Locally symmetric varieties and holomorphic symplectic manifolds

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An *irreducible holomorphic symplectic manifold* or *IHS manifold* is a compact complex manifold X such that $\pi_1(X) = 1$ and $H^{2,0}(X) = \mathbb{C}\sigma$, where σ is an everywhere nondegenerate 2-form. These coincide with the *compact hyperkähler manifolds*, and the two names are often used interchangeably.

We give here a quick and selective overview of IHS manifolds and their moduli: see [11] for an extended version, and for full references. The basic facts are due to Bogomolov [3], Beauville [1] and Huybrechts [14].

If X is an IHS manifold then K_X is trivial, but X need not be algebraic in general. The complex dimension is necessarily even: we put $\dim_{\mathbb{C}} X = 2n$.

The basic example of an IHS manifold is a K3 surface. Other examples, with $n \geq 2$, are quite scarce.

List 1 Types of IHS manifolds

> Type $\mathrm{K3}^{[n]}$. *The Hilbert scheme* $\mathrm{Hilb}^n S$ *of any* K3 *surface* S*, parametrising length* n *zero-dimensional subschemes of* S*, is an IHS manifold: so are deformations of such Hilbert schemes. These are called IHS manifolds of type* $\mathrm{K3}^{[n]}$.
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> *Type* Km^{n+1}. *For any abelian surface* A*, consider the Hilbert scheme* $\mathrm{Hilb}^{n+1}(A)$ *and its Albanese map* $\alpha\colon \mathrm{Hilb}^{n+1}(A) \to A$*, which on the open part where the subscheme consists of* $n+1$ *distinct closed points in* A *is simply the map that takes their sum. The fibre* $\alpha^{-1}(0)$*, the generalised Kummer manifold, is an IHS manifold: again, so are deformations of the fibre. These are called IHS manifolds of type* Km^{n+1}.
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> Type OG10. *A special construction due to O'Grady [23]. We consider a particular* 10*-dimensional moduli space of coherent sheaves on a* K3 *surface. Such a moduli space automatically has an everywhere*

nondegenerate 2-form on its smooth locus, but is in general singular. In this special case with invariants are chosen so that the singularities admit a crepant resolution, and then the desingularised moduli space also has a nondegenerate 2-form and is thus an IHS manifold. Again there are further deformations of the resulting space, and the manifolds that arise are called IHS manifolds of type OG10.

Type OG6. *Another special construction, also due to O'Grady [24]. Instead of a* K3 *surface one starts with an abelian surface. The dimension of the moduli space of sheaves for which the crepant resolution exists is* 6*, and again there is a further deformation. These are called IHS manifolds of type OG6.*

No other deformation types are known, and one may reasonably wonder whether there are any.

There is also a list, also quite short, of projective constructions that yield IHS manifolds. I believe the following list to be complete at the time of the Kinosaki meeting.

List 2 Constructions of IHS manifolds

(a) *The Fano variety $F(X)$ of lines on a smooth cubic 4-fold X [2]: type* $\mathrm{K3}^{[2]}$.

(b) *The variety* $\mathrm{VSP}(F_6^3, 10)$ *parametrising representations of a cubic form in six variables as a sum of ten cubes [16]: type* $\mathrm{K3}^{[2]}$.

(c) *The examples of Debarre and Voisin [5], contained in* $\mathrm{Gr}(6, 10)$*: type* $\mathrm{K3}^{[2]}$.

(d) *The examples of Lehn, Lehn, Sorger and van Straten [19]: type* $\mathrm{K3}^{[4]}$.

(e) *Double EPW sextics [25]: type* $\mathrm{K3}^{[2]}$.

(f) *EPW cubes [15]: type* $\mathrm{K3}^{[2]}$.

(g) *The family given in [18]: type OG10.*

Additionally, [21] gives a projective construction of some generalised Kummer varieties, but not of deformations of them. The construction in List 2(g) also gives only a special family of varieties.

IHS manifolds have many good properties. Their deformations are unobstructed, and they are, loosely speaking, determined by their second cohomology. For K3 surfaces, this is true in a very precise way: if S is a K3 surface then $H^2(S, \mathbb{Z})$ comes with an integral quadratic form (the cup product) and the Hodge structure on $H^2(S)$ completely determines S.

For IHS manifolds X (of complex dimension n) in general, the cup product does not do this but instead we define the Beauville-Bogomolov-Fujiki form (or *BBF form*) q_X on $H^2(X)$ as follows. We normalise σ so that $\int_X (\sigma\bar{\sigma})^n = 1$ and we define q_X by

$$\gamma q_x(\alpha) = \frac{n}{2}\int_X \alpha^2(\sigma\bar{\sigma})^{n-1} + (1-n)\left(\int_X \alpha\sigma^{n-1}\bar{\sigma}^n\right)\left(\int_X \bar{\alpha}\sigma^n\bar{\sigma}^{n-1}\right),$$

where γ is a constant, uniquely defined by the requirement that q_X is to be integer-valued and indivisible on $H^2(X,\mathbb{Z})$, of signature $(3, b_2-3)$. It is shown in [1] that such a γ exists.

The BBF form is not known to be even in general, but it is in fact even for all IHS manifolds in List 1. It is related to the cup product: it satisfies $\alpha^{2n} = c_F q_X(\alpha)^n$, for a constant c_F called the *Fujiki constant*.

The numerical type $\mathcal{N}$ of an IHS manifold X is the pair (Λ, c_F) consisting of the isometry class Λ of the BBF form and the Fujiki constant. They are $c_F = (2n)!/(n!2^n)$ for type $\mathrm{K3}^{[n]}$, $c_F = (n+1)(2n)!/(n!2^n)$ for type $\mathrm{Km}^{[n+1]}$, $c_F = 945$ for type OG10 and $c_F = 60$ for type OG6.

We denote by U the hyperbolic plane, $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and by E_8 the unique irreducible positive definite unimodular even lattice or rank 8.

As abstract lattices, the lattices $H^2(X,\mathbb{Z}), q_X$ are $L_{\mathrm{K3}} := 3U \oplus 2E_8(-1)$ for K3 surfaces; $L_{\mathrm{K3}} \oplus \langle -2(n-1)\rangle$ for type $\mathrm{K3}^{[n]}$ with $n > 1$; $L_{\mathrm{K3}} \oplus A_2(-1)$ for type OG10; $3U \oplus \langle -2(n+1)\rangle$ for type $\mathrm{Km}^{[n+1]}$; and $3U \oplus 2\langle -2\rangle$ for type OG6.

As usual, it is better for moduli to consider polarised varieties. A polarisation is, strictly speaking, the Néron-Severi class of an ample line bundle $\mathcal{L}$ on X. We relax this slightly in two ways. Firstly, since X is simply-connected, we may as well take the polarisation to be $H = c_1(\mathcal{L}) \in H^2(X,\mathbb{Z})$; secondly, we allow semi-ample line bundles (these are sometimes called semi-polarisations). This is all right because if an IHS manifold carries a semi-ample line bundle then it does also carry an ample line bundle, i.e. it is in fact itself projective. Note that by no means all IHS manifolds are projective, as we already know from K3 surfaces.

A polarisation is thus a semi-positive $(1,1)$ class in $h \in H^2(X,\mathbb{Z})$. We assume that it is primitive, and we set $q_X(h) = 2d$ (since in all cases that we know of, q_X is an even form).

We fix the numerical type $\mathcal{N}$ and an abstract lattice $L_{\mathcal{N}}$ in the isometry class Λ, as above. A *marking* of an IHS manifold X of numerical type $\mathcal{N}$ is an isomorphism of lattices $\varphi\colon (H^2(X,\mathbb{Z}), q_X) \to L_{\mathcal{N}}$. Then a choice of polarisation H gives a class $h = \varphi(H) \in L_{\mathcal{N}}$ with $h^2 = 2d$, and we may consider its orthogonal complement $L_h = h^{\perp}_{L_{\mathcal{N}}}$.

For any indefinite lattice L we define the *period domain*

$$\Omega_L := \{[x] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (x,x) = 0,\ (x,\bar{x}) > 0\}.$$

If X is a marked IHS manifold of numerical type $\mathcal{N}$, then the *period point* of X is $\varphi(\sigma) \in \Omega_{L_{\mathcal{N}}}$. Now we may forget the marking by taking the quotient

$O(L_{\mathcal{N}})\backslash\Omega_{L_{\mathcal{N}}}$ and the Torelli theorem says that this is the moduli space of IHS manifolds of numerical type $\mathcal{N}$. Unfortunately "the" Torelli theorem is not actually quite true, and neither is $O(L_{\mathcal{N}})$ the correct group to use, but in any case we cannot expect good results for algebraic geometry by doing this, if only because most IHS manifolds are themselves not algebraic.

Nevertheless, the idea is not fundamentally wrong. One way to set it right is to work with polarised varieties from the beginning: fix a class $h \in L_{\mathcal{N}}$ and define a polarised marking of a polarised IHS manifold (X, H) to be an isomorphism $\varphi\colon H^2(X, \mathbb{Z}) \to L_{\mathcal{N}}$ such that $\varphi(H) = h$. Then $\varphi(\sigma) \in L_h$, so we work with L_h instead of $L_{\mathcal{N}}$.

Several things change, most of them for the better, when we do this.

- L_h has signature $(2, b_2 - 3)$, so Ω_{L_h} has two connected components and we have to pick one of them, calling it $\mathcal{D}_{L_h}$.

- We now have one more piece of numerical data to record: the $O(L)$-orbit $[h]$ of h. For K3 surfaces this is determined by $2d$, so one may speak of the moduli space of K3surfaces of degree $2d$ (or genus $g = d + 1$). In general there are more possibilities.

- The Torelli theorem in its simplest form fails: non-isomorphic IHS manifolds can have the same period point. The next guess is that the period point determines the birational class of X, but that is wrong too. However, a sufficiently good Torelli theorem for our purposes, due to Verbitsky, does hold. The main effect is that we get a map

$$\mathcal{M}_{\mathcal{N},[h]} \longrightarrow O^*(L_h)\backslash\mathcal{D}_{L_h}$$

  from the moduli space of $[h]$-polarised IHS manifolds of numerical type $\mathcal{N}$ (which exists and is quasi-projective by GIT), but the moduli space may have many components. Nevertheless the map is generically finite, and birational and surjective when restricted to each component.

- The group $O^*(L_h)$ here denotes a yet unspecified subgroup of $O(L_h)$ of finite index. The statement in the previous item is true if we take $O^*(L_h) = O^+(L_h)$ (the stabiliser of the chosen component $\mathcal{D}_{L_h}$), but it is desirable to lift the map as far as possible, i.e. to choose as small a group $O^*(L_h)$ as possible.

The correct choice of $O^*(L_h)$ is therefore the group $\mathrm{Mon}^2(L_h)$, which is the subgroup generated by all possible monodromy actions: in other words, the isometries of geometric origin. In actual fact this depends only on $\mathcal{N}$, in the sense that any element of $\mathrm{Mon}^2(L_h)$ must (by definition) lift to $O(L)$ (and stabilise h) and the group of such lifts coincides with the subgroup of $O(L)$ generated by monodromy actions.

With this background, we are in a position to list some problems.

List 3 Problems about IHS manifolds

(a) *Can we find more deformation or numerical types of IHS manifolds, beyond List 1? It is now twenty years since the last one.*

(b) *Alternatively, can we prove (perhaps only for some low dimensions) that there are no more? Opinion differs about this one. I personally think that there are no more at least in dimension 4, but the evidence is not very compelling and I have been informed that at least one expert takes the opposite view.*

(c) *Can we identify* $\mathrm{Mon}^2(\mathcal{N})$*? This is currently answered in full – "currently" because if we were to find IHS manifolds outside List 1, the question would arise again for them. The cases of* $\mathrm{K3}^{[n]}$ *and* $\mathrm{Km}^{[n+1]}$ *were settled by Markman, with modifications by Mongardi (see [22]), and the O'Grady cases have very recently been worked out by Mongardi, Rapagnetta and Saccà for OG6 and by Onorati for OG10.*

(d) *Can we say anything about the birational class of* $\mathcal{M}_{\mathcal{N},[h]}$*?*

The case of List 3(d) that has had most attention is, naturally, that of K3 surfaces, where $[h]$ is determined by d. The current state of knowledge is that the compactified moduli space $\bar{\mathcal{F}}_d$ of degree $2d$ polarised K3 surfaces (which is of dimension 19) is uniruled (that is, has Kodaira dimension $\kappa = -\infty$) for $d \leq 12$ and also for $d = 15, 16, 17, 19$ and 21. Most of these results are either classical or due to Mukai: see the references in [11]. The case $d = 21$ is settled in [7], which appeared on the morning of this talk. In some cases more precise statements, such as unirationality or rationality, are known. It is also claimed in [17] that $\bar{\mathcal{F}}_d$ is uniruled for $d = 33$ and $d = 36$, but I am not alone in being unable to understand the argument sufficiently. By contrast ([8], and see also [26]), $\bar{\mathcal{F}}_d$ is of general type if $d > 61$, and also for $d = 46, 50, 52, 54, 57, 58$ and 60; and $\kappa \geq 0$ if $d \geq 40$ except perhaps for $d = 41, 44, 45, 47$.

For the higher-dimensional cases, we have to distinguish different orbits of h of the same degree. In the simplest cases (which are the only ones we shall need) there are at most two possibilities, which are called split and non-split [10]. The non-split case arises only for $d \equiv 3 \bmod 4$. The names refer to the shape of the lattices L_h, which consist of a unimodular part and a part of small rank, which may or may not split into rank 1 pieces.

The known families in List 2 are all rational, so in those cases, if the families are of maximal dimension the corresponding moduli spaces are unirational. This applies, for example, to the moduli spaces $\mathcal{M}_{\mathrm{K3}^{[2]},d,\mathrm{n-spl}}$ of $\mathrm{K3}^{[2]}$ type IHS manifolds with non-split polarisation of degree d for $d = 3$ (List 2(a)) and $d = 11$ (List 2(c)). In the other direction, by methods similar to those used for K3 surfaces it is known that $\mathcal{M}_{\mathrm{K3}^{[2]},d,\mathrm{spl}}$ is of general type for $d \geq 12$ and has $\kappa \geq 0$ for $d = 9$ and $d = 11$: see [10]. Similarly, $\mathcal{M}_{OG10,d,\mathrm{spl}}$ is of general type for any d that is not a power of 2.

The case of generalised Kummers presents some extra difficulties. Very recently [4] it has been shown that $\mathcal{M}_{\mathrm{Km}^{[3]},d,\mathrm{spl}}$ is of general type if $d > 252288$ and $d = 3(a^2 + b^2 + c^2)$ with $\mathrm{hcf}(6, abc/\,\mathrm{hcf}(a, b, c)) \neq 1$.

All these moduli spaces are, or dominate, orthogonal modular varieties. An *orthogonal modular variety* is a compactified quotient $X = \overline{\Gamma\backslash\mathcal{D}_L}$, where L is a lattice of signature $(2, k)$ and Γ is an arithmetic subgroup of $\mathrm{O}_{\mathbb{R}}(L) \cong \mathrm{O}(2, k)$. Ma, extending [9], shows in [20] that almost all such varieties with $k \geq 9$ are of general type. The method is to interpret modular forms for Γ as differential forms on X: this is the approach used by Tai [27] to prove similar results for moduli spaces of principally polarised abelian varieties, which are symplectic modular varieties coming from the symplectic group $\mathrm{Sp}(2g, \mathbb{R})$.

In general these differential forms have poles along the compactifying divisor (one says "at the cusps") and along the ramification locus of the quotient map, arising from torsion in Γ. By a result in [8], corrected and extended in [20], the singularities of X are canonical if $k \geq 9$, and there are so many high-weight modular forms for Γ that many of the corresponding differential forms have to extend without poles to the boundary divisors and the ramification divisors.

Slightly more precisely, the space of weight w modular forms is of dimension $O(w^k)$, by Hirzebruch proportionality: the obstruction to extending a pluricanonical form to a divisor consists of $O(w)$ residues, each of which is a section in a bundle on the divisor whose space of sections is $O(w^{k-1})$. Consequently there will be pluricanonical forms as long as the coefficients are favourable, which is the case for most Γ: for instance, when $d \gg 0$ in the cases above.

The first difficulty when one tries to use this approach to tackle the case of generalised Kummers is that $k = 4$, so one must also control the non-canonical singularities and estimate the obstruction that they give to extending pluricanonical forms to a resolution of singularities. The second difficulty is that the number of boundary divisors may be large. To overcome this, one often uses the low-weight cusp form trick: instead of looking at arbitrary modular forms of weight w, one finds a cusp form f_0 of weight $w_0 < k$ and looks at forms divisible by a high power of f_0. There are fewer of these, but the corresponding differential forms automatically extend to the boundary divisors.

The low-weight cusp form trick first appeared in [12] and when it is available it usually gives better results that the asymptotic method. In [8] and elsewhere, such a cusp form is obtained from quasi-pullback of the Borcherds form Φ, which is a weight 12 cusp form associated to $\mathrm{O}(2, 26)$. Because quasi-pullback increases the weight, this can only work for $14 \leq$

$k \leq 25$, but that includes the interesting cases of K3, $\mathrm{K3}^{[n]}$ and OG10 ($k = 19, 20, 21$ respectively). It does not include $\mathrm{Km}^{[n+1]}$ and OG6, with $k = 4$ and $k = 5$; nor does it include the case $k = 3$, which, because of the exceptional isogeny $\mathrm{Sp}(4, \mathbb{R}) \to \mathrm{SO}(2, 3)$, corresponds to moduli of polarised abelian surfaces.

In such cases it is sometimes possible to construct a suitable cusp form by lifting Jacobi forms. This is the approach taken in [4] and this is where the arithmetic conditions on d are needed.

A very similar situation arises in the case $k = 3$, corresponding to polarised abelian surfaces. The most important such moduli space is the space $\mathcal{A}_p$ of $(1, p)$-polarised abelian surfaces (of level 1). The degree p is usually assumed to be prime, to avoid some combinatorial complications. Again there are geometric families that show uniruledness for some small values of p (e.g. [13]): in the other direction, the best result known is due to Erdenberger [6], who showed that $\mathcal{A}_p$ is of general type unless $p \in \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\}$.

When [6] appeared on the arXiv, John McKay immediately pointed out that this is the list of supersingular primes `http://www.oeis.org/A002267`, the primes that divide the order of the Monster: the same list, recognised by the same person, led to the Moonshine conjectures and the famous work of Borcherds (yielding, among many other things, the Borcherds form Φ). We still do not have a completely satisfactory explanation for this coincidence, although it is not completely mysterious. It is quite likely that it is an artefact of the proof: it could perfectly well be that, for example, $\mathcal{A}_{71}$ is in fact of general type.

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