ENRIQUES $2n$-FOLDS AND ANALYTIC TORSION – A SUMMARY

KEN-ICHI YOSHIKAWA

1. Introduction – Borcherds Φ-function

This note is a brief summary of our talk in Kinosaki Algebraic Geometry Symposium 2019. We report a recent progress on a generalization of Borcherds Φ-function to higher dimension. For the details, we refer the reader to the forthcoming paper [47].

In 1996, Borcherds proved the following:

Theorem 1.1 (Borcherds [16])**.** *The moduli space of Enriques surfaces is quasi-affine.*

This theorem has the following application to the family of Enriques surfaces.

Corollary 1.2. *Every family of Enriques surfaces without singular fibers over a compact connected complex space is isotrivial.*

Borcherds proved Theorem 1.1 by constructing an automorphic form Φ nowhere vanishing on the moduli space of Enriques surfaces. This remarkable automorphic form is called the *Borcherds Φ-function* or the *Borcherds-Enriques form.* In many respects, Φ is similar to the Dedekind η-function and is viewed as its generalization to Enriques surfaces (cf. [45], [46]). In this way, on the moduli space of compact Kähler manifolds with torsion canonical bundle of low dimension, we often have a nice automorphic form such as the Dedekind η-function and the Borcherds Φ-function. So it is very natural to seek for their generalizations in higher dimension. In this note, we explain such a generalization to a class of compact Kähler manifolds of even dimension $2n$. These manifolds, which we call *simple Enriques $2n$-folds*, are higher dimensional analogues of Enriques surfaces introduced and studied independently by Boissière-Nieper-Wißkirchen-Sarti [12] and Oguiso-Schröer [35].

Contrary to its geometric nature, the original construction of Φ due to Borcherds is not geometric. Indeed, it is obtained from the denominator formula for certain generalized Kac-Moody Lie algebra [16] or Borcherds products [17]. On the other hand, it is possible to construct Φ from *analytic torsion* of Enriques surfaces [44]. In this note, we explain how this third approach can be generalized to construct a function on the moduli space of simple Enriques $2n$-folds. We will discuss the following topics:

- a holomorphic torsion invariant for *simple Enriques $2n$-folds*,
- applications of the invariant to families of simple Enriques $2n$-folds,
- the automorphy of the holomorphic torsion invariant,
- some explicit formulas for the invariant as a function on the moduli space.

2. Simple Enriques $2n$-folds

Let us recall the Beauville-Bogomolov decomposition theorem.

Definition 2.1. Let X be a compact connected Kähler manifold. Then

(1) X is Calabi-Yau $\iff K_X \cong \mathcal{O}_X$ and $h^q(\mathcal{O}_X) = 0$ $(0 < q < \dim X)$.
(2) X is hyperkähler $\iff \pi_1(X) = \{1\}$ and $H^0(\Omega_X^2)$ is generated by a holomorphic symplectic form.

Theorem 2.2 (Beauville [5], Bogomolov [14])**.** *Let Y be a compact connected Ricci-flat Kähler manifold. Then there is a finite étale covering $X \to Y$ such that*

$$X \cong T \times \prod U_i \times \prod S_j,$$

where T is a complex torus, U_i is a simply connected Calabi-Yau manifold, and S_j is a hyperkähler manifold.

In this note, since we are interested in generalizations of Enriques surfaces in higher dimension and their holomorphic torsion invariant, we focus on compact Kähler manifolds of *even dimension* with torsion canonical bundle. Unfortunately, in even dimension, it turns out that the holomorphic analytic torsion (of the trivial bundle) is trivial for the building blocks in the Beauville-Bogomolov decomposition theorem. [1] However, as in the case of Enriques surfaces, it turns out that the holomorphic torsion of their étale quotient is non-trivial in general. Among those compact Kähler manifolds of even dimension with torsion canonical bundle, we focus on the simplest ones.

Definition 2.3. A compact connected Kähler $2n$-fold Y is *simple Enriques* if $\pi_1(Y) \neq \{1\}$ and its universal covering $\widetilde{Y}$ is either Calabi-Yau or hyperkähler. The covering degree of $\widetilde{Y} \to Y$ is called the *index* of Y. Y is said to be of Calabi-Yau (resp. hyperkähler) type if $\widetilde{Y}$ is Calabi-Yau (resp. hyperkähler).

We remark that this class of manifolds have already been introduced and studied by two groups of authors: Boissière-Nieper-Wißkirchen-Sarti [12] introduced *Enriques varieties* and Oguiso-Schröer [35] introduced *Enriques manifolds.*

Fact 2.4 (Boissière-Nieper-Wißkirchen-Sarti [12], Oguiso-Schröer [35])**.** *Let Y be a simple Enriques $2n$-fold with index d. Then the following hold:*

(1) $\pi_1(Y) \cong \mathbb{Z}/d\mathbb{Z}$.
(2) $K_Y^{\otimes d} \cong \mathcal{O}_Y$ *and* $K_Y^{\otimes i} \not\cong \mathcal{O}_Y$ $(0 < i < d)$.
(3) *If Y is of Calabi-Yau type, then $d = 2$.*
(4) *If Y is of hyperkähler type, then $d|(n+1)$.*

3. Analytic torsion of Calabi-Yau and hyperkähler manifolds

Let us recall the notion of analytic torsion (of the trivial line bundle). Let (X, γ) be a compact Kähler manifold. Let $\zeta_q(s)$ be the spectral zeta function of the Hodge-Kodaira Laplacian $\square_q = (\bar{\partial} + \bar{\partial}^*)^2$ acting on the $(0, q)$-forms on X:

$$\zeta_q(s) := \sum_{\lambda \in \sigma(\square_q) \setminus \{0\}} \lambda^{-s} \dim E(\square_q, \lambda),$$

[1] In odd dimension, the analytic torsion of the trivial line bundle of a Calabi-Yau manifold is non-trivial. Moreover, in arbitrary dimension, one can construct another holomorphic torsion invariant called the BCOV invariant [20], [18], [19], [49], [50]. In this note, we will not discuss BCOV invariants.

where $\sigma(\Box_q) \subset \mathbb{R}_{\geq 0}$ is the set of eigenvalues of $\Box_q$ and $E(\Box_q, \lambda)$ is the eigenspace of $\Box_q$ corresponding to the eigenvalue λ. By the ellipticity of $\Box_q$, $\sigma(\Box_q)$ is a discrete subset of $\mathbb{R}$ and $E(\Box_q, \lambda)$ is of finite dimensional for all $\lambda \in \sigma(\Box_q)$. It is classical that $\zeta_q(s)$ extends to a meromorphic function on $\mathbb{C}$ and is holomorphic at $s = 0$.

Definition 3.1 (Ray-Singer [38])**.** The analytic torsion of (X, γ) is defined as

$$\tau(X, \gamma) := \exp\{-\sum_{q\geq 0}(-1)^q q\, \zeta_q'(0)\}.$$

Recall that for a positive-definite Hermitian matrix H of finite dimension, one has

$$\log\det H = -\left.\frac{d}{ds}\right|_{s=0} \operatorname{Tr} H^{-s}.$$

After this formula, the regularized determinant of the Laplacian $\Box_q$ is defined as

$$\det \Box_q := \exp(-\zeta_q'(0)).$$

Hence the analytic torsion $\tau(X, \gamma)$ can be expressed as the product

$$\tau(X, \gamma) = \prod (\det \Box_q)^{(-1)^q q}.$$

One of the reasons why the analytic torsion is so interesting rests on the following theorem of Bismut-Gillet-Soulé known as the *curvature formula*.

Theorem 3.2 (Bismut-Gillet-Soulé [8], [9], [10])**.** *For a proper, surjective, locally Kähler, smooth morphism $f\colon X \to S$ between complex manifolds endowed with a fiberwise Kähler metric $h_{X/S}$ on the relative tangent bundle TX/S, one has*

$$-dd^c \log\tau + \sum(-1)^q c_1(R^q f_*\mathcal{O}_X, h_{L^2}) = [f_*\mathrm{Td}(TX/S, h_{X/S})]^{(1,1)}.$$

Here $c_1(R^q f_*\mathcal{O}_X, h_{L^2})$ is the first Chern form of the locally free sheaf $R^q f_*\mathcal{O}_X$ on S endowed with the L^2-metric, $\mathrm{Td}(TX/S, h_{X/S})$ is the Todd form of $(TX/S, h_{X/S})$, and $[f_*\mathrm{Td}(TX/S, h_{X/S})]^{(1,1)}$ is the degree $(1,1)$-component of its fiber integral.

We remark that this theorem is a special case of the general curvature formula for Quillen metrics. See [8], [9], [10] for more details.

Up to a universal constant, the analytic torsion of a flat elliptic curve with normalized area 1 is given by the value of the Petersson norm of the Dedekind η-function evaluated at its period [38]. Similarly, up to a universal constant, the analytic torsion of a Ricci-flat Enriques surface with normalized volume 1 is given by the value of the Petersson norm of the Borcherds Φ-function evaluated at its period [44]. However, the analytic torsion of a flat complex torus is given by the constant function 1, and the same is true for Ricci-flat $K3$ surfaces.

In higher and even dimension, we have the following result for the building blocks of the Beauville-Bogomolov decomposition theorem. Let X be a complex torus or a Calabi-Yau manifold or a hyperkähler manifold of dimension $2n$. Let η be a nonzero canonical form, i.e., nowhere vanishing holomorphic volume form on X. Let γ be a Kähler form on X. Let $c_1(X, \gamma)$ be the first Chern form of (TX, γ) and let $\mathrm{Td}(X, \gamma)$ be the Todd form of (TX, γ).

Theorem 3.3 ([47])**.** *The analytic torsion of (X, γ) is given by*

$$\tau(X, \gamma) = \exp\left\{-\frac{1}{2}\int_X \log\left(\frac{\eta \wedge \overline{\eta}}{\gamma^{2n}/(2n)!} \cdot \frac{\mathrm{Vol}(X, \gamma)}{\|\eta\|_{L^2}^2}\right) \frac{\mathrm{Td}(X, \gamma)}{\mathrm{Td}(c_1(X, \gamma))}\right\}.$$

In particular, if γ is Ricci-flat, then $\tau(X, \gamma) = 1$.

This theorem says that in even dimension the analytic torsion does not give any interesting function on the moduli space for the building blocks of the Beauville-Bogomolov decomposition theorem.

Remark 3.4. In contrast to Theorem 3.3, the BCOV invariant of a Calabi-Yau n-fold is non-trivial for $n \neq 2$. However, for complex tori and hyperkähler manifolds, the BCOV invariant is again trivial [18]. If we replace a principally polarized abelian variety (p.p.a.v.) of dimension $g > 1$ by its theta divisor, then the holomorphic torsion of the theta divisor is given by the Petersson norm of a Siegel modular form characterizing the Andreotti-Mayer locus evaluated at the period of the p.p.a.v [43].

4. Holomorphic torsion invariant of Enriques $2n$-folds

Let Y be a simple Enriques $2n$-fold of index d. Let $\varXi$ be a pluricanonical form of weight d, i.e., a nowhere vanishing holomorphic section of $K_Y^{\otimes d}$. Then $|\varXi|^{\frac{2}{d}} := |\varXi \otimes \overline{\varXi}|^{\frac{1}{d}}$ is a volume form on Y. Let γ be a Kähler form on Y. Define a Bott-Chern term $A(Y,\gamma)$ by

$$A(Y,\gamma) := \exp\left\{\frac{1}{2}\int_Y \log\left(\frac{|\varXi|^{\frac{2}{d}}}{\gamma^{2n}/(2n)!}\cdot\frac{\mathrm{Vol}(Y,\gamma)}{\|\varXi\|^{2/d}_{L^{\frac{2}{d}}}}\right)\frac{\mathrm{Td}(Y,\gamma)}{\mathrm{Td}(c_1(Y,\gamma))}\right\}.$$

If $p\colon \widetilde{Y} \to Y$ is the universal covering, then $A(Y,\gamma) = \tau(\widetilde{Y}, p^*\gamma)^{-1/d}$.

Definition 4.1. With the same notation as above, define

$$\tau_{\mathrm{Enr}}(Y) := \begin{cases} \tau(Y,\gamma)\mathrm{Vol}(Y,\gamma)^{\frac{d-1}{d}}A(Y,\gamma) \\ \tau(Y,\gamma)\mathrm{Vol}(Y,\gamma)^{\frac{(n+1)(d-1)}{2nd}}A(Y,\gamma) \end{cases} \quad \text{if } Y \text{ is of } \begin{cases} \text{Calabi-Yau type} \\ \text{hyperkähler type.} \end{cases}$$

By the anomaly formula for Quillen metrics [8], [9], [10], we have the following:

Theorem 4.2 ([47])**.** *$\tau_{\mathrm{Enr}}(Y)$ is independent of the choice of a Kähler metric γ on Y. In particular, $\tau_{\mathrm{Enr}}(Y)$ is an invariant of Y, and τ_{Enr} is viewed as a function on the moduli space of simple Enriques $2n$-folds.*

When $n = 1$, τ_{Enr} is given by the Petersson norm of the Borcherds Φ-function [44]. To understand the nature of τ_{Enr}, let us explain some of its basic properties.

Let Y be a simple Enriques $2n$-fold of index d. Let $\mathrm{Def}(Y)$ be the Kuranishi space of Y, which is smooth by Bogomolov-Tian-Todorov [15], [39], [40]. Let $f\colon (\mathfrak{Y}, Y) \to (\mathrm{Def}(Y), [Y])$ be the universal deformation of Y. Let $\varXi$ be a relative pluricanonical form of weight d for f, i.e., $\varXi$ is a nowhere vanishing section of $f_*K^{\otimes d}_{\mathfrak{Y}/\mathrm{Def}(Y)}$.

Definition 4.3. The Weil-Petersson form is the Kähler form on $\mathrm{Def}(Y)$ defined as

$$\omega_{\mathrm{WP}} := -dd^c \log \|\varXi\|^{2/d}_{L^{2/d}}.$$

Theorem 4.4 ([47])**.** *$\log\tau_{\mathrm{Enr}}$ is a strictly plurisubharmonic function on $\mathrm{Def}(Y)$ such that*

$$dd^c \log\tau_{\mathrm{Enr}} = \nu_{n,d}\,\omega_{\mathrm{WP}},$$

where

$$\nu_{n,d} := \begin{cases} \frac{d-1}{d} = \frac{1}{2} \\ \frac{(n+1)(d-1)}{2nd} \end{cases} \quad \textit{if } Y \textit{ is of } \begin{cases} \text{Calabi-Yau type} \\ \text{hyperkähler type.} \end{cases}$$

This formula follows from the curvature formula for Quillen metrics [8], [9], [10] (or its equivariant version [27]):

$$-dd^c \log \tau + \sum (-1)^q c_1(R^q f_* \mathcal{O}_{\mathfrak{Y}}, h_{L^2}) = \left[f_* \mathrm{Td}(T\mathfrak{Y}/S, h_{\mathfrak{Y}/S})\right]^{(1,1)}$$

applied to the Kuranishi family $f\colon \mathfrak{Y} \to S = \mathrm{Def}(Y)$.

By Theorem 4.4, it is easy to deduce the following generalization of Corollary 1.2 in higher dimension.

Corollary 4.5. *Every family of simple Enriques $2n$-folds without singular fibers over a compact complex space is isotrivial.*

In Corollary 4.5, the total space of the family is not assumed to be Kähler. Hence the result holds true even when the total space of the family is non-Kähler.

Another basic property of the invariant τ_{Enr} is its regularity for one parameter degenerating families of simple Enriques $2n$-folds.

Theorem 4.6 ([47])**.** *Let $f\colon \mathcal{Y} \to C$ be a family of $2n$-folds over a compact Riemann surface C, whose general fibers are simple Enriques. If $0 \in C$ is a point of the discriminant locus, i.e., $f^{-1}(0)$ is singular, then there exists $\alpha \in \mathbb{Q}$ such that*

$$\log \tau_{\mathrm{Enr}}(Y_s) = \alpha \log |s|^2 + O\left(\log(-\log|s|)\right) \qquad (s \to 0),$$

where s is a local parameter of C centered at 0.

This theorem is obtained by applying the Bismut-Lebeau embedding theorem [11] for Quillen metrics to the family of embedings $Y_s \hookrightarrow \mathcal{Y}$. When the degeneration $f\colon (\mathcal{Y}, Y_0) \to (S, 0)$ is semistable, i.e., Y_0 is a *reduced* simple normal crossing divisor, α can explicitly be evaluated as the integral of certain characteristic classes attached to the critical locus. It is also possible to compute the value α by using the embedding formula for equivariant Quillen metrics [7].

As in the case of BCOV invariant [20], [18], we propose the following:

Conjecture 4.7. τ_{Enr} is a birational invariant of simple Enriques $2n$-folds. Namely, if Y and Y' are birational simple Enriques $2n$-folds, then

$$\tau_{\mathrm{Enr}}(Y) = \tau_{\mathrm{Enr}}(Y').$$

5. Quasi-affinity of the moduli space: polarized case

Let us recall the following result, which is a special case of a very general result of Viehweg:

Theorem 5.1 (Viehweg [42])**.**

1. *There is a coarse moduli space of* polarized *simple Enriques $2n$-folds of index d with Hilbert polynomial h. Let $\mathcal{M}$ be a component of the moduli space. Then $\mathcal{M}$ is quasi-projective.*
2. *There exists an ample line bundle λ on $\mathcal{M}$ such that λ is identified with the "direct image of the d-th tensor power of the relative canonical bundle of the universal family".*

In general, there is no universal family of polarized simple Enriques $2n$-folds over $\mathcal{M}$. However, we have its substitute in an appropriate sense [42].

Theorem 5.2 ([47])**.** *There exists a natural number $N \in \mathbb{N}$ such that $\lambda^N \cong \mathcal{O}_{\mathcal{M}}$. In particular, $\mathcal{M}$ is quasi-affine.*

Corollary 5.3. *$\mathcal{M}$ does not contain compact subvarieties of positive dimension.*

We remark that this corollary follows also from Theorem 4.4. The proof of Theorem 5.2 is based on the Grothendieck-Riemann-Roch theorem and is very similar to Pappas' proof of the quasi-affinity of the moduli space of Enriques surfaces [37].

By the curvature theorem and the regularity theorem, we have the following:

Theorem 5.4 ([47]). *There exist $N \in \mathbb{Z}_{>0}$, a flat $U(1)$-bundle $\chi \in H^1(\mathcal{M}, U(1))$ and a* nowhere vanishing *holomorphic section σ of $\lambda^{2N\nu_{n,d}} \otimes \chi$ defined on $\mathcal{M}$ such that*

$$\tau_{\mathrm{Enr}}^{-2N} = \|\sigma\|^2.$$

Moreover, there exists a compactification $\overline{\mathcal{M}}$ of $\mathcal{M}$ with the following properties:

(i) *$\overline{\mathcal{M}}$ is a normal projective variety.*
(ii) *λ^N extends to an ample line bundle on $\overline{\mathcal{M}}$.*
(iii) *Set $B := \overline{\mathcal{M}} \setminus \mathcal{M}$. Then there exist a flat $U(1)$-bundle $\overline{\chi} \in H^1(\overline{\mathcal{M}} \setminus (\mathrm{Sing}\,\overline{\mathcal{M}} \cap B), U(1))$ with $\overline{\chi}|_{\mathcal{M}} = \chi$ and a (possibly meromorphic) section $\overline{\sigma}$ of $\lambda^{2N\nu_{n,d}} \otimes \overline{\chi}$ defined on $\overline{\mathcal{M}} \setminus (\mathrm{Sing}\,\overline{\mathcal{M}} \cap B)$ such that*

$$\overline{\sigma}|_{\mathcal{M}} = \sigma.$$

Remark 5.5. If $\overline{\chi} \in H^1(\overline{\mathcal{M}}, U(1))$ in Theorem 5.4, then it is possible to prove that $\overline{\sigma}$ is defined on $\overline{\mathcal{M}}$, from which the quasi-affinity of $\mathcal{M}$ follows. Hence we have an analytic proof of Theorem 5.2 in this case. In general, we do not know if $\overline{\chi}$ is extended to an element of $H^1(\overline{\mathcal{M}}, U(1))$.

Question 5.6. Does there always exist a compactification $\overline{\mathcal{M}}$ of $\mathcal{M}$ to which χ extends as a flat $U(1)$-bundle and to which λ^N extends as an ample line bundle?

6. Automorphy of τ_{Enr}

Let Y be a simple Enriques $2n$-fold of *hyperkähler type with index* 2. Let $X = \widetilde{Y}$ be its universal covering hyperkähler manifold.

Fact 6.1 (Beauville [5], Fujiki [21]). *There is an integral non-degenerate symmetric bilinear form $(\cdot,\cdot)_X$ on $H^2(X,\mathbb{Z})$ and a constant $c_X \in \mathbb{Q}_{>0}$ such that*

(1) $\int_X \lambda^{2n} = c_X \cdot (\lambda,\lambda)_X^n \quad (\forall\, \lambda \in H^2(X,\mathbb{Z}))$.
(2) $(\sigma,\sigma)_X = 0$ *and* $(\sigma,\overline{\sigma})_X > 0$ *for all* $\sigma \in H^0(X,\Omega_X^2)$.
(3) $\mathrm{sign}\,(\cdot,\cdot)_X = (3, b_2 - 3)$.

The pair $(H^2(X,\mathbb{Z}), (\cdot,\cdot)_X)$ is called the Beauville-Bogomolov-Fujiki lattice *(BBF lattice for short).*

In what follows, we make the following:

Assumption 6.2.

$$\mathrm{Aut}_0(X) := \ker\left\{\mathrm{Aut}(X) \to \mathrm{Aut}\left(H^2(X,\mathbb{Z})\right)\right\} = \{1\} \quad \text{and} \quad b_2(X) \geq 5.$$

Let Λ be a fixed abstract lattice isometric to the BBF lattice of X. An isometry of lattices $\alpha\colon H^2(X,\mathbb{Z}) \to \Lambda$ is called a marking of X and the pair (X,α) is called a marked hyperkähler manifold. By Huybrechts [24], there exists a coarse moduli space of marked hyperkähler manifolds with BBF lattice Λ. By Markman [31], under the assumption $\mathrm{Aut}_0(X) = \{1\}$, this is a fine moduli space, i.e., there exists a universal marked family.

Let $\iota\colon X \to X$ be the involution such that $X/\iota = Y$ and let $I := \alpha \circ \iota^* \circ \alpha^{-1}$ be the involution on Λ induced by ι. Set

$$M := \{x \in \Lambda;\ I(x) = x\}, \qquad N := \{x \in \Lambda;\ I(x) = -x\}.$$

Then $\mathrm{sign}(M) = (1, r(M) - 1)$ and $\mathrm{sign}(N) = (2, r(N) - 2)$.

Definition 6.3. The period domain for simple Enriques $2n$-folds deformation equivalent to Y is the Hermitian domain of type IV defined by

$$\Omega_N := \{[\sigma] \in \mathbb{P}(N \otimes \mathbb{C});\ (\sigma, \sigma)_N = 0, \quad (\sigma, \overline{\sigma})_N > 0\}.$$

The period of $(Y, \alpha) = (X, \iota, \alpha)$ is defined as

$$\alpha(H^0(X, \Omega_X^2)) \in \Omega_N.$$

Recall that there exists a moduli space of (unpolarized) Enriques surfaces. Similarly, under Assumption 6.2, it is possible to construct a moduli space of *(unpolarized)* simple Enriques $2n$-folds of hyperkähler type by using the existence of a universal marked family of hyperkähler manifolds. In contrast to the fact that the moduli space of Enriques surfaces is Hausdorff, the resulting moduli space of simple Enriques $2n$-folds of hyperkähler type is non-Hausdorff in general if $n > 1$. Two simple Enriques $2n$-folds Y and Y' of hyperkähler type are said to be inseparable if $\mathrm{Def}(Y) \cap \mathrm{Def}(Y') \neq \emptyset$ as set germs. We define a relation $\sim$ on the moduli space of simple Enriques $2n$-folds of hyperkähler type by $Y \sim Y'$ if Y and Y' are inseparable. In [25], Joumaah proved that the relation $\sim$ is an equivalence relation. The quotient of the moduli space of simple Enriques $2n$-folds by the equivalence relation $\sim$ is called the Hausdorff reduction of the moduli space. Joumaah has proved the following result. (For the notion of MBM classes, we refer the reader to [1], [2], [3]. For a lattice L, $O(L)$ denotes its automorphism group.)

Theorem 6.4 (Joumaah [25]). *The Hausdorff reduction of the moduli space of simple Enriques $2n$-folds of hyperkähler type deformation equivalent to Y is given by the modular variety of orthogonal type*

$$(\Omega_N - \mathcal{D}_N^{\mathrm{MBM}})/\Gamma_{N,[\mathcal{K}]},$$

where $\mathcal{D}_N^{\mathrm{MBM}}$ is a divisor on Ω_N determined by the MBM classes in N, $\Gamma_{N,[\mathcal{K}]} \subset O(N)$ is a subgroup of finite index, and $[\mathcal{K}]$ is a datum encoding the deformation equivalence class of (X, ι).

Remark 6.5. Precisely speaking, Joumaah dealt with the moduli space of pairs consisting of a hyperkähler manifold of $K3^{[n]}$-type and an anti-symplectic involution on it. Thanks to [1], [2], [3], [41], [30], [4], his proof still works in the case of moduli space of those simple Enriques $2n$-folds satisfying Assumption 6.2.

Remark 6.6. By [25], there is a chamber structure of the positive cone of M induced by the MBM classes in M. Then $\mathcal{K}$ is one of the chambers, and there is a finite-index subgroup $\Gamma_M \subset O(M)$ such that $[\mathcal{K}]$ is the Γ_M-orbit of the chamber $\mathcal{K}$ ([25]).

Theorem 6.7 ([47]). *There exist an integer $\nu \in \mathbb{Z}_{>0}$ and a (possibly meromorphic) automorphic form $\Phi_{N,[\mathcal{K}]}$ on Ω_N^+ for $\Gamma_{N,[\mathcal{K}]}$ of weight $\nu(n+1)/4$ such that*

$$\tau_{\mathrm{Enr},[\mathcal{K}]}^{-\nu} = \left\|\Phi_{N,[\mathcal{K}]}\right\|^2, \qquad \mathrm{Supp}\left(\mathrm{div}\,\Phi_{N,[\mathcal{K}]}\right) \subset \mathcal{D}_N^{\mathrm{MBM}}.$$

Question 6.8. Without Assumption 6.2, it is not clear for us if the invariant τ_{Enr} descends to a function on the space of periods. Does Theorem 6.7 still hold true without Assumption 6.2?

Since the line bundle of automorphic forms is an ample line bundle on the modular variety $\Omega_N/\Gamma_{N,[\mathcal{K}]}$ by Baily-Borel, we have the following generalization of Borcherds' theorem [16] (cf. Theorem 1.1).

Theorem 6.9 ([47])**.** *The Hausdorff reduction of the moduli space of simple Enriques $2n$-folds satisfying Assumption 6.2 is quassi-affine.*

By Theorems 6.7 and 6.9, $\Phi_{N,[\mathcal{K}]}$ may be viewed as a generalization of the Borcherds Φ-function in higher dimension. At least as a working hypothesis, it makes sense to conjecture that $\Phi_{N,[\mathcal{K}]}$ possesses some nice properties similar to those of the Borcherds Φ-function.

Conjecture 6.10 (elliptic modularity)**.** $\Gamma_{N,[\mathcal{K}]}$ contains the stable orthogonal group $\widetilde{O}(N) := \ker\{O(N) \to O(N^\vee/N, q_N)\}$, and $\Phi_{N,[\mathcal{K}]}$ is a Borcherds product. Namely, there is an elliptic modular form of type ρ_N, the Weil representation attached to N, and weight $(4 - r(N))/2$, whose Borcherds lift is $\Phi_{N,[\mathcal{K}]}$.

Conjecture 6.11 (reflectivity)**.** $\Phi_{N,[\mathcal{K}]}$ is holomorphic and reflective, i.e., $\mathrm{div}(\Phi_{N,[\mathcal{K}]})$ is an effective divisor, whose support is contained in the ramification divisor of the modular projection $\Omega_N \to \Omega_N/\Gamma_{N,[\mathcal{K}]}$.

We refer the reader to [22] for a characterization of the ramification divisor of the modular projection in terms of lattices. Recall the following theorem of Ma:

Theorem 6.12 (Ma [28])**.** *Let L be a lattice of signature $(2, n)$. If $n > 26$, then there are no reflective modular forms for any finite-index subgroup of $O(L)$.*

Combining Conjecture 6.11 and Theorem 6.12, we make the following:

Conjecture 6.13. For any simple Enriques $2n$-fold Y of index 2 of hyperkähler type with $\mathrm{Aut}_0(\widetilde{Y}) = \{1\}$, one has the following uniform upper bound of the dimension of the Kuranishi space of Y:

$$\dim \mathrm{Def}(Y) \leq 26.$$

If Conjecture 6.11 holds true, then so does Conjecture 6.13 by Theorem 6.12. Hence Conjecture 6.13 is a conjectural consequence of Conjecture 6.11 and Theorem 6.12.

Recall that when $\Phi_{N,[\mathcal{K}]}$ is holomorphic and its divisor is expressed as

$$\mathrm{div}\,\Phi_{N,[\mathcal{K}]} = \sum_{d \in \Delta_N^{\mathrm{MBM}}} c(d)\, d^\perp, \qquad c(d) \in \mathbb{Z}_{\geq 0},$$

then the slope of $\Phi_{N,[\mathcal{K}]}$ is defined as

$$\mathrm{slope}(\Phi_{N,[\mathcal{K}]}) := \max_{d \in \Delta_N^{\mathrm{MBM}}} \frac{c(d)}{\mathrm{wt}(\Phi_{N,[\mathcal{K}]})},$$

where $\mathrm{wt}(\Psi)$ denotes the weight of an automorphic form Ψ.

Conjecture 6.14 (boundedness of slope)**.** There is a uniform upper bound of $\mathrm{slope}(\Phi_{N,[\mathcal{K}]})$. Namely, there exists an absolute constant $C > 0$ such that for any deformation type of simple Enriques $2n$-folds satisfying Assumption 6.2, one has

$$0 < \mathrm{slope}(\Phi_{N,[\mathcal{K}]}) \leq C.$$

We recall another theorem of Ma:

Theorem 6.15 (Ma [29])**.** *For any $C > 0$, up to a scaling, the number of isometry classes of lattices of rank ≥ 6 carrying a reflective modular form with slope bounded from above by C is finite.*

Combining Conjecture 6.14 and Theorem 6.15, we make the following:

Conjecture 6.16. Up to a scaling, there are only finitely many possibilities of the anti-invariant sublattice N with $\operatorname{rank} N \geq 6$.

If both of Conjectures 6.11 and 6.14 hold true, then so does Conjecture 6.16 by Theorem 6.15. Hence Conjecture 6.16 is a conjectural consequence of Conjectures 6.11 and 6.14 and Theorem 6.15.

Question 6.17. Does some of O'Grady's 10-dimensional hyperkähler manifolds or their deformations admit an Enriques involution, i.e., a fixed-point-free and anti-symplectic involution? More generally, does a deformation of a hyperkähler manifold X with $\mathrm{Aut}_0(X) = \{1\}$ and $\dim X = 2(2n+1)$ always admit an Enriques involution?

Question 6.18. For a simple Enriques $2n$-fold Y and its universal covering X, one has the relation

$$b_2(Y) = b_2^-(X) = \text{anti-invariant subspace of } H^2(X,\mathbb{C}).$$

Is there any bound of $b_2(X)$ in terms of $b_2(Y)$? For all Enriques $2n$-folds known so far [35], [36], one always has

$$b_2(X) \leq 2b_2(Y).$$

Does this inequality hold true in general? If the answer is affirmative and if Conjecture 6.11 holds, then we will have $b_2(X) \leq 56$ and hence $\dim \mathrm{Def}(X) \leq 54$.

7. Some simple Enriques $2n$-folds of Calabi-Yau type

It is classical that the universal covering $K3$ surface of a generic Enriques surface is isomorphic to a $(2,2,2)$-complete intersection of $\mathbb{P}^5$. By using this projective model, it is possible to give an algebraic expression of the Borcherds Φ-function [26]. Replacing the Borcherds Φ-function by the invariant τ_{Enr}, we can generalize this result to higher dimension.

7.1. **Simple Enriques $2n$-folds of Boissière-Nieper-Wißkirchen-Sarti.** Let $A = (A_1,\dots,A_{m+1}), B = (B_1,\dots,B_{m+1}) \in \mathrm{Sym}(m+1,\mathbb{C}) \otimes \mathbb{C}^{m+1}$, where A_i, B_j are complex $(m+1)\times(m+1)$-symmetric matrices. Let $Q(x,A_i) = {}^t x A_i x$ and $Q(y,B_j) = {}^t y B_j y$ be the quadratic forms associated with A_i and B_j, respectively. Define a $(2,\dots,2)$-complete intersection of $\mathbb{P}^{2m+1}$

$$X_{(A,B)} := \left\{(x,y) \in \mathbb{P}^{2m+1};\, Q(x,A_i) + Q(y,B_i) = 0 \quad (1 \leq i \leq m+1)\right\}.$$

If A and B are sufficiently general, then $X_{(A,B)}$ is a Calabi-Yau manifold of dimension m. Define

$$Y_{(A,B)} := X_{(A,B)}/\iota, \qquad \iota(x,y) := (x,-y).$$

Fact 7.1. *Let $R(A)$ be the resultant of the system of quadratics $Q(x,A_1),\dots,Q(x,A_{m+1})$. Then the following hold:*

(1) $X^{\iota}_{(A,B)} \neq \emptyset \iff R(A)R(B) = 0.$

(2) *If A and B are sufficiently general, $R(A)R(B) \neq 0$, and m is even, then $Y_{(A,B)}$ is a simple Enriques m-fold of Calabi-Yau type.*

Theorem 7.2 ([47]). *For even m, there is a constant C_m depending only on m such that for any $A, B \in \mathrm{Sym}(m+1, \mathbb{C}) \otimes \mathbb{C}^{m+1}$ sufficiently general with $R(A)R(B) \neq 0$,*

$$\tau_{\mathrm{Enr}}\left(Y_{(A,B)}\right)^{-2^{m+1}} = C_m \left|R(A)R(B)\right| \left|\int_{X_{(A,B)}} \omega_{(A,B)} \wedge \overline{\omega}_{(A,B)}\right|^{2^m}.$$

Here $\omega_{(A,B)}$ is the canonical form on $X_{(A,B)}$ defined as the residue of the system of $m+1$-quadric polynomials

$$Q(x, A_1) + Q(y, B_1), \ldots, Q(x, A_{m+1}) + Q(y, B_{m+1}).$$

Remark 7.3. Since $\tau_{\mathrm{Enr}}(Y_{A,B}) = C\,\|\Phi(Y_{A,B})\|^{-1/4}$ when $m = 2$ ([44]), where $\|\Phi(Y_{A,B})\|$ is the Petersson norm of the Borcherds Φ-function evaluated at its period, Theorem 7.2 is exactly [26, Theorem 1.1] when $m = 2$. In this sense, Theorem 7.2 is a generalization of [26, Theorem 1.1] in higher dimension.

7.2. **Enriques varieties parametrized by configuration space.** Let us consider the special case of (A, B), where all of the quadric equations are of diagonal type. Let g be an *even* positive integer. Let $M_{m,n}(\mathbb{C})$ be the complex $m \times n$-matrices.

For $N = (\mathbf{n}_1, \ldots, \mathbf{n}_{2g+2}) \in M_{g+1,2g+2}(\mathbb{C})$, $\mathbf{n}_i \in \mathbb{C}^{g+1}$ $(1 \leq i \leq 2g+2)$, set

$$X_N := \{[x] \in \mathbb{P}^{2g+1};\ \sum_{i=1}^{2g+2} x_i^2 \mathbf{n}_i = \mathbf{0}\}.$$

When N is sufficiently general, X_N is a Calabi-Yau g-fold.

For $J = \{j_1 < \cdots < j_{g+1}\} \subset \{1, \ldots, 2g+2\}$, let J^c be the complement of J and let $\langle J \rangle$ be the corresponding partition

$$\langle J \rangle := J \amalg J^c = \{1, \ldots, 2g+2\}.$$

Hence $\langle J \rangle = \langle J^c \rangle$. For each partition $\langle J \rangle$, define

$$Y_{N,\langle J \rangle} := X_N / \iota_{\langle J \rangle}, \qquad \iota_{\langle J \rangle}(x_J, x_{J^c}) := (x_J, -x_{J^c}).$$

Fact 7.4. (1) *If N and N' lie in the same orbit of $\mathrm{GL}(\mathbb{C}^{g+1}) \times (\mathbb{C}^*)^{2g+2}$, then $X_N \cong X_{N'}$.*
(2) *$X_N^{\iota_{\langle J \rangle}} = \emptyset$ iff $\Delta_{\langle J \rangle}(N) := \det(\mathbf{n}_{j_1}, \ldots, \mathbf{n}_{j_{g+1}}) \det(\mathbf{n}_{j_1^c}, \ldots, \mathbf{n}_{j_{g+1}^c}) \neq 0$.*
(3) *$Y_{N,\langle J \rangle}$ is a simple Enriques g-fold of Calabi-Yau type for all $\langle J \rangle$ iff none of the $(g+1) \times (g+1)$-minors of N vanish.*

Theorem 7.5 ([47]). *For all $N \in M^o_{g+1,2g+2}(\mathbb{C}) := M_{g+1,2g+2}(\mathbb{C}) \setminus \bigcup_{\langle J \rangle} \mathrm{div}(\Delta_{\langle J \rangle})$ and $\langle J \rangle$, the following equality holds:*

$$\tau_{\mathrm{Enr}}\left(Y_{N,\langle J \rangle}\right)^{-2^{g+1}} = C_g \left|\Delta_{\langle J \rangle}(N) \int_{X_N} \omega_N \wedge \overline{\omega}_N\right|^{2^g}.$$

Here C_g is the same constant as in Theorem 7.2.

Corollary 7.6 ([47]). *For all $N \in M^o_{g+1,2g+2}(\mathbb{C})$ and partitions $\langle J \rangle, \langle J' \rangle$,*

$$\left\{\tau_{\mathrm{Enr}}\left(Y_{N,\langle J \rangle}\right) / \tau_{\mathrm{Enr}}\left(Y_{N,\langle J' \rangle}\right)\right\}^{-2} = \left|\Delta_{\langle J \rangle}(N) / \Delta_{\langle J' \rangle}(N)\right|.$$

In particular, if N is sufficiently general, then $Y_{N,\langle J \rangle} \not\cong Y_{N,\langle J' \rangle}$ for all distinct partitions $\langle J \rangle$, $\langle J' \rangle$. Hence X_N has at least $\binom{2g+2}{g+1}/2$ distinct Enriques structures.

7.3. **Simple Enriques $2n$-folds associated with hyperelliptic curves.** Let

$$\lambda = (\lambda_1, \ldots, \lambda_{2g+2}) \in \mathbb{C}^{2g+2} \setminus \mathrm{div}(\Delta),$$

where $\Delta(\lambda) := \prod_{i<j}(\lambda_j - \lambda_i)$ is the difference product. Define a hyperelliptic curve of genus g with level 2 structure

$$C_\lambda := \{(x, y) \in \mathbb{C}^2;\ y^2 = (x - \lambda_1) \cdots (x - \lambda_{2g+2})\}.$$

Define a complex $(g + 1) \times (2g + 2)$-matrix of Vandermonde type

$$M(\lambda) := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{2g+2} \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{2g+2}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^g & \lambda_2^g & \cdots & \lambda_{2g+2}^g \end{pmatrix}.$$

For $M(\lambda) = (M_1(\lambda), M_2(\lambda))$, $M_1(\lambda), M_2(\lambda) \in M_{g+1}(\mathbb{C})$, define

$$M(\lambda)^\vee := ({}^tM_1(\lambda)^{-1}, {}^tM_2(\lambda)^{-1}).$$

Fact 7.7 (Mumford [32])**.** *There is a one-to-one correspondence between the partitions $\{\langle J\rangle\}$ and the non-vansihing even theta constants on C_λ.*

Under this correspondence, write $\theta_{\langle J\rangle}(\Omega_\lambda)$ for the non-vanishing even theta constant on C_λ corresponding to the partition $\langle J\rangle$, where $\Omega_\lambda \in \mathfrak{S}_g$ is the period of C_λ with respect to a certain symplectic basis of $H_1(C_\lambda, \mathbb{Z})$ (cf. [32]).

Theorem 7.8 ([47])**.** *There is a constant C_g depending only on g such that for all $\lambda \in \mathbb{C}^{2g+2} \setminus \mathrm{div}(\Delta)$,*

$$\tau_{\mathrm{Enr}} \left(Y_{M(\lambda)^\vee,\langle J\rangle}\right)^{-1} = C_g \left\|\theta_{\langle J\rangle}(\Omega_\lambda)\right\|^2,$$

where $\|\theta_{\langle J\rangle}(\Omega_\lambda)\|$ is the Petersson norm of the theta constant $\theta_{\langle J\rangle}(\Omega_\lambda)$.

8. Simple Enriques $2n$-folds associated with Enriques surfaces

In [35], [36], Oguiso-Schröer have constructed three series of simple Enriques $2n$-folds. We give a formula for τ_{Enr} for those simple Enriques $2n$-folds.

8.1. **Simple Enriques $2n$-folds of Oguiso-Schröer I.** Let S be an Enriques surface and let $\widetilde{S}$ be the universal covering $K3$ surface of S. Let $X := \mathrm{Hilb}^n(\widetilde{S})$ be the Hilbert scheme of zero dimensional subschemes of $\widetilde{S}$ of length n. Let $\epsilon\colon X \to \widetilde{S}^n/\mathfrak{S}_n$ be the symplectic resolution of the symmetric product $\widetilde{S}^n/\mathfrak{S}_n$ (Hilbert-Chow morphism).

Fact 8.1 (Beauville [5])**.** *X is a hyperkähler $2n$-fold with the following properties.*

(1) $b_2(X) = 23$.
(2) $(H^2(X, \mathbb{Z}), q_{\mathrm{BBF}}) \cong \mathbb{L}_{K3} \oplus \langle -2(n-1)\rangle$.
(3) *$\langle -2(n-1)\rangle$ is generated by the class $E/2$, where E is the exceptional divisor of $\epsilon\colon X \to \widetilde{S}^n/\mathfrak{S}_n$.*
(4) $\mathrm{Aut}_0(X) = \{1\}$.

Here $\mathbb{L}_{K3} := \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{E}_8 \oplus \mathbb{E}_8$ is the $K3$-lattice and $\langle k\rangle$ is the one-dimensional lattice $(\mathbb{Z}, kx^2)$.

Let $\iota\colon \widetilde{S} \to \widetilde{S}$ be the non-trivial covering transformation such that $S = \widetilde{S}/\iota$. Let $\widetilde{\iota}\colon X \to X$ be the involution induced by ι. Let $H^2(X,\mathbb{Z})_\pm$ be the ±1-eigenlattice of the $\widetilde{\iota}$-action on $H^2(X,\mathbb{Z})$. Set

$$M := \mathbb{U}(2) \oplus \mathbb{E}_8(2) \oplus \langle -2(n-1)\rangle, \qquad N := \mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{E}_8(2).$$

Fact 8.2 (Oguiso-Schröer [35], [36]). (1) *If n is odd, then $Y := X/\widetilde{\iota}$ is a simple Enriques $2n$-fold of hyperkähler type.*
(2) *$H^2(X,\mathbb{Z})_+ \cong M$ and $H^2(X,\mathbb{Z})_- \cong N$.*

Theorem 8.3 ([47]). *There is a constant C_n depending only on an odd n such that*

$$\tau_{\mathrm{Enr}}\left(\mathrm{Hilb}^n(\widetilde{S})/\widetilde{\iota}\right) = C_n\,\|\Phi(S)\|^{-\frac{n+1}{8}}$$

for every Enriques surface S, where $\|\Phi(S)\|$ is the Petersson norm of the Borcherds Φ-function evaluated at the period of S.

8.2. **Simple Enriques $2n$-folds of Oguiso-Schröer II.** As before, let S be an Enriques surface and let $\widetilde{S}$ be the universal covering $K3$ surface of S. Let

$$\mathbb{M}_{L3} := \mathbb{U}(-1) \oplus \mathbb{L}_{K3}$$

be the Mukai lattice such that $H(\widetilde{S},\mathbb{Z}) \cong \mathbb{M}_{K3}$. Let $v = (v_0, v_1, v_2) \in H(\widetilde{S},\mathbb{Z})$, $v_i \in H^{2i}(\widetilde{S},\mathbb{Z})$ and let H be an ample line bundle on $\widetilde{S}$. Let $M_H(v)$ be the moduli space of H-stable torsion free coherent sheaves E on $\widetilde{S}$ with Mukai vector v, i.e.,

$$v(E) := \mathrm{ch}(E)\sqrt{\mathrm{Td}(\widetilde{S})} = v.$$

By Mukai [33], if v_1 is a primitive vector of $H(\widetilde{S},\mathbb{Z})$ with $v^2 \geq 0$ and H is sufficiently general, then $M_H(v)$ is a hyperkähler manifold of dimension $v^2 + 2$. By Mukai [33] and O'Grady [34], there is a Hodge isometry of lattices

$$\theta\colon v^\perp \cap H(\widetilde{S},\mathbb{Z}) \to H^2(M_H(v),\mathbb{Z}).$$

By Beauville [6], Hassett-Tschinkel [23], Yoshioka [48], $M_H(v)$ is deformation equivalent to $\mathrm{Hilb}^{v^2/2+1}(\widetilde{S})$ such that $\mathrm{Aut}_0(M_H(v)) = \{1\}$ and $b_2(M_H(v)) = 23$.

Recall that $\iota\colon \widetilde{S} \to \widetilde{S}$ is the non-trivial covering transformation such that $S = \widetilde{S}/\iota$. When H and v are ι-invariant, ι lifts to an involution $\widetilde{\iota}$ on $M_H(v)$.

Fact 8.4 (Oguiso-Schröer [35], [36]). *If v is ι-invariant and $\chi(E)$ is odd, then $\widetilde{\iota}$ is free from fixed points and $M_H(v)/\widetilde{\iota}$ is a simple Enriques $2n$-fold with $n = (v^2+2)/2$ such that*

$$\theta((v^\perp \cap H(\widetilde{S},\mathbb{Z}))_-) = \theta(H^2(\widetilde{S},\mathbb{Z})_-) = H^2(M_H(v),\mathbb{Z})_- \cong H^2(S,\mathbb{Z})_- \cong \mathbb{U}\oplus\mathbb{U}(2)\oplus\mathbb{E}_8(2).$$

Theorem 8.5 ([47]). *There is a constant $C_{[\mathcal{K}]}$ depending only on the deformation type $[\mathcal{K}]$ of $M_H(v)/\widetilde{\iota}$ such that for every Enriques surface S*

$$\tau_{\mathrm{Enr}}\left(M_{(S,H)}(v)/\widetilde{\iota}\right) = C_{[\mathcal{K}]}\,\|\Phi(S)\|^{-\frac{v^2+4}{16}}.$$

8.3. **Simple Enriques $2n$-folds of Oguiso-Schröer III.**

Fact 8.6 (Oguiso-Schröer [35], [36]). *For an Enriques surface S, $\mathrm{Hilb}^n(S)$ is a simple Enriques $2n$-fold of Calabi-Yau type.*

Theorem 8.7 ([47]). *There is a constant C_n' depending only on $n > 1$ such that*

$$\tau_{\mathrm{Enr}}\left(\mathrm{Hilb}^n(S)\right) = C_n'\,\|\Phi(S)\|^{-\frac{n}{4}}.$$

Problem 8.8. Determine the following universal constants:

$$\tau_{\mathrm{Enr}}(S)/\|\Phi(S)\|^{-1/4}, \quad \tau_{\mathrm{Enr}}\left(\mathrm{Hilb}^n(\widetilde{S})/\widetilde{\iota}\right)/\tau_{\mathrm{Enr}}(S)^{\frac{n+1}{2}}, \quad \tau_{\mathrm{Enr}}\left(\mathrm{Hilb}^n(S)\right)/\tau_{\mathrm{Enr}}(S)^n.$$

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Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, JAPAN

Email address: `yosikawa@math.kyoto-u.ac.jp`