A DESCRIPTION OF PRIME DIVISORS BY ARCS AND ITS APPLICATIONS

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Abstract. In the talk of the conference at Kinosaki Symposium, we introduced our project "construction of the second bridge between singularities in characteristic 0 and those in positive characteristic". We reduce the problem into one conjecture and show some affirmative cases in which the conjecture hold. We also show some applications of the conjecture.

1. Introduction

For studies of singularities in characteristic 0, there are many tools; resolutions of the singularities, Bertini's theorem (generic smoothness), many kinds of vanishing theorems, etc., which are not available for singularities in positive characteristic. So, in order to avoid these difficulties, one possible way is to reduce our problems in positive characteristic case into the problem in characteristic 0.

The first bridge between characteristic 0 and positive characteristic is already established is in the following form:

$$\left\{\begin{array}{c}\text{singularities in birational geometry}\\ \text{in characteristic } 0\end{array}\right\} \Leftrightarrow \left\{\begin{array}{c}F\text{-singularities in characteristic } p\\ \text{for } \textbf{“general”}\ \ p\end{array}\right\}$$

Here, birational singularities are, for example, rational, log-terminal, log-canonical singularities, et al. and F-singularities are F-rational, F-regular, F-pure singularities, et al.. This correspondence is studied by many people (K-i. Watanabe, N. Hara, K. Smith, S. Takagi, K. Schwede et al.) and yields many interesting phenomena.

However, note that this correspondence does not state anything about a correspondence between singularities in characteristic 0 and those in characteristic $p > 0$ for **specific** p.

Instead of thinking of F-singularities in positive characteristic, we think of singularities in birational geometry in positive characteristic and also in characteristic 0. Our challenge is construct a bridge in the following direction:

$$\left\{\begin{array}{c}\text{singularities in birational geometry}\\ \text{in characteristic } 0\\ \text{with the similar properties}\\ \text{as the one given in characteristic } p\end{array}\right\} \Leftarrow \left\{\begin{array}{c}\text{singularities in birational geometry}\\ \text{in characteristic } p>0\\ \text{for a } \textbf{“given”}\ \ p\end{array}\right\}$$

When the bridge is successfully constructed, we can reduce many statements in positive characteristic to those in characteristic 0 (see, Applications 4.1, 4.3, 4.5, 4.6, 4.7). In this short note, we will describe the final step of the construction as a conjecture and give some affirmative cases for the conjecture to hold.

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2. Terminologies

Our object to study here is a pair $(A, \mathfrak{a}^e)$ consisting of the affine space A of dimension $N > 0$ over an algebraically closed field k. and a "multiideal" $\mathfrak{a}^e = \mathfrak{a}_1^{e_1} \cdots \mathfrak{a}_s^{e_s}$ on A with the exponent $e = \{e_1, \ldots, e_s\} \subset \mathbb{R}_{>0}$, where $\mathfrak{a}_i$'s are nonzero coherent ideal sheaf on A. We introduce a method "lifting to characteristic 0" which constructs objects in characteristic 0 from objects in positive characteristic, with preserving certain properties.

Definition 2.1. Let S be an integral domain of characteristic 0, i.e., the canonical homomorphism $\mathbb{Z} \to S$ of rings is injective. For a prime number $p \in \mathbb{Z}$, we denote the canonical projection by $\Phi_p : S \to S \otimes_{\mathbb{Z}} \mathbb{Z}/(p)$.

(1) (Basic case) For $\tilde{f} \in S$ and $f \in S \otimes_{\mathbb{Z}} \mathbb{Z}/(p)$, we say that $\tilde{f}$ is a *lifting to characteristic* 0 *(or just a lifting)* of f, if $\Phi_p(\tilde{f}) = f$. In this case we also write $\tilde{f}(\text{mod } p) = f$.

(2) Let $\mathbf{f} = \{f_1, f_2, \ldots, f_r\}$ be a set of elements of an integral domain R of characteristic $p > 0$. Let $\tilde{\mathbf{f}} = \{\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_r\}$ be a set of elements of an integral domain $\widetilde{R}$ of characteristic 0.

We say that $\tilde{\mathbf{f}}$ is a *lifting to characteristic* 0 *(or just a lifting)* of $\mathbf{f}$ and write $\tilde{\mathbf{f}}(\text{mod } p) = \mathbf{f}$, if the following holds:

(a) there exists a subring $S \subset \widetilde{R}$ and an injective homomorphism $\iota : S \otimes_{\mathbb{Z}} \mathbb{Z}/(p) \hookrightarrow R$ of rings;

(b) Identify the ring $S \otimes_{\mathbb{Z}} \mathbb{Z}/(p)$ and its image by the injection ι. The inclusions $\tilde{\mathbf{f}} \subset S$ and $\mathbf{f} \subset S \otimes_{\mathbb{Z}} \mathbb{Z}/(p)$ hold with the following relations:

$$\tilde{f}_i(\text{mod } p) = f_i \quad \text{for every} \quad i = 1, 2, \ldots, r.$$

(3) Let $\mathfrak{a} \subset R$ be a nonzero proper ideal of a ring R of characteristic $p > 0$. Let $\widetilde{\mathfrak{a}} \subset R$ be an ideal of a ring $\widetilde{R}$ of characteristic 0. We say that $\widetilde{\mathfrak{a}}$ is a *lifting to characteristic* 0 *(or just a lifting)* of $\mathfrak{a}$ and write $\widetilde{\mathfrak{a}}(\text{mod } p) = \mathfrak{a}$ if we can take generators $\tilde{\mathbf{f}} = \{\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_r\}$ of $\widetilde{\mathfrak{a}}$ and $\mathbf{f} = \{f_1, f_2, \ldots, f_r\}$ of $\mathfrak{a}$ such that $\tilde{\mathbf{f}}(\text{mod } p) = \mathbf{f}$.

Lemma 2.2 ([1]). *For a finite subset $\{a_1, \ldots, a_s\} \subset k$ of a field k of characteristic $p > 0$, there exists a finite subset $\{\tilde{a}_1, \ldots, \tilde{a}_s\} \subset \mathbb{C}$ such that $\{\tilde{a}_1, \ldots, \tilde{a}_s\}(\text{mod } p) = \{a_1, \ldots, a_s\}$.*

Based on this lemma, we obtain the following:

Corollary 2.3. *For a finite subset $\{f_1, \ldots, f_s\} \subset k[x_1, \ldots, x_N]$ of a field k of characteristic $p > 0$, there exists a finite subset $\{\tilde{f}_1, \ldots, \tilde{f}_s\} \subset \mathbb{C}[x_1, \ldots, x_N]$ such that $\{\tilde{f}_1, \ldots, \tilde{f}_s\}(\text{mod } p) = \{f_1, \ldots, f_s\}$.*

Next, let us remind us some basic terminologies of singularities in birational geometry.

Definition 2.4. The log discrepancy of such a pair at a prime divisor E over A is defined as

$$a(E; A, \mathfrak{a}^e) := k_E - \sum_{i=1}^{s} e_i \mathrm{val}_E \mathfrak{a}_i + 1,$$

where k_E is the coefficient of the relative canonical divisor $K_{\overline{A}/A}$ at E.
Here $\varphi : \overline{A} \to A$ is a partial resolution such that E appears on $\overline{A}$.

We say that the pair $(A, \mathfrak{a}^e)$ is log canonical (resp. canonical) at a point $0 \in A$ if

$$a(E; A, \mathfrak{a}^e) \geq 0, \quad (\text{resp.} \geq 1) \tag{1}$$

holds for every exceptional prime divisor E over A with the center containing $0 \in A$.

Definition 2.5. Let $(A, \mathfrak{a}^e)$ and $0 \in A$ as above. Then the minimal log discrepancy is defined as follows:

(1) When $\dim A \geq 2$,

$$\mathrm{mld}(0; A, \mathfrak{a}^e) = \inf\{a(E; A, \mathfrak{a}^e) \mid E : \text{ prime divisor with the center } 0\}.$$

(2) When $\dim A = 1$, define $\mathrm{mld}(0; A, \mathfrak{a}^e)$ by the same definitions as above if the right hand side of the above definition is non-negative and otherwise define $\mathrm{mld}(0; A, \mathfrak{a}^e) = -\infty$.

Then, it is well known that
"$(A, \mathfrak{a}^e)$ is log canonical at $0 \in A$ if and only if $\mathrm{mld}(0; A, \mathfrak{a}^e) \geq 0$."

Definition 2.6. Let A, N, and e as above and $x \in A$ a closed point. We say that a prime divisor E over A with the center $\{x\}$ computes $\mathrm{mld}(x; A, \mathfrak{a}^e)$, if

$$\begin{cases} a(E; A, \mathfrak{a}^e) = \mathrm{mld}(x; A, \mathfrak{a}^e), & \text{when } \mathrm{mld}(x; A, \mathfrak{a}^e) \geq 0 \\ a(E; A, \mathfrak{a}^e) < 0, & \text{when } \mathrm{mld}(x; A, \mathfrak{a}^e) = -\infty \end{cases}$$

Remark 2.7. If there is a log resolution of $(A, \mathfrak{a}^e)$ in a neighborhood of x, or if e is a set of rational numbers, then a prime divisor computing $\mathrm{mld}(x; A, \mathfrak{a}^e)$ exists. Otherwise, the existence of such a divisor is not known in general.

3. The conjecture and some affirmative cases

Let k be an algebraically closed field of characteristic $p > 0$.

Definition 3.1. Let E be a prime divisor over $A = \mathbb{A}_k^N$ with $N > 1$ with the center at the origin $0 \in A$. We say that E has a partner in characteristic 0, if there exists a prime divisor $\widetilde{E}$ over $\widetilde{A} = \mathbb{A}_{\mathbb{C}}^N$ with the center at the origin $0 \in \widetilde{A}$ such that

$$k_E = k_{\widetilde{E}}$$

and for every $f \in k[x_1, \ldots, x_N]$ there exists a lifting $\tilde{f} \in \mathbb{C}[x_1, \ldots, x_N]$ of f satisfying

$$\mathrm{val}_E f = \mathrm{val}_{\widetilde{E}} \tilde{f}.$$

Now we can state our main conjecture:

Conjecture 3.2. *Every prime divisor over A with the center at 0 has a partner in characteristic 0.*

Next we show some affirmative cases for the conjecture to hold. One rather trivial example is as follows:

Example 3.3. A toric divisor with the center at 0 has a partner in characteristic 0. Here, toric divisor means a divisor over $\mathbb{A}_k^N$ corresponding to a integer vector $\overline{\mathfrak{p}} \in \mathbb{Z}_{\geq 0}^N$ in terms of toric geometry.

Proof. Actually, we can take a partner $\widetilde{E}$ on $\widetilde{A}$ as the toric divisor corresponding to the same vector $\overline{\mathfrak{p}}$ as of E and for a given polynomial $f \in k[x_1, \ldots, x_N]$ we can take $\tilde{f} \in \mathbb{C}[x_1, \ldots, x_N]$ as any lifting whose monomials are the same as the ones appearing in f. □

Example 3.4. The following E_i $(i \in \mathbb{N})$ all have partners in characteristic 0.
Let $\mathfrak{b} \subset k[x_1, \ldots, x_N]$ be a homogeneous prime ideal and $Z := Z(\mathfrak{b}) \subset A$.

(1) Let $\varphi_1 : A_1 \to A$ be the blow-up at the origin $0 \in A$ and E_1 the exceptional divisor. As E_1 is a toric divisor, it is obvious that it has a partner.

(2) Let $C_1 = E_1 \cap Z_1$ with the reduced structure, where Z_1 is the proper transform of Z on A_1. Let $\varphi_2 : A_2 \to A_1$ be the blow up at the center C_1. Let E_2 be the unique prime divisor of $\varphi_2^{-1}(C_1)$ dominant to C_1.

(3) Assume E_n $(n \leq i-1)$ are constructed. Let $C_{i-1} = E_{i-1} \cap Z_{i-1}$ with the reduced structure, where Z_{i-1} is the proper transform of Z on A_{i-1}. Let $\varphi_i : A_i \to A_{i-1}$ be the blow up at the center C_{i-1}. Let E_i be the unique prime divisor of $\varphi_i^{-1}(C_{i-1})$ dominant to C_{i-1}.

Proof. First, we define $\tilde{\mathfrak{b}} \subset \mathbb{C}[x_1, \ldots, x_N]$ as follows:

Let $c = \mathrm{ht}\mathfrak{b}$. Take c elements $g_1, \ldots, g_c$ of $\mathfrak{b}$ such that these are a regular sequence of the local ring $k[\mathbf{x}]_{\mathfrak{b}}$. Here, the notation $\mathbf{x}$ stands for N variables $x_1, \ldots, x_N$. Then any liftings $\tilde{g}_1, \ldots, \tilde{g}_c$ are also regular sequence in the neighborhood of $\mathfrak{b}$ in an appropriate $\mathbb{Z}$-algebra S as in Definition 2.1. Here, we take these liftings so that monomials in $\tilde{g}_i$ are the same as ones appearing in g_i (such a lifting is called a simple lifting).

Then, there is a minimal prime ideal $\tilde{\mathfrak{b}}$ of the ideal $(\tilde{g}_1, \ldots, \tilde{g}_c)$ with $\mathrm{ht}\tilde{\mathfrak{b}} = c$ and its reduction modulo p coincides with $\mathfrak{b}$ at the local ring of $\mathfrak{b}$. By making use of $\tilde{\mathfrak{b}}$ we obtain the prime divisors $\widetilde{E}_i$ $(i \in \mathbb{N})$ in the same way as E_i's. Then, it is clear that

$$k_{E_i} = k_{\widetilde{E}_i} = (N-1) + c(i-1).$$

On the other hand, for every homogeneous element $g \in k[\mathbf{x}]$ we can take a lifting $\tilde{g} \in \mathbb{C}[\mathbf{x}]$ such that

$$\mathrm{mult}_0 g = \mathrm{mult}_0 \tilde{g} \tag{2}$$

$$\mathrm{mult}_{\mathfrak{b}} g = \mathrm{mult}_{\tilde{\mathfrak{b}}} \tilde{g}. \tag{3}$$

Now, by (2) and (3), we obtain

$$\mathrm{val}_{E_1} g = \mathrm{mult}_0 g = \mathrm{mult}_0 \tilde{g} = \mathrm{val}_{\widetilde{E}_1} \tilde{g}$$
$$\mathrm{val}_{E_2} g = \mathrm{val}_{E_1} g + \mathrm{mult}_{\mathfrak{b}} g = \mathrm{val}_{\widetilde{E}_1} \tilde{g} + \mathrm{mult}_{\tilde{\mathfrak{b}}} \tilde{g} = \mathrm{val}_{\widetilde{E}_2} \tilde{g}$$
$$\cdots$$
$$\mathrm{val}_{E_i} g = \mathrm{val}_{E_{i-1}} g + \mathrm{mult}_{\mathfrak{b}} g = \mathrm{val}_{\widetilde{E}_{i-1}} \tilde{g} + \mathrm{mult}_{\tilde{\mathfrak{b}}} \tilde{g} = \mathrm{val}_{\widetilde{E}_i} \tilde{g}.$$

This shows that a homogeneous element g of $k[\mathbf{x}]$ is lifted to an element of $\mathbb{C}[\mathbf{x}]$ with the required condition with respect to E_i and $\widetilde{E}_i$ for each i. (in such a case, we say that "g is well liftable with respect to E_i and $\widetilde{E}_i$").

Next we show that for every polynomial $f \in k[\mathbf{x}]$ is well liftable with respect to E_i and $\widetilde{E}_i$ for each i. As we know well liftability of any f with respect to E_1 and $\widetilde{E}_1$, we may assume that $i \geq 2$. If the initial term $\mathrm{in} f$ of f is not contained in $\mathfrak{b}$, then the proper transform of $Z' := Z(f)$ on A_1 does not contain the center C_1 of the second blow up $A_2 \to A_1$. Let $\tilde{f}$ be a simple lifting, then the proper transform of $\tilde{Z}' := Z(\tilde{f})$ does not contain the center $\widetilde{C_1}$ of the second blow up $\widetilde{A}_2 \to \widetilde{A}_1$. Then for every $i \geq 2$, we have

$$\mathrm{val}_{E_i} f = \mathrm{val}_{E_1} f = \mathrm{val}_{\widetilde{E}_1} \tilde{f} = \mathrm{val}_{\widetilde{E}_i} \tilde{f}.$$

If the initial term $\mathrm{in} f = g_1 \in \mathfrak{b}$, let $f_1 := f - g_1$. Then we have $\mathrm{mult}_0 f_1 > \mathrm{mult}_0 f$. If $\mathrm{in} f_1 \notin \mathfrak{b}$, then f_1 is well liftable by the discussion above and f is the sum of two well liftable polynomials f_1 and g_1. On the other hand, we can prove

$$\mathrm{val}_{E_i} f = \min\{\mathrm{val}_{E_i} f_1, \mathrm{val}_{E_i} g_1\},$$

which yields that f is also well liftable.

If the initial term $\mathrm{in} f_1 \in \mathfrak{b}$, then for f_1 we do the same procedure as in the previous discussion for f. As the degree of f is finite, this procedure terminates at a finite stage. ☐

4. Applications

In this section, we show some application under the hypothesis that the conjecture 3.2 holds. All statements below are proved in [1].

Application 4.1 ([1])**.** *Assume Conjecture 3.2 holds. Let $A = \mathbb{A}^N_k$ be defined over an algebraically closed field k of characteristic $p > 0$ and $0 \in A$ the origin. Let $\mathfrak{a}$ and e be as in the beginning of the second section. Assume that there exists a prime divisor E over A computing* $\mathrm{mld}(0; A, \mathfrak{a}^e)$*, then there exist coherent ideal sheaves $\widetilde{\mathfrak{a}}_i$ on $\widetilde{A} = \mathbb{A}^N_{\mathbb{C}}$ satisfying*

$$\mathrm{mld}(0; A, \mathfrak{a}^e) = \mathrm{mld}(0; \widetilde{A}, \widetilde{\mathfrak{a}}^e),$$

where the origin of $\widetilde{A}$ is also denoted by 0. More precisely, we obtain $\widetilde{\mathfrak{a}}_i \pmod p = \mathfrak{a}_i$, and there is a divisor $\widetilde{E}$ over $\widetilde{A}$ such that $\widetilde{E}$ computes $\mathrm{mld}(0; \widetilde{A}, \widetilde{\mathfrak{a}}^e)$ *and satisfies* $\widetilde{E} \pmod p = E$.

Here, we introduce Mustaţă-Nakamura Conjecture (MN-Conjecture for short). It is stated in [4] under more general settings, but we use it only for the following special case.

Conjecture 4.2 (MN-Conjecture: $M_{N,e}$)**.** *Let $A = \mathbb{A}^N_k$ be defined over an algebraically closed field and let $0 \in A$ be the origin. Given a finite subset $e \subset \mathbb{R}_{>0}$, there is a positive integer $\ell_{N,e}$ (depending on N and e) such that for every multiideal $\mathfrak{a}^e$ on A with the exponent e, there is a prime divisor E that computes* $\mathrm{mld}(0; A, \mathfrak{a}^e)$ *and satisfies $k_E \le \ell_{N,e}$.*

This conjecture is proved to hold for $N = 2$ and for the case $\mathfrak{a}_i$'s are monomial ideals with N arbitrary ([4] for characteristic 0 and [2] and [1] for positive characteristic). But it is not yet proved in general even in characteristic 0.

Application 4.3 ([1])**.** *Assume Conjecture 3.2 holds. If MN-Conjecture holds in characteristic 0, then it also holds in any positive characteristic base field. It also guarantees the existence of a prime divisor computing* mld *for positive characteristic case, which is not known in general.*

There is another conjecture called ACC conjecture which is shown to be equivalent to $M_{N,e}$ for every finite subset $e \subset \mathbb{R}_{>0}$ by [3] in characteristic 0.

Conjecture 4.4 (ACC Conjecture: A_N)**.** *Let A, N and 0 be as above. For every fixed DCC set $J \subset \mathbb{R}_{>0}$, the set*

$$\{\mathrm{mld}(0; A, \mathfrak{a}^e) \mid e \subset J, (A, \mathfrak{a}^e) \text{ is log canonical at } 0\}$$

satisfies ACC.

Application 4.5 ([1])**.** *Assume Conjecture 3.2 holds. If ACC conjecture holds in characteristic 0, then it also holds in any positive characteristic.*

The following is an application by K. Shibata ([1, Appendix]).

Application 4.6 ([1])**.** *Assume that Conjecture 3.2 holds. Let $A = \mathbb{A}^N_k$ be defined over an algebraically closed base field k of arbitrary characteristic and $\mathfrak{m}$ the maximal ideal defining the origin 0 of A. Let μ be a positive integer.*

Then, there is a positive integer $L_{N,\mu}$ (depending only on N and μ) such that for every ideal $\mathfrak{a}$ with $\mathfrak{m}^\mu \subset \mathfrak{a}$, $\mathrm{lct}(0; A, \mathfrak{a})$ *is computed by a prime divisor E over A with $k_E \le L_{N,\mu}$.*

The following is provided by M. Mustaţă:

Application 4.7 ([1]). *Assume Conjecture 3.2 holds. Let* $\mathfrak{a}$ *be an ideal in* $k[x_1, \ldots, x_N]$, *where* k *is an algebraically closed field of characteristic* $p > 0$. *Then either there is a divisor* E *with center 0 that computes* $c = lct(0; A, \mathfrak{a})$ *or* c *lies in the set* T_{N-1} *of log canonical thresholds in characteristic 0, for ideals in* $k[x_1, \ldots, x_{N-1}]$. *In any case, we get that the log canonical threshold of every ideal in positive characteristic is a rational number.*

Remark 4.8. All statements of Applications in this section hold for $N = 2$ (see [2]).

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