\boldsymbol{l}'', \boldsymbol{l}]^1, \boldsymbol{l}']^1 = 0, \tag{B.17}$$

$$[[\boldsymbol{l}, \boldsymbol{l}']^2, \boldsymbol{l}'']^2 + (-1)^{l(l'+l'')}[[\boldsymbol{l}', \boldsymbol{l}'']^2, \boldsymbol{l}]^2 + (-1)^{l'(l''+l)}[[\boldsymbol{l}'', \boldsymbol{l}]^2, \boldsymbol{l}']^2 = 0, \tag{B.18}$$

where $\boldsymbol{l}'' = \sum_t \boldsymbol{l}''\,|^{2t}$. Due to the Jacobi identity of the commutator and the decomposition in Eq. (B.14), the identity

$$\begin{aligned}
&[[\boldsymbol{l}, \boldsymbol{l}']^1, \boldsymbol{l}'']^2 + [[\boldsymbol{l}, \boldsymbol{l}']^2, \boldsymbol{l}'']^1 \\
&\quad + (-1)^{l(l'+l'')}[[\boldsymbol{l}', \boldsymbol{l}'']^1, \boldsymbol{l}]^2 + (-1)^{l(l'+l'')}[[\boldsymbol{l}', \boldsymbol{l}'']^2, \boldsymbol{l}]^1 \\
&\qquad + (-1)^{l''(l+l')}[[\boldsymbol{l}'', \boldsymbol{l}]^1, \boldsymbol{l}']^2 + (-1)^{l''(l+l')}[[\boldsymbol{l}'', \boldsymbol{l}]^2, \boldsymbol{l}']^1 = 0
\end{aligned} \tag{B.19}$$

also holds. Note that in the pure NS sector,

$$[\boldsymbol{l}|^0, \boldsymbol{l}'|^0]^1 = [\boldsymbol{l}|^0, \boldsymbol{l}'|^0]|^0 = [\boldsymbol{l}|^0, \boldsymbol{l}'|^0], \tag{B.20a}$$

$$[\boldsymbol{l}|^0, \boldsymbol{l}'|^0]^2 = 0. \tag{B.20b}$$

Appendix C. A proof of cyclicity

We first note that if $\boldsymbol{b}$ is cyclic with respect to ω_{l} then $\boldsymbol{b}$ is cyclic with respect to ω_{s}:

$$\begin{aligned}\omega_{\mathrm{s}}(\Phi_1, b_n(\Phi_2, \ldots, \Phi_{n+1})) &= \omega_{\mathrm{l}}(\xi\Phi_1, b_n(\Phi_2, \ldots, \Phi_{n+1}))\\ &= (-1)^{|\Phi_1|}\omega_{\mathrm{l}}(b_n(\xi\Phi_1, \Phi_2, \ldots, \Phi_n), \Phi_{n+1})\\ &= -(-1)^{|\Phi_1|}\omega_{\mathrm{l}}(\xi b_n(\Phi_1, \Phi_2, \ldots, \Phi_n), \Phi_{n+1})\\ &= -(-1)^{|\Phi_1|}\omega_{\mathrm{s}}(b_n(\Phi_1, \Phi_2, \ldots, \Phi_n), \Phi_{n+1}).\end{aligned} \tag{C.1}$$

Here, the second equality comes from the assumption, and we used $[\eta, b_n] = 0$ after inserting $\{\eta, \xi\} = 1$ in front of b_n in the third equality. Therefore it is sufficient to prove that if $\boldsymbol{B}$ is cyclic with respect to ω_{l}, then $\pi_1\boldsymbol{b} = \pi_1\boldsymbol{B}\hat{\boldsymbol{F}}$ is also cyclic with respect to ω_{l}.

We use mathematical induction with respect to the number of inputs. Since $b_2 = B_2$, it is cyclic with respect to ω_{l} from the assumption. Next assume that b_n for $2 \le n \le k$ is cyclic with respect to ω_{l}. Using the relation in Eq. (A.13) we find that

$$\begin{aligned}&\omega_{\mathrm{l}}\Big(\varphi_a, b_{k+1}(\varphi_1 \wedge \cdots \wedge \varphi_k \wedge \varphi_b)\Big) = \omega_{\mathrm{l}}\Big(\varphi_a, \pi_1\boldsymbol{B}\hat{\boldsymbol{F}}(\varphi_1 \wedge \cdots \wedge \varphi_k \wedge \varphi_b)\Big)\\ &= \sum_{p=1}^{k}\sum_{\sigma}(-1)^{\epsilon(\sigma)}\omega_{\mathrm{l}}\Big(\varphi_a, \pi_1\boldsymbol{B}\big(\hat{\boldsymbol{F}}(\varphi_{\sigma(1)} \wedge \cdots \wedge \varphi_{\sigma(p)}) \wedge \pi_1\hat{\boldsymbol{F}}(\varphi_{\sigma(p+1)} \wedge \cdots \wedge \varphi_{\sigma(k)} \wedge \varphi_b)\big)\Big)\\ &= -\sum_{p=1}^{k}\sum_{\sigma}(-1)^{|\varphi_a|+\epsilon(\sigma)}\omega_{\mathrm{l}}\Big(\pi_1\boldsymbol{B}\Big(\varphi_a \wedge \hat{\boldsymbol{F}}\big(\varphi_{\sigma(1)} \wedge \cdots \wedge \varphi_{\sigma(p)}\big)\Big),\\ &\quad \pi_1\hat{\boldsymbol{F}}\big(\varphi_{\sigma(p+1)} \wedge \cdots \wedge \varphi_{\sigma(k)} \wedge \varphi_b\big)\Big)\\ &= -(-1)^{|\varphi_a|}\omega_{\mathrm{l}}\Big(\pi_1\boldsymbol{B}\Big(\varphi_a \wedge \hat{\boldsymbol{F}}\big(\varphi_1 \wedge \cdots \wedge \varphi_k\big)\Big), \varphi_b\Big)\\ &\quad -\sum_{p=1}^{k}\sum_{\sigma}(-1)^{|\varphi_a|+\epsilon(\sigma)}\omega_{\mathrm{l}}\Big(\pi_1^1\boldsymbol{B}\Big(\varphi_a \wedge \hat{\boldsymbol{F}}\big(\varphi_{\sigma(1)} \wedge \cdots \wedge \varphi_{\sigma(p)}\big)\Big),\\ &\quad \Xi\pi_1^1\boldsymbol{b}\left(\varphi_{\sigma(p+1)} \wedge \cdots \wedge \varphi_{\sigma(k)} \wedge \varphi_b\right)\Big),\end{aligned} \tag{C.2}$$

where in the last equality we used Eq. (3.13).

Using the cyclicity of b_n for $n \le k$, the second term further becomes

$$\begin{aligned}&-\sum_{p=1}^{k}\sum_{\sigma}(-1)^{|\varphi_a|+\epsilon(\sigma)}\omega_{\mathrm{l}}\Big(\pi_1^1\boldsymbol{B}\Big(\varphi_a \wedge \hat{\boldsymbol{F}}\big(\varphi_{\sigma(1)} \wedge \cdots \wedge \varphi_{\sigma(p)}\big)\Big),\\ &\quad \Xi\pi_1^1\boldsymbol{b}\left(\varphi_{\sigma(p+1)} \wedge \cdots \wedge \varphi_{\sigma(k)} \wedge \varphi_b\right)\Big)\\ &= -\sum_{p=1}^{k}\sum_{\sigma}(-1)^{|\varphi_a|+\epsilon(\sigma)}\omega_{\mathrm{l}}\Big(\pi_1\boldsymbol{b}\left(\Xi\pi_1^1\boldsymbol{B}\left(\varphi_a \wedge \hat{\boldsymbol{F}}(\varphi_{\sigma(1)} \wedge \cdots \wedge \varphi_{\sigma(p)})\right)\right.\\ &\quad \left.\wedge\, \varphi_{\sigma(p+1)} \wedge \cdots \wedge \varphi_{\sigma(k)}\right), \varphi_b\Big)\end{aligned}$$

$$= \sum_{p=1}^{k} \sum_{\sigma} (-1)^{|\varphi_a|+\epsilon(\sigma)} \omega_1 \Big(\pi_1 \boldsymbol{B} \hat{\boldsymbol{F}}, (\pi_1 \hat{\boldsymbol{F}}^{-1} \left(\varphi_a \wedge \hat{\boldsymbol{F}}(\varphi_{\sigma(1)} \wedge \cdots \wedge \varphi_{\sigma(p)}) \right) \wedge \varphi_{\sigma(p+1)} \wedge \cdots \wedge \varphi_{\sigma(k)} \Big), \varphi_b \Big). \tag{C.3}$$

Here, for $0 < p < k$ the relation

$$\begin{aligned} 0 &= \pi_1 \hat{\boldsymbol{F}}^{-1} \hat{\boldsymbol{F}} \big(\varphi_a \wedge \varphi_{\sigma(1)} \wedge \cdots \wedge \varphi_{\sigma(p)} \big) \\ &= \pi_1 \hat{\boldsymbol{F}}^{-1} \Big(\varphi_a \wedge \hat{\boldsymbol{F}} \big(\varphi_{\sigma(1)} \wedge \cdots \wedge \varphi_{\sigma(p)} \big) \Big) \\ &\quad + \sum_{q=0}^{p} \sum_{\tau} (-1)^{\epsilon(\tau)} \pi_1 \hat{\boldsymbol{F}}^{-1} \Big(\pi_1 \hat{\boldsymbol{F}} (\varphi_a \wedge \varphi_{\tau(\sigma(1))} \wedge \cdots \wedge \varphi_{\tau(\sigma(q))}) \\ &\qquad \wedge \hat{\boldsymbol{F}} (\varphi_{\tau(\sigma(q+1))} \wedge \cdots \wedge \varphi_{\tau(\sigma(p))}) \Big) \end{aligned} \tag{C.4}$$

holds. Then we have

$$\begin{aligned} &\sum_{p=1}^{k} \sum_{\sigma} (-1)^{|\varphi_a|+\epsilon(\sigma)} \pi_1 \hat{\boldsymbol{F}}^{-1} \Big(\varphi_a \wedge \hat{\boldsymbol{F}} \big(\varphi_{\sigma(1)} \wedge \cdots \wedge \varphi_{\sigma(p)} \big) \Big) \wedge \varphi_{\sigma(p+1)} \wedge \cdots \wedge \varphi_{\sigma(k)} \\ &= -\sum_{p=1}^{k} \sum_{\sigma} \sum_{q=0}^{p} \sum_{\tau} (-1)^{|\varphi_a|+\epsilon(\sigma)+\epsilon(\tau)} \\ &\qquad \times \pi_1 \hat{\boldsymbol{F}}^{-1} \Big(\pi_1 \hat{\boldsymbol{F}} \Big(\varphi_a \wedge \varphi_{\tau(\sigma(1))} \wedge \cdots \wedge \varphi_{\tau(\sigma(q))} \Big) \\ &\qquad\qquad \wedge \hat{\boldsymbol{F}} \Big(\varphi_{\tau(\sigma(q+1)))} \wedge \cdots \wedge \varphi_{\tau(\sigma(p))} \Big) \Big) \wedge \varphi_{\sigma(p+1)} \wedge \cdots \wedge \varphi_{\sigma(k)} \\ &= -\sum_{p=1}^{k} \sum_{\sigma} (-1)^{|\varphi_a|+\epsilon(\sigma)} \hat{\boldsymbol{F}}^{-1} \Big(\pi_1 \hat{\boldsymbol{F}} \big(\varphi_a \wedge \varphi_{\sigma(1)} \wedge \cdots \wedge \varphi_{\sigma(p)} \big) \wedge \hat{\boldsymbol{F}} \big(\varphi_{\sigma(p+1)} \wedge \cdots \wedge \varphi_{\sigma(k)} \big) \Big), \end{aligned} \tag{C.5}$$

where we used Eq. (A.15) in the last equality. Substituting this into the expression in Eq. (C.3), we can show that b_{k+1} is cyclic with respect to ω_l:

$$\begin{aligned} &\omega_l \Big(\varphi_a, b_{k+1}(\varphi_1, \ldots, \varphi_k, \varphi_b) \Big) \\ &= -(-1)^{|\varphi_a|} \omega_l \Big(\pi_1 \boldsymbol{B} \Big(\varphi_a \wedge \hat{\boldsymbol{F}} \big(\varphi_1 \wedge \cdots \wedge \varphi_k \big) \Big), \varphi_b \Big) \\ &\quad - \sum_{p=1}^{k-1} \sum_{\sigma} (-1)^{|\varphi_a|+\epsilon(\sigma)} \omega_l \Big(\pi_1^1 \boldsymbol{B} \Big(\pi_1 \hat{\boldsymbol{F}} \big(\varphi_a \wedge \varphi_{\sigma(1)} \wedge \cdots \wedge \varphi_{\sigma(p)} \big) \\ &\qquad \wedge \hat{\boldsymbol{F}} \big(\varphi_{\sigma(p+1)} \wedge \cdots \wedge \varphi_{\sigma(k)} \big) \Big), \varphi_b \Big) \\ &= -\sum_{p=0}^{k-1} \sum_{\sigma} (-1)^{|\varphi_a|+\epsilon(\sigma)} \omega_l \Big(\pi_1^1 \boldsymbol{B} \Big(\pi_1 \hat{\boldsymbol{F}} \big(\varphi_a \wedge \varphi_{\sigma(1)} \wedge \cdots \wedge \varphi_{\sigma(p)} \big) \end{aligned}$$

$$\wedge\ \hat{\boldsymbol{F}}\Big(\varphi_{\sigma(p+1)}\ \wedge\ \cdots\ \wedge\ \varphi_{\sigma(k)}\Big)\Big),\varphi_b\Big)$$
$$= -(-1)^{|\varphi_a|}\omega_{\rm l}\Big(\pi_1\boldsymbol{B}\hat{\boldsymbol{F}}(\xi\Phi_a\ \wedge\ \Phi_1\ \wedge\ \cdots\ \wedge\ \Phi_k),\Phi_b\Big)$$
$$= -(-1)^{|\varphi_a|}\omega_{\rm l}\Big(b_{k+1}(\varphi_a,\varphi_1,\ldots,\varphi_k),\varphi_b\Big). \tag{C.6}$$

Hence, b_n is cyclic with respect to $\omega_{\rm l}$ for arbitrary n. That is, $\boldsymbol{b}$ is cyclic with respect to $\omega_{\rm l}$.

Appendix D. A proof of the identity in Eq. (5.10)

Define

$$\hat{\boldsymbol{g}}(t_0)\ = \vec{\mathcal{P}}\exp\left(\int_{t_0}^1 dt\boldsymbol{\lambda}(t)\right), \tag{D.1}$$

with $\boldsymbol{\lambda}(t)\equiv\boldsymbol{\lambda}^{[0]}(t)|^0$, which implies that

$$\partial_{t_0}\hat{\boldsymbol{g}}(t_0)\ =\ \boldsymbol{\lambda}(t_0)\hat{\boldsymbol{g}}(t_0). \tag{D.2}$$

Using Eqs. (A.17) and (A.18), we can show that

$$\begin{aligned}
&\partial_{t_0}\omega_{\rm l}(\pi_1\hat{\boldsymbol{g}}(t_0)\boldsymbol{l}_1(e^{\wedge\Phi}),\pi_1\hat{\boldsymbol{g}}(t_0)\boldsymbol{l}_2(e^{\wedge\Phi}))\\
&=\ \omega_{\rm l}(\pi_1\boldsymbol{\lambda}(t_0)\hat{\boldsymbol{g}}(t_0)(e^{\wedge\Phi}\wedge\pi_1\boldsymbol{l}_1(e^{\wedge\Phi})),\pi_1\hat{\boldsymbol{g}}(t_0)(e^{\wedge\Phi}\wedge\pi_1\boldsymbol{l}_2(e^{\wedge\Phi})))\\
&\ +\omega_{\rm l}(\pi_1\hat{\boldsymbol{g}}(t_0)(e^{\wedge\Phi}\wedge\pi_1\boldsymbol{l}_1(e^{\wedge\Phi})),\pi_1\boldsymbol{\lambda}(t_0)\hat{\boldsymbol{g}}(t_0)(e^{\wedge\Phi}\wedge\pi_1\boldsymbol{l}_2(e^{\wedge\Phi}))\\
&=\ \omega_{\rm l}(\pi_1\boldsymbol{\lambda}(t_0)(e^{\wedge g(\Phi)}\wedge g_{l_1}(e^{\wedge\Phi})),g_{l_2}(e^{\wedge\Phi}))\\
&\ +\omega_{\rm l}(g_{l_1}(e^{\wedge\Phi}),\pi_1\boldsymbol{\lambda}(t_0)(e^{g(\Phi)}\wedge g_{l_2}(e^{\wedge\Phi}))\\
&=\ 0,
\end{aligned} \tag{D.3}$$

where

$$g(\Phi)\ =\ \pi_1\hat{\boldsymbol{g}}(t_0)(e^{\wedge\Phi}),\qquad g_{l_i}(e^{\wedge\Phi})\ =\ \pi_1\hat{\boldsymbol{g}}(t_0)(e^{\wedge\Phi}\wedge\pi_1\boldsymbol{l}_i(e^{\wedge\Phi})),\quad (i=1,2). \tag{D.4}$$

In the last equality we used the fact that the gauge products $\boldsymbol{\lambda}$ represent the degree-even coderivation cyclic with respect to $\omega_{\rm l}$:

$$\omega_{\rm l}(\Phi_1,\lambda_n(\Phi_2\wedge\cdots\wedge\Phi_{n+1}))\ =\ (-1)^{|\Phi_1|}\omega_{\rm l}(\lambda_n(\Phi_1\wedge\cdots\wedge\Phi_n),\Phi_{n+1}). \tag{D.5}$$

Therefore the quantity

$$\omega_{\rm l}(\pi_1\hat{\boldsymbol{g}}(t_0)\boldsymbol{l}_1(e^{\wedge\Phi}),\pi_1\hat{\boldsymbol{g}}(t_0)\boldsymbol{l}_2(e^{\wedge\Phi})) \tag{D.6}$$

is independent of t_0, and in particular,

$$\omega_{\rm l}(\pi_1\hat{\boldsymbol{g}}(0)\boldsymbol{l}_1(e^{\wedge\Phi}),\pi_1\hat{\boldsymbol{g}}(0)\boldsymbol{l}_2(e^{\wedge\Phi}))\ =\ \omega_{\rm l}(\pi_1\hat{\boldsymbol{g}}(1)\boldsymbol{l}_1(e^{\wedge\Phi})\pi_1\hat{\boldsymbol{g}}(1)\boldsymbol{l}_2(e^{\wedge\Phi})). \tag{D.7}$$

Since $\hat{\boldsymbol{g}}(0)=\hat{\boldsymbol{g}}$ and $\hat{\boldsymbol{g}}(1)=\mathbb{I}_{\mathcal{SH}}$, this is nothing but the identity in Eq. (5.10).

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