— REPORT FOR KINOSAKI CONFERENCE —
SASAKI-EINSTEIN RATIONAL HOMOLOGY 5-SPHERES

JOONYEONG WON

A Riemannian manifold (M, g) is called Sasakian if its conical metric $\bar{g} = r^2 g + dr^2$ is a Kähler metric on the cone $C(M) = M \times \mathbb{R}^+$. Sasakian metrics, which are defined on odd dimensional manifolds, can be considered as an odd dimensional counterpart of Kähler metrics, which are defined on even dimensional manifolds. Also Riemannian manifold is called contact if conical metric is simplectic. Contact manifold is odd-dimensional counterpart of symplectic manifold. If the metric g satisfies the Einstein condition, i.e., $\mathrm{Ric}_g = \lambda g$ for some constant λ, then the metric g is called Einstein. The Sasakian manifold M is isometrically embedded into $C(M)$ by $M = M \times \{1\} \hookrightarrow C(M)$. The cone $C(M)$ is equipped with an integrable complex structure J since it is Kähler. The canonical vector field $r\partial_r$ defines the Reeb vector field ξ on M through the integrable complex structure, i.e., $\xi := J(r\partial_r)$. Sasakian manifolds can be classified into three types according to the Reeb foliation $\mathcal{F}_\xi$ given by the Reeb vector field ξ. If the orbits of the Reeb vector field ξ are all closed, then ξ integrates to an isometric S^1-action on M. Since ξ vanishes nowhere, the action is locally free. If the action is free, then the Sasakian structure is said to be regular. If not, then it is said to be quasi-regular. On the other hand, if the orbits of the Reeb vector field ξ are not all closed, then it is said to be irregular. In the regular or the quasi-regular case, the space of leaves of the Reeb foliation $\mathcal{F}_\xi$ is a compact Kähler manifold or orbifold, respectively. Furthermore, if M is Sasaki-Einstein, then it becomes a Kähler-Einstein manifold or orbifold. Indeed, the classification of $(2n-1)$-dimensional quasi-regular Sasaki-Einstein manifolds is closely related to the study of $(n-1)$-dimensional Kähler-Einstein Fano orbifolds.

It is not an easy task to determine whether a given Fano orbifold admits an orbifold Kähler-Einstein metric. However, the seminal work of Chen, Donaldson, Sun and Tian on existence of Kähler-Einstein metrics on Fano manifolds and their K-stability has opened wide a new gate to an area where existence of Kähler-Einstein metrics can be determined in purely algebraic ways. Since then, the result has been gradually being developed toward log $\mathbb{Q}$-Fano varieties. Indeed, Li, Tian and Wang proved that the result of Chen, Donaldson, Sun and Tian also holds for log $\mathbb{Q}$-Fano varieties with a mild assumption. The following theorem is a simplified version of their result that allows us to immediately utilize it for our purpose.

Theorem 0.1. Let S be a del Pezzo surface with quotient singularities and D be a prime divisor on S. Suppose that $-(K_S + \frac{m-1}{m}D)$ is ample for a positive integer m. If $(S, \frac{m-1}{m}D)$ is uniformly K-stable, then S has a Kähler-Einstein edge metric with angle $\frac{2\pi}{m}$ along D.

There are a few algebro-geometric methods known to us that can verify K-stability in concrete cases. The α-invariant originally introduced by Tian is one of the ways. The original definition of the α-invariant was given in an analytic way. There is however an algebro-geometric way to define the α-invariant over an arbitrary field of characteristic zero.

Definition 0.2. Let (X, Δ) be a log $\mathbb{Q}$-Fano variety. The α-invariant of (X, Δ) is defined by the number

$$\alpha(X,\Delta) = \sup\left\{\lambda \in \mathbb{Q} \,\middle|\, \begin{array}{l}\text{the log pair } (X, \Delta + \lambda D) \text{ is log canonical for every}\\ \text{effective } \mathbb{Q}\text{-divisor } D \text{ numerically equivalent to } -(K_X + \Delta).\end{array}\right\}.$$

The α-invariant plays a role in Kähler geometry by giving a sufficient condition for existence of orbifold Kähler-Einstein metrics.

Theorem 0.3. Let (X, Δ) be a Fano orbifold. If

$$\alpha(X,\Delta) > \frac{\dim(X)}{\dim(X)+1},$$

then (X, Δ) admits an orbifold Kähler-Einstein metric.

It quite often occurs that the α-invariant cannot determine existence of an orbifold Kähler-Einstein metric on a given Fano orbifold.

Recently Fujita and Odaka introduced a new algebro-geometric way to test K-stability of log $\mathbb{Q}$-Fano varieties. This supplies another method to check existence of orbifold Kähler-Einstein metrics.

To explain the method of Fujita and Odaka, let (X, Δ) be a $\mathbb{Q}$-factorial log pair with Kawamata log terminal singularities, $Z \subset X$ a closed subvariety and D an effective $\mathbb{Q}$-divisor on X. The log canonical threshold of D along Z on the log pair (X, Δ) is the number given by

$$c_Z(X,\Delta;D) = \sup\left\{\lambda \,\middle|\, \text{the log pair } (X, \Delta+\lambda D) \text{ is log canonical along } Z.\right\}.$$

Since log canonicity is a local property,

$$c_Z(X,\Delta;D) = \inf_{p\in Z}\left\{c_p(X,\Delta;D)\right\}.$$

If $X = \mathbb{C}^n$, $\Delta = 0$, and $D = (f = 0)$, where f is a polynomial defined over $\mathbb{C}^n$, then we also use the notation $c_0(f)$ for the log canonical threshold of D at the origin, instead of $c_0(X, 0; D)$.

Definition 0.4. Let (X, Δ) be a log $\mathbb{Q}$-Fano variety and let m be a positive integer such that the plurianticanonical linear system $|-m(K_X + \Delta)|$ is non-empty. Set $\ell_m = h^0(X, \mathcal{O}_X(-m(K_X + \Delta)))$. For a section s in $\mathrm{H}^0(X, \mathcal{O}_X(-m(K_X+\Delta)))$, we denote the effective divisor of the section s by $D(s)$. If ℓ_m sections $s_1, \ldots, s_{\ell_m}$ form a basis of the space $\mathrm{H}^0(X, \mathcal{O}_X(-m(K_X + \Delta)))$, then the $\mathbb{Q}$-divisor

$$D := \frac{1}{\ell_m}\sum_{i=1}^{\ell_m}\frac{1}{m}D(s_i)$$

is said to be of m-basis type with respect to the log $\mathbb{Q}$-Fano variety (X, Δ). For a positive integer m, we set

$$\delta_m(X,\Delta) = \inf_{\substack{D:\\ m\text{-basis type}}} c_X(X,\Delta;D).$$

We set $\delta_m(X, \Delta) = 0$ if $|-m(K_X + \Delta)|$ is empty. The δ-invariant of (X, Δ) is defined by the number

$$\delta(X,\Delta) = \limsup_m \delta_m(X,\Delta).$$

The δ-invariant turns out to provide a necessary and sufficient criterion for uniform K-stability.

Theorem 0.5. Let (X, Δ) be a log $\mathbb{Q}$-Fano variety. Then (X, Δ) is uniformly K-stable if and only if $\delta(X, \Delta) > 1$.

This potent criterion has been put into practice for smooth del Pezzo surfaces, and therein its effectiveness has been presented.

The development of the theory on quasi-regular Sasaki-Einstein metrics has followed that of the theory on Kähler-Einstein metrics on Fano varieties.

Now we have been strongly reinforced by new technologies for detecting Kähler-Einstein Fano orbifolds, in particular, the δ-invariant method, so it would be natural to expect that many hidden Sasaki-Einstein manifolds can be detected by the new methods. Indeed, the classification of simply connected Sasaki-Einstein rational homology 5-spheres can be completed by applying the δ-invariant method to certain hypersurfaces in 3-dimensional weighted projective spaces.

The main result is the complete classification of simply connected Sasaki-Einstein rational homology 5-spheres. Before we state the Main Theorem, let us explain how closed simply connected spin 5-manifolds are classified.

Theorem 0.6. For a positive integer m, there is a unique closed simply connected 5-dimensional manifold M_m with $\mathrm{H}_2(M_m, \mathbb{Z}) = \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$ that admits a spin structure. Furthermore, a closed simply connected 5-dimensional manifold M that admits a spin structure is of the form

$$M = k(S^2 \times S^3)\#M_{m_1}\# \dots \#M_{m_r},$$

where $k(S^2 \times S^3)$ is the k-fold connected sum of $S^2 \times S^3$ for a non-negative integer k and m_i is a positive integer greater than 1 with m_i dividing m_{i+1}.

We denote by kM_m the k-fold connected sum of M_m. Since a simply connected Sasaki-Einstein manifold must be spin, Smale's classification of simply connected 5-manifolds will be enough for our purpose. Thus Smale manifolds can be considered in three types.

- Torsion free
- Rational homology sphere
- Mixed type

Main Theorem. For each positive integer $n \geq 4$, the rational homology 5-sphere nM_2 admits a Sasaki-Einstein metric.

Together with the works of Boyer, Galicki, Kollár and Nakamaye, the Main Theorem completes the classification of simply connected rational homology 5-spheres that admit Sasaki-Einstein metrics.

Theorem 0.7. A simply connected rational homology 5-sphere admits a (quasi-regular) Sasaki-Einstein metric if and only if it is one of the following:

(1) the 5-sphere S^5;
(2) M_r, where r is a positive integer with $r \geq 2$ not divisible by 30;
(3) $2M_5$;
(4) $2M_4$;
(5) $2M_3, 3M_3, 4M_3$;
(6) nM_2, where $n \geq 2$.

Remark 0.8. Regular Sasaki-Einstein metrics on simply-connected 5-manifolds are completely classified. In particular, the 5-sphere is the only simply-connected regular Sasaki-Einstein rational homology 5-sphere. No irregular Sasaki-Einstein structure exists on a simply connected rational homology 5-sphere.

The proof of the Main Theorem is based on the method introduced by Kobayashi and developed by Boyer, Galicki and Kollár. Our new ingredient added to this method is to use the δ-invariant instead of the α-invariant. Even though it is difficult to compute or estimate both the invariants in general, a few methods have been developed well enough so that δ-invariants can be estimated effectively on surfaces with at worst quotient singularities.

Let X be a quasi-smooth hypersurface in a weighted projective space $\mathbb{P}(\mathbf{w}) = \mathbb{P}(a_0, a_1, \ldots, a_n)$ defined by a quasi-homogeneous polynomial $F(z_0, z_1, \ldots, z_n)$ in variables $z_0, \ldots, z_n$ with weights $\mathrm{wt}(z_i) = a_i$. The equation $F(z_0, z_1, \ldots, z_n) = 0$ also defines a hypersurface $\widehat{X}$ in $\mathbb{C}^{n+1}$ smooth outside the origin. The link of X is defined by the intersection

$$L_X = S^{2n+1}_{\mathbf{w}} \cap \widehat{X},$$

where $S^{2n+1}_{\mathbf{w}}$ is the unit sphere centred at the origin in $\mathbb{C}^{n+1}$ with the Sasakian structure induced from the weight $\mathbf{w} = (a_0, a_1, \ldots, a_n)$. This is a smooth compact manifold of dimension $2n-1$. It is simply-connected if $n \geq 3$. The situation can be diagrammed as follows

$$\begin{array}{ccc} L_X & \hookrightarrow & S^{2n+1}_{\mathbf{w}} \\ \downarrow & & \downarrow \\ X & \hookrightarrow & \mathbb{P}(\mathbf{w}) \end{array}$$

where the horizontal arrows are Sasakian and Kählerian embeddings, respectively, and the vertical arrows are S^1 orbibundles and orbifold Riemannian submersions.

Put $m = \gcd(a_1, \ldots, a_n)$. Suppose that $m > 1$ and $\gcd(a_0, a_1, \ldots, a_{i-1}, \widehat{a_i}, a_{i+1}, \ldots, a_n) = 1$ for each $i = 1, \ldots, n$. Also set $b_0 = a_0$ and $b_i = \frac{a_i}{m}$ for $i = 1, \ldots, n$. We also suppose that $\deg_{\mathbf{w}}(F) - \sum a_i < 0$. In other words, X is a Fano orbifold.

There is a quasi-homogeneous polynomial $G(x_0, \ldots, x_n)$ in variables $x_0, \ldots, x_n$ with weights $\mathrm{wt}(x_i) = b_i$ such that $F(z_0, z_1, \ldots, z_n) = G(z_0^d, z_1, \ldots, z_n)$. The equation $G(x_0, \ldots, x_n) = 0$ defines a well-formed quasi-smooth hypersurface Y in $\mathbb{P}(b_0, b_1, \ldots, b_n)$. Denote by D the divisor on Y cut by $x_0 = 0$.

Lemma 0.9. If there is a Kähler-Einstein edge metric on Y with angle $\frac{2\pi}{m}$ along the divisor D, then there is a Sasaki-Einstein metric on the link L_X of X.

We now consider a specific quasi-smooth hypersurface X_n of degree $4n+2$ in $\mathbb{P}(2, 2, 2n, 2n+1)$, where n is a positive integer. We use quasi-homogeneous coordinates x, y, z, w with weights $\mathrm{wt}(x) = \mathrm{wt}(y) = 2$, $\mathrm{wt}(z) = 2n$ and $\mathrm{wt}(w) = 2n+1$. By suitable coordinate changes, X_n may be assumed to be given by

$$w^2 - z^2x - zr_{n+1}(x, y) - r_{2n+1}(x, y) = 0,$$

where r_{n+1} and r_{2n+1} are homogeneous polynomials of degrees $n+1$ and $2n+1$, respectively, in the variables x, y. Note that either r_{n+1} contains y^{n+1} or r_{2n+1} contains y^{2n+1} due to the quasi-smoothness of X_n.

Let Y_n be the hypersurface in $\mathbb{P}(1, 1, n, 2n+1)$ defined by

$$w - z^2x - zr_{n+1}(x, y) - r_{2n+1}(x, y) = 0,$$

where we use the same notation for quasi-homogeneous coordinates as in $\mathbb{P}(2, 2, 2n, 2n+1)$, abusing the notation. Let C_w be the curve in Y_n that is cut out by the equation $w = 0$. Then the curve C_w is reduced and irreducible. The log pair

$$\left(Y_n, \frac{1}{2}C_w\right) \tag{0.10}$$

is a log del Pezzo surface that works for the Main Theorem.

Lemma 0.11. The link of the surface X_n is nM_2.

It has long been known that the equation is a candidate that yields a Sasaki-Einstein metric on nM_2. The reason why this candidate had not been able to be confirmed as a Sasaki-Einstein metric producer on nM_2 is that we did not have any method to determine whether $\left(Y_n, \frac{1}{2}C_w\right)$ admits an orbifold Kähler-Einstein metric. In particular, the α-invariant method is not sharp enough to do this job. Indeed, $\alpha\left(Y_n, \frac{1}{2}C_w\right)$ is at most $\frac{2}{3}$, which is too small to apply Theorem 0.3. However, the δ-invariant is decisive, so that it allows us to determine existence of orbifold Kähler-Einstein metric on $\left(Y_n, \frac{1}{2}C_w\right)$ through its uniform K-stability.

It follows from Lemmas that for the proof of the Main Theorem it is enough to show that $(Y_n, \frac{1}{2}C_w)$ possesses an orbifold Kähler-Einstein metric. Previous theorems imply that the following assertion completes the proof of the Main Theorem.

Theorem 0.12. For each $n \geq 4$,

$$\delta\left(Y_n, \frac{1}{2}C_w\right) \geq \frac{8n+8}{8n+7}.$$

For the detailed proofs of main theorem, we refer readers to the paper [1].

Finally we recall open problems about mixed types in the book–Sasakian geometry by Boyer, Galicki.

Let M be a Sasakian manifold.

1. Suppose $b_2(M) > 9$. M is Sasaki-Einstein if and only if $H_2(M, \mathbb{Z})_{tor} = 0$.
2. $k(S^2 \times S^3)\#M_m$ admits Sasaki-Einstein structure for all $0 \leq k \leq 8$ and for all $m > 2$ giving 31 missing cases.

References

[1] J. Park, J. Won, "Simply connected Sasaki-Einstein rational homology 5-spheres", arXiv:1905.13304

Joonyeong Won
Center for Mathematical challenges, Korea Institute for Advanced Study
85 Hoegiro Dongdaemun-gu, Seoul,02455, Korea
leonwon@kias.re.kr