TUTORIAL ON LOCAL INTERSECTION INEQUALITIES

IVAN CHELTSOV

Abstract. We survey several local intersection inequalities for effective $\mathbb{Q}$-divisors on smooth surfaces, and present their global applications as examples and exercises.

Let S be a normal surface, let P be a smooth point of the surface S, and let

$$\boxed{D=\sum_{i=1}^{r}a_iC_i}$$

where $C_1,\ldots,C_r$ are distinct irreducible curves on S, and $a_1,\ldots,a_r$ are non-negative rational numbers. Let $f\colon\widetilde{S}\to S$ be a birational morphism such that $\widetilde{S}$ is smooth, let $E_1,\ldots,E_n$ be exceptional curves of the morphism f, and let $\widetilde{C}_1,\ldots,\widetilde{C}_r$ be proper transforms of the curves $C_1,\ldots,C_r$ on the surface $\widetilde{S}$, respectively. Then

$$\boxed{K_{\widetilde{S}}+\sum_{i=1}^{r}a_i\widetilde{C}_i+\sum_{i=1}^{n}b_iE_i\sim_{\mathbb{Q}}f^*\big(K_S+D\big)}$$

for some rational numbers $b_1,\ldots,b_n$. Suppose also that all curves $\widetilde{C}_1,\ldots,\widetilde{C}_r$ are smooth, and the divisor

$$\boxed{E_1+\cdots+E_n+\widetilde{C}_1+\cdots+\widetilde{C}_r}$$

has at most simple normal crossing singularities.

Definition. The log pair (S,D) is log canonical at P if the following conditions hold:

(1) $a_i\leqslant 1$ for every $i\in\{1,\ldots,r\}$ such that $P\in C_i$;
(2) $b_j\leqslant 1$ for every $j\in\{1,\ldots,n\}$ such that $f(E_j)=P$.

Exercise 1. Show that this definition does not depend on the choice of f.

If $\boxed{\mathrm{mult}_P(D)>2}$ then (S,D) is not log canonical at P.

Question. Which simple conditions on D imply that (S,D) is log canonical at P?

Here is an easy answer to this question:

Exercise 2. Show that the inequality

$$\boxed{\mathrm{mult}_P\big(D\big)\leqslant 1}$$

implies that the log pair (S,D) is log canonical at the point P.

Let us show how to use this simple criterion.

We assume that all considered varieties are projective, normal and defined over complex numbers.

Example 3 ([3, Lemma 1.7.9])**.** Let $S = \mathbb{P}^1 \times \mathbb{P}^1$. Suppose that D has bi-degree (a, b), where $a \in \mathbb{Q}$ and $b \in \mathbb{Q}$ such that $a \leqslant 1$ and $b \leqslant 1$. Then (S, D) is log canonical at P. Indeed, let L_1 and L_2 be the curves of bi-degree $(1, 0)$ and $(0, 1)$ such that $P = L_1 \cap L_2$. Then $(S, aL_1 + bL_2)$ is log canonical at P. Moreover, if $L_1 \not\subseteq \mathrm{Supp}(D)$, then

$$1 \geqslant b = L_1 \cdot D \geqslant \Big(L_1 \cdot D\Big)_P \geqslant \mathrm{mult}_P\big(L_1\big)\mathrm{mult}_P\big(D\big) = \mathrm{mult}_P\big(D\big),$$

so that (S, D) is log canonical at P as required. Similarly, if $L_2 \not\subseteq \mathrm{Supp}(D)$, then

$$1 \geqslant a = L_2 \cdot D \geqslant \Big(L_2 \cdot D\Big)_P \geqslant \mathrm{mult}_P\big(L_2\big)\mathrm{mult}_P\big(D\big) = \mathrm{mult}_P\big(D\big),$$

so that (S, D) is log canonical at P. Hence, we may assume that $L_1 = C_1$ and $L_2 = C_2$. Put

$$\boxed{\mu = \min\left(\frac{a_1}{a}, \frac{a_2}{b}\right)}$$

which gives $\mu \leqslant 1$, since $a_1 \leqslant a$ and $a_2 \leqslant b$. Moreover, if $\mu = 1$, then $a_1 = a$ and $a_2 = b$, which implies that $D = aL_1 + bL_2$, so that the pair (S, D) is log canonical at P as required. Therefore, we may assume that $\mu < 1$, which implies that $r \geqslant 3$. Let

$$\boxed{D' = \frac{a_1 - \mu a}{1 - \mu} L_1 + \frac{a_2 - \mu b}{1 - \mu} L_2 + \sum_{i=3}^{r} \frac{a_i}{1 - \mu} C_i}$$

so that either $L_1 \not\subseteq \mathrm{Supp}(D')$ or $L_2 \not\subseteq \mathrm{Supp}(D')$. Moreover, we also have $D' \sim_{\mathbb{Q}} D$, so that the log pair (S, D') is log canonical at P (we just proved this). But

$$\boxed{D = (1 - \mu)D' + \mu(aL_1 + bL_2)}$$

and $(S, aL_1 + bL_2)$ is log canonical at P. This implies that (S, D) is log canonical at P.

Exercise 4 ([16, Lemma 4.8])**.** Let $S = \mathbb{F}_1$, let C and L be irreducible curves on the surface S such that $C^2 = -1$, $C \cdot L = 1$ and $L^2 = 0$. Suppose that

$$D \sim_{\mathbb{Q}} aC + bL,$$

where $a \in \mathbb{Q}$ and $b \in \mathbb{Q}$ such that $a \leqslant 1$ and $b \leqslant 1$. Prove that (S, D) is log canonical at P.

Exercise 5 ([4, Lemma 3.1])**.** Let S be a smooth hypersurface of degree 6 in $\mathbb{P}(1, 1, 2, 3)$. Suppose that $D \sim_{\mathbb{Q}} -K_S$. Prove that $(S, \frac{5}{6}D)$ is log canonical.

Unfortunately, the application scope of this simple multiplicity-criterion is very limited. To expand its application scope, we suppose that $r \geqslant 2$ and write

$$\boxed{\Delta = \sum_{i=2}^{r} a_i C_i}$$

so that $D = a_1 C_1 + \Delta$.

Theorem 6 ([6, Theorem 7])**.** If $a_1 \leqslant 1$, $P \in C_1$, the curve C_1 is smooth at P, and

$$\boxed{\Big(C_1 \cdot \Delta\Big)_P \leqslant 1}$$

then the log pair (S, D) is log canonical at P.

Let us show how to apply this result.

Example 7 ([13, Theorem 3.3])**.** Let S be a surface of degree 4 in $\mathbb{P}(1,1,1,2)$ that has at most isolated ordinary double points, and let $\pi\colon S\to\mathbb{P}^2$ be the double cover induced by the projection $\mathbb{P}(1,1,1,2)\dashrightarrow\mathbb{P}^2$, and let R be the quartic in $\mathbb{P}^2$ that is the ramification curve of the morphism π. Suppose that $D\sim_{\mathbb{Q}}-K_S$ and (S,D) is not log canonical at P. We claim that $\pi(P)\in R$. Indeed, suppose that $\pi(P)\notin R$. Let us seek for a contradiction. Let $g\colon\widehat{S}\to S$ be the blow up of the surface S at the point P. Then

$$\boxed{K_{\widehat{S}}+\widehat{D}+\Big(\mathrm{mult}_P(D)-1\Big)E\sim_{\mathbb{Q}}g^*\big(K_S+D\big)\sim_{\mathbb{Q}}0}$$

where $\widehat{D}$ is the proper transform of the divisor D on the surface $\widehat{S}$, and E is the exceptional curve of the morphism g. Then the log pair

$$\boxed{\Big(\widehat{S},\widehat{D}+\big(\mathrm{mult}_P(D)-1\big)E\Big)}$$

is not log canonical at some point $Q\in E$. Hence, our original criterion gives

$$\boxed{\mathrm{mult}_P\big(D\big)+\mathrm{mult}_Q\big(\widehat{D}\big)>2}$$

but $|-K_S|$ contains a curve Z such that its proper transform $\widehat{Z}$ via g passes through Q. Note that $P\in Z$ and Z is smooth at the point P, so that (S,Z) is log canonical at P. Thus, arguing as in Example 3, we may assume that the support of D does not contain at least one irreducible component of the curve Z. If the curve Z is irreducible, then

$$2-\mathrm{mult}_P\big(D\big)=2-\mathrm{mult}_P\big(Z\big)\mathrm{mult}_P\big(D\big)=\widehat{Z}\cdot\widehat{D}\geqslant\mathrm{mult}_Q\big(\widehat{Z}\big)\mathrm{mult}_Q\big(\widehat{D}\big)=\mathrm{mult}_Q\big(\widehat{D}\big),$$

which contradicts the inequality we proved earlier. Thus, the curve Z must be reducible. We may then write $Z=Z_1+Z_2$, where Z_1 and Z_2 are irreducible smooth rational curves. Without loss of generality we may assume that the curve Z_2 is not contained in $\mathrm{Supp}(D)$. Then $P\in Z_1$, because otherwise $P\in Z_2$, so that

$$1=D\cdot Z_2\geqslant\mathrm{mult}_P\big(Z_2\big)\mathrm{mult}_P\big(D\big)=\mathrm{mult}_P\big(D\big)>1.$$

Similarly, we see that Z_1 is contained in $\mathrm{Supp}(D)$, so that we may assume that $C_1=Z_1$. Let k be the number of singular points of the surface S that are contained in C_1. Then

$$1=Z_2\cdot D=a_1Z_2\cdot C_1+Z_2\cdot\Delta\geqslant a_1\big(Z_2\cdot C_1\big)=\Big(2-\frac{k}{2}\Big)a_1,$$

which gives $\boxed{a_1\leqslant\dfrac{2}{4-k}\leqslant 1}$ because $k\leqslant 2$ (why?). Recall that the log pair

$$\Big(\widehat{S},a_1\widehat{C}_1+\widehat{\Delta}+\big(\mathrm{mult}_P(D)-1\big)E\Big)$$

is not log canonical at Q, where $\widehat{C}_1$ and $\widehat{\Delta}$ are the proper transforms of C_1 and Δ on the surface $\widehat{S}$, respectively. Then $a_1>\frac{2}{4-k}$, since

$$\Big(2-\frac{k}{2}\Big)a_1=\widehat{C}_1\cdot\Big(\widehat{\Delta}+\big(\mathrm{mult}_P(D)-1\big)E\Big)>1$$

by Theorem 6. This is a contradiction.

Exercise 8 ([4, Lemma 4.1])**.** Let $\overline{S}$ be a hypersurface of degree 6 in $\mathbb{P}(1,1,2,3)$ that has at most Du Val singular points of type A_1 or A_2, and let $\nu\colon S\to\overline{S}$ be its minimal resolution of singularities. Suppose that $D\sim_{\mathbb{Q}}-K_S$. Show that $(S,\frac{2}{3}D)$ is log canonical at P.

Exercise 9 ([13, Theorem 3.1])**.** Let $\overline{S}$ be a hypersurface of degree 6 in $\mathbb{P}(1,1,2,3)$ that has at most Du Val singularities, let O be its singular point of type D_4, let $\overline{C}$ be the unique curve in the linear system $|-K_{\overline{S}}|$ that passes through O, and let $\nu\colon S\to\overline{S}$ be the minimal resolution of singularities of the surface $\overline{S}$. Suppose that $\nu(P)=O$. Suppose also that

$$D\sim_{\mathbb{Q}}-K_S,$$

and the support of the divisor D does not contain the proper transform of the curve $\overline{C}$. Prove that the log pair (S,D) is log canonical at P.

Under an additional constrained on Δ, the assertion of Theorem 6 can be improved.

Theorem 10 ([10, Lemma 3.5])**.** Suppose that $a_1\leqslant 1$, $P\in C_1$, and C_1 is smooth at P. If $\mathrm{mult}_P(\Delta)\leqslant 1$ and

$$\boxed{\Big(C_1\cdot\Delta\Big)_P\leqslant 2-a_1}$$

then the log pair (S,D) is log canonical at P.

Let us show how to apply this result.

Example 11 ([17, Remark 2.11])**.** Let S be a smooth surface of degree 6 in $\mathbb{P}(1,1,2,3)$. Suppose that $D\sim_{\mathbb{Q}}-K_S$, and C_1 is a curve in the pencil $|-K_S|$ that passes through P. Suppose also that $a_1\leqslant\frac{1}{3}$, and C_1 is singular at P. Then $(S,\frac{3}{2}D)$ is log canonical at P. Indeed, suppose that $(S,\frac{3}{2}D)$ is not log canonical at P. Let us seek for a contradiction. Let $m=\mathrm{mult}_P(\Delta)$. Then $m\leqslant\frac{1-a_1}{2}$, since

$$1=D\cdot C_1=\big(a_1C_1+\Delta\big)\cdot C\geqslant a_1+2m.$$

Let $g\colon\widehat{S}\to S$ be the blow up of the point P, let E be the exceptional curve of blow up g, let $\widehat{C}_1$ and $\widehat{\Delta}$ be the proper transforms of C_1 and Δ on the surface $\widehat{S}$, respectively. Then

$$\boxed{\left(\widehat{S},\frac{3a_1}{2}\widehat{C}+\frac{3}{2}\widehat{\Delta}+\Big(3a_1+\frac{3}{2}m-1\Big)E\right)}$$

is not log canonical at some point $Q\in E$. Note that $3a_1+\frac{3}{2}m-1<1$. But

$$E\cdot\widehat{\Delta}=m\leqslant\frac{1-a_1}{2}\leqslant\frac{1}{2}.$$

Thus, we have $Q\in E\cap\widehat{C}$ by Theorem 6. On the other hand, we have

$$\boxed{\mathrm{mult}_Q\left(\frac{3}{2}\widehat{\Delta}+\Big(3a_1+\frac{3}{2}m-1\Big)E\right)\leqslant 3(a_1+m)-1\leqslant 1}$$

so that we can apply Theorem 10 to our log pair at Q. This gives

$$\frac{9}{2}a_1-\frac{1}{2}=\widehat{C}\cdot\left(\frac{3}{2}\widehat{\Delta}+\Big(3a_1+\frac{3}{2}m-1\Big)E\right)>2-\frac{3}{2}a_1,$$

so that $a_1>\frac{5}{12}$, which is a contradiction.

Exercise 12 ([17, § 4.1]). Let S be a smooth cubic surface in $\mathbb{P}^3$, let $g\colon \widehat{S} \to S$ be the blow up of the point P, and let $\widehat{D}$ be the proper transform of the divisor D on the surface $\widehat{S}$. Suppose that P is an Eckardt point of the surface S, and C_1, C_2 and C_3 are the lines in the surface S that passes through P. Suppose that $D \sim_{\mathbb{Q}} -K_S$, and

$$\boxed{\mathrm{mult}_P\big(D\big) + \mathrm{mult}_Q\big(\widehat{D}\big) \leqslant \frac{17}{9}}$$

for every point Q in the g-exceptional curve. Suppose also that $a_1 \leqslant \frac{5}{9}$, $a_2 \leqslant \frac{5}{9}$ and $a_3 \leqslant \frac{5}{9}$. Prove that the log pair $(S, \frac{6}{5}D)$ is log canonical at P (cf. Exercise 5).

Some problems requires very special analogues of Theorems 6 and 10.

Exercise 13 ([8, Lemma 25]). Let $m = \mathrm{mult}_P(\Delta)$, and let $x \in \mathbb{Q}$ such that $0 \leqslant x \leqslant 1$. Suppose that $P \in C_1$, the curve C_1 is smooth at P, $a_1 \leqslant \frac{1}{3} + \frac{x}{2}$ and $m \leqslant 1 + \frac{x}{2} - a_1$. Suppose also that

$$\boxed{\Big(C_1 \cdot \Delta\Big)_P \leqslant 1 - \frac{x}{2} + a_1}$$

so that $m \leqslant 1 - \frac{x}{2} + a_1$. Show that (S, D) is log canonical at P.

In many generalizations of Theorems 6 and 10, we have to deal with two special curves among $C_1, \ldots, C_r$. Because of this, we assume that $r \geqslant 3$, and we let

$$\boxed{\Omega = \sum_{i=3}^{r} a_i C_i}$$

so that $D = a_1C_1 + a_2C_2 + \Omega$.

Theorem 14 ([9, Corollary 1.29]). Suppose that $P \in C_1 \cap C_2$, the curves C_1 and C_2 are smooth at the point P, and they intersect transversally at P. Suppose also that

$$\boxed{\frac{2m-2}{m+1}a_1 + \frac{2}{m+1}a_2 \leqslant 1}$$

for some integer $m \geqslant 3$. If

$$\boxed{\Big(C_1 \cdot \Omega\Big)_P \leqslant 2a_1 - a_2}$$

and

$$\boxed{\Big(C_2 \cdot \Omega\Big)_P \leqslant \frac{m}{m-1}a_2 - a_1}$$

then (S, D) is log canonical at P.

Let us show how to apply this result.

Example 15 ([9, Theorem 4.1]). Let $\overline{S}$ be a hypersurface of degree 6 in $\mathbb{P}(1,1,2,3)$ that has Du Val singularities, and let $\nu\colon S \to \overline{S}$ be it minimal resolution of singularities. Suppose that $D \sim_{\mathbb{Q}} -K_S$, and $\nu(P)$ is a Du Val singular point of the surface $\overline{S}$ of type A_3. Then (S, D) is log canonical at P. Indeed, suppose that (S, D) is not log canonical at P. Let us seek for a contradiction. We may assume that C_1, C_2 and C_3 are ν-exceptional curves with the following intersection form

$\bullet$	C_1	C_2	C_3
C_1	-2	1	0
C_2	1	-2	1
C_3	0	1	-2

Write $\Lambda = a_3C_3 + \cdots + a_rC_r$. Then $D\cdot C_1 = D\cdot C_2 = D\cdot C_3 = 0$, so that

$$\boxed{\begin{cases} 2a_1 - a_2 = \Lambda\cdot C_1 \geqslant 0\\ 2a_2 - a_1 - a_3 = \Lambda\cdot C_1 \geqslant 0\\ 2a_3 - a_2 = \Lambda\cdot C_1 \geqslant 0\end{cases}}$$

which implies that $a_1>0$, $a_2>0$ and $a_3>0$. Let $\overline{Z}$ be the unique curve in $|-K_{\overline{S}}|$ that passes through $\nu(P)$, and let Z be its proper transform on the surface S. Then

$$\boxed{Z + C_1 + C_2 + C_3 \sim -K_S}$$

and Z is irreducible. Arguing as in Example 3, we may assume that $Z \not\subseteq \mathrm{Supp}(\Lambda)$. Then

$$0 \leqslant Z\cdot\Lambda = 1 - a_1 - a_2,$$

so that $\boxed{a_1 + a_2 \leqslant 1}$ which (together with previous inequalities for a_1, a_2 and a_3) gives

$$\begin{cases} a_1 \leqslant \dfrac{3}{4},\\ a_2 \leqslant 1,\\ a_3 \leqslant \dfrac{3}{4}.\end{cases}$$

If $P \in C_1 \setminus C_2$, then Theorem 6 gives

$$2a_1 - a_2 = \Lambda\cdot C_1 > 1,$$

which gives a contradiction with previously obtained inequalities, so that $P \notin C_1\setminus C_2$. Similarly, we see that $P\notin C_3\setminus C_2$. Using the same approach, we obtain $P\notin C_2\setminus(C_1\cup C_2)$. Thus, without loss of generality, we may assume that $P = C_1\cap C_2$. Then

$$\boxed{\Big(\Lambda\cdot C_1\Big)_P \leqslant \Lambda\cdot C_1 = 2a_1 - a_2}$$

and $a_1 + \frac{1}{2}a_2 \leqslant 1$. Thus, applying Theorem 14 with $m=3$, we get

$$\boxed{2a_2 - a_1 - a_3 = \Lambda\cdot C_2 \geqslant \Big(\Lambda\cdot C_2\Big)_P > \frac{3}{2}a_2 - a_1}$$

which immediately leads to a contradiction.

Exercise 16 ([9, Lemma 4.6])**.** Let $\overline{S}$ be a surface of degree 6 in $\mathbb{P}(1,1,2,3)$ that has one singular point, which is a singular point of type A_4, and let $\nu\colon S\to\overline{S}$ be the minimal resolution of singularities. Suppose that $D\sim_{\mathbb{Q}} -K_S$. Show that $(S,\frac{4}{5})$ is log canonical at P.

Theorem 14 has very limited application scope. It can be generalized as follows:

Theorem 17 ([9, Theorem 1.28])**.** Suppose that $P \in C_1 \cap C_2$, the curves C_1 and C_2 are smooth at the point P, and they intersect transversally at P. Suppose also that there are non-negative rational numbers α, β, A, B, M, and N such that

$$\begin{cases} \alpha a_1 + \beta a_2 \leqslant 1, \\ A(B-1) \geqslant 1, \\ \alpha(A+M-1) \geqslant A^2(B+N-1)\beta, \\ \alpha(1-M) + A\beta \geqslant A, \\ M \leqslant 1, \\ N \leqslant 1. \end{cases}$$

Suppose that $2M + AN \leqslant 2$ or $\alpha(B+1-MB-N) + \beta(A+1-AN-M) \geqslant AB-1$. If

$$\boxed{\Big(C_1 \cdot \Omega\Big)_P \leqslant M + Aa_1 - a_2}$$

and

$$\boxed{\Big(C_1 \cdot \Omega\Big)_P \leqslant N + Ba_2 - a_1}$$

then the log pair (S, D) is log canonical at P.

This result has more applications than Theorem 14 (see [5] for examples).

Example 18 ([12, Lemma 4.9])**.** Let S be a smooth cubic surface in $\mathbb{P}^3$, and let T_P be its plane section that is singular at the point P. Suppose that $T_P = L + C$, where L is a line, and C is a conic that intersects L transversally at P. Suppose also that $D \sim_{\mathbb{Q}} -K_S$. Then (S, D) is log canonical at P. Indeed, suppose that (S, D) is not log canonical at P. Let us seek for a contradiction. Arguing as in Example 3, we may assume that either the line L or the conic C (or both) is not contained in $\mathrm{Supp}(D)$. If $L \not\subseteq \mathrm{Supp}(D)$, then

$$1 = L \cdot D \geqslant \Big(L \cdot D\Big)_P \geqslant \mathrm{mult}_P(D) > 1,$$

which is a contradiction. So, we see that L is contained in $\mathrm{Supp}(D)$, so that $C \not\subseteq \mathrm{Supp}(D)$. We may assume that $C_1 = L$. Let $m = \mathrm{mult}_P(\Delta)$. Then

$$2 = C \cdot D = 2a_1 + \Delta \cdot C \geqslant 2a_1 + m.$$

Let $g\colon \widehat{S} \to S$ be the blow up of the point P, let E be the g-exceptional curve, let $\widehat{C}_1$ be the proper transform on $\widehat{S}$ of the line C_1, and let $\widehat{\Delta}$ be the proper transform of the divisor Δ. Then $(\widehat{S}, a_1\widehat{C}_1 + (a_1 + m - 1)E + \widehat{\Delta})$ is not log canonical at some point $Q \in E$, since

$$K_{\widehat{S}} + a_1\widehat{C}_1 + \widehat{\Delta} + \big(a_1 + m - 1\big)E \sim_{\mathbb{Q}} g^*\big(K_S + D\big).$$

Moreover, it follows from Example 7 that $Q = \widehat{C}_1 \cap E$. Now, applying Theorem 17 with

$$\begin{cases} M = 1, \\ N = 0, \\ A = 1, \\ B = 2, \\ \alpha = 1, \\ \beta = 1, \end{cases}$$

we get

$$\boxed{m = \widehat{\Delta}\cdot E \geqslant \Big(\widehat{\Delta}\cdot E\Big)_P > 1 + (a_1 + m - 1) - a_1 = m}$$

or

$$\boxed{1 + a_1 - m = \widehat{\Delta}\cdot \widehat{C}_1 \geqslant \Big(\widehat{\Delta}\cdot \widehat{C}_1\Big) > 2a_1 - (a_1 + m - 1) = 1 + a_1 - m}$$

where we used E and $\widehat{C}_1$ as the curves C_1 and C_2 in Theorem 17, respectively.

The assertion proved in Example 18 can be reproved using the following result:

Theorem 19 ([6, Theorem 13]). Suppose that $P \in C_1 \cap C_2$, the curves C_1 and C_2 are smooth at P, and they intersect transversally at P. Suppose that $\mathrm{mult}_P\big(\Omega\big) \leqslant 1$. If

$$\boxed{\Big(C_1\cdot\Omega\Big)_P \leqslant 2\big(1 - a_2\big)}$$

and

$$\boxed{\Big(C_2\cdot\Omega\Big)_P \leqslant 2\big(1 - a_1\big)}$$

then the log pair (S, D) is log canonical at P.

Let us show how to apply Theorem 19.

Example 20 ([12, Lemma 4.8]). Let S be a smooth cubic surface in $\mathbb{P}^3$, and let T_P be its plane section that is singular at the point P. Suppose that $D \sim_{\mathbb{Q}} -K_S$, and

$$\boxed{T_P = L_1 + L_2 + L_3}$$

where L_1, L_2, L_3 are lines such that $L_1 \cap L_2 = P \notin L_3$. Then (S, D) is log canonical at P. Indeed, suppose that (S, D) is not log canonical at P. Let us seek for a contradiction. Arguing as in Example 3, we may assume that one line among L_1, L_2 and L_3 is not contained in the support of the divisor D. If $L_1 \not\subseteq \mathrm{Supp}(D)$, then

$$1 = L_1\cdot D \geqslant \mathrm{mult}_P\big(L_1\big)\mathrm{mult}_P\big(D\big) = \mathrm{mult}_P\big(D\big) > 1,$$

which is absurd. This shows that $L_1 \subseteq \mathrm{Supp}(D)$. Similarly, we see that $L_2 \subseteq \mathrm{Supp}(D)$. Thus, the line L_3 is not contained in $\mathrm{Supp}(D)$. We may assume that $C_1 = L_1$ and $C_2 = L_2$. Then $\boxed{a_1 + a_2 \leqslant 1}$ since

$$1 = L_3\cdot D = L_3\cdot\Big(a_1C_1 + a_2C_2 + \Omega\Big) = a_1 + a_2 + L_3\cdot\Omega \geqslant a_1 + a_2.$$

Put $m = \mathrm{mult}_P(\Omega)$. Then

$$\begin{cases} 1 = C_1\cdot\Big(a_1C_1 + a_2C_2 + \Omega\Big) = -a_1 + a_2 + C_1\cdot\Omega \geqslant -a_1 + a_2 + m, \\ 1 = C_2\cdot\Big(a_1C_1 + a_2C_2 + \Omega\Big) = a_1 - a_2 + C_2\cdot\Omega \geqslant a_1 - a_2 + m, \end{cases}$$

which implies that $m \leqslant 1$. Thus, we can apply Theorem 19 to (S, D). This gives

$$\boxed{1 + a_1 - a_2 = C_1\cdot\Omega \geqslant \Big(C_1\cdot\Omega\Big) > 2\big(1 - a_2\big)}$$

or

$$\boxed{1 - a_1 + a_2 = C_2\cdot\Omega \geqslant \Big(C_2\cdot\Omega\Big) > 2\big(1 - a_1\big)}$$

so that $a_1 + a_2 > 1$. But we proved already that $a_1 + a_2 \leqslant 1$.

Exercise 21 ([12, Corollary 1.13]). Let S be a smooth cubic surface in $\mathbb{P}^3$, and let T_P be the plane section of the surface S that is singular at the point P. Suppose that $D \sim_{\mathbb{Q}} -K_S$, and at least one irreducible component of the cubic curve T_P is not contained in $\mathrm{Supp}(D)$. Show that (S, D) is log canonical at the point P.

Exercise 22 ([11, Theorem 4.1]). Let S be a smooth surface in $\mathbb{P}(1,1,1,2)$ of degree 4. Suppose that D is ample. Let

$$\boxed{\lambda = \frac{2}{3}\frac{(-K_S \cdot D)}{D^2}}$$

and suppose that $-K_S - \lambda D$ is nef. Prove that $(S, \lambda D)$ is log canonical at P.

Exercise 23 ([1, Theorem 1.2]). Let S be a smooth hypersurface in $\mathbb{P}^3$ of degree 4, let T_P be its hyperplane section that is singular at P, let

$$\boxed{\lambda = \sup\Big\{\mu \in \mathbb{Q} \ \Big|\ \text{the log pair } \big(S, \mu T_P\big) \text{ is log canonical at } P\Big\}}$$

and suppose that $D \sim_{\mathbb{Q}} T_P$. Prove that $(S, \lambda D)$ is log canonical at P.

Exercise 24 ([2, Theorem 4.1]). Let S be a smooth hypersurface in $\mathbb{P}^3$ of degree $d \geqslant 3$, and let T_P be its plane section that is singular at the point P. Suppose that $D \sim_{\mathbb{Q}} T_P$. Let $\lambda = \frac{2}{d}$. Prove that $(S, \lambda D)$ is log canonical at P.

Theorem 19 has other applications (see [13, 7]). Let us present one of them.

Example 25. Let $\mathcal{C}$ be a smooth curve of bi-degree $(1,2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ that is given by

$$\boxed{x_1 x_2^2 = y_1 y_2^2}$$

and let L_λ be the curve given by $x_1 = \lambda y_1$, so that L_∞ is the curve given by $y_1 = 0$. Let $\pi\colon S \to \mathbb{P}^1 \times \mathbb{P}^1$ be blow up of $k \geqslant 4$ points in the curve $\mathcal{C}$ such that no two of them are contained in one curve L_λ for every $\lambda \in \mathbb{C} \cup \{\infty\}$. Suppose that C_1 is the proper transform of a curve L_μ for some $\mu \in \mathbb{C} \setminus 0$ such that L_μ contain no points blown up by π, and C_2 is the proper transform of the curve $\mathcal{C}$, and $P = C_1 \cap C_2$. Suppose that

$$\boxed{1 > a_2 \gg \frac{1}{2} \geqslant a_1 \geqslant 0}$$

and $D \sim_{\mathbb{Q}} -K_S$. Then the log pair (S, D) is log canonical at P. Suppose that it is not. Let us seek for a contradiction. Let $\beta = 1 - a_2$ and $m = \mathrm{mult}_P(\Omega)$. Then $a_1 + m - \beta < 1$, because

$$1 \gg 2\beta = C_1 \cdot \Omega \geqslant \Big(C_1 \cdot \Omega\Big)_P \geqslant m.$$

Let $g\colon \widehat{S} \to S$ be the blow up of the point P, let E be the g-exceptional curve, let $\widehat{C}_1$ be the proper transform of the curve C_1 via g, let $\widehat{C}_2$ the proper transform of the curve C_2, and let $\widehat{\Omega}$ be the proper transform of the divisor Ω. Then the log pair

$$\boxed{\Big(\widehat{S}, a_1\widehat{C}_1 + (1-\beta)\widehat{C}_2 + (a_1 + m - \beta)E + \widehat{\Omega}\Big)}$$

is not log canonical at some point $Q \in E$. We have $Q \in \widehat{C}_1 \cup \widehat{C}_2$, since otherwise

$$2\beta \geqslant m = E \cdot \widehat{\Omega} \geqslant \Big(E \cdot \widehat{\Omega}\Big)_Q > 1$$

by Theorem 6. Similarly, we have $Q \in \widehat{C}_2$, since otherwise Theorem 6 gives

$$1 \gg 2\beta \geqslant E \cdot \widehat{\Omega} \geqslant \Big(E \cdot \widehat{\Omega}\Big)_Q > 1 - a_1 \geqslant \frac{1}{2}.$$

Since $\mathrm{mult}_Q(\widehat{\Omega}) \leqslant m < 1$, we can apply Theorem 19. Either it gives

$$\boxed{2\beta \geqslant m = E \cdot \widehat{\Omega} > 2\beta}$$

or it gives

$$\boxed{2 - \beta(k-4) - 2a_1 - m = \widehat{C} \cdot \widehat{\Omega} > 2(1 - a_1 - m + \beta)}$$

so that $2\beta \geqslant m > (k-2)\beta$. In both cases, we obtain a contradiction.

Finally, let us present two local inequalities discovered in [15].

Theorem 26 ([15, Theorem 3.4])**.** Suppose that $P \in C_1 \cap C_2$, the curves C_1 and C_2 are smooth at the point P, and they intersect transversally at P. Let $m = \mathrm{mult}_P(\Omega)$. Suppose that $a_1 \leqslant 1$, $a_2 \leqslant 1$ and $m \leqslant 1$. Suppose that

$$\boxed{\Big(C_1 \cdot \Omega\Big)_P \leqslant 2 - a_1 - a_2}$$

in the case when $a_1 + m > 1$. Suppose also that

$$\boxed{\Big(C_2 \cdot \Omega\Big)_P \leqslant \frac{m}{a_2 + m - 1}(1 - a_1) - 1 + a_2}$$

in the case when $a_2 + m > 1$. Then (S, D) is log canonical at P.

Proof. By Theorem 6, we may assume that $a_2 < 1$. We will use an inductive argument. Let $g\colon \widehat{S} \to S$ be the blow up of the point P, let E be the g-exceptional curve, let $\widehat{C}_1$ be the proper transform of the curve C_1 via g, let $\widehat{C}_2$ the proper transform of the curve C_2, and let $\widehat{\Omega}$ be the proper transform of the divisor Ω. Then the log pair

$$\boxed{\Big(\widehat{S}, a_1\widehat{C}_1 + a_2\widehat{C}_2 + (a_1 + a_2 + m - 1)E + \widehat{\Omega}\Big)}$$

is not log canonical at some point $Q \in E$. If $a_1 + m \leqslant 1$, then $a_1 + a_2 + m - 1 \leqslant a_2 \leqslant 1$. Similarly, if $a_1 + m > 1$, then

$$a_1 + a_2 + m - 1 \leqslant a_1 + a_2 + \Big(C_1 \cdot \Omega\Big)_P - 1 \leqslant 1.$$

Observe also that $\widehat{C}_1 \cap \widehat{C}_1 \cap E = \varnothing$.

Arguing as in Example 25, we see that $Q = \widehat{C}_1 \cap E$ or $Q = \widehat{C}_2 \cap E$, because $m \leqslant 1$. Moreover, if $Q = \widehat{C}_1 \cap E$, then applying Theorem 6 twice, we see that

$$1 < \Big(\widehat{C}_1 \cdot \big((a_1 + a_2 + m - 1)E + \widehat{\Omega}\big)\Big)_Q = a_1 + a_2 + m - 1 + \Big(\widehat{C}_1 \cdot \widehat{\Omega}\Big)_Q = a_1 + a_2 - 1 + \Big(C_1 \cdot \Omega\Big)_Q$$

and

$$1 < \Big(E \cdot \big(a_1\widehat{C}_1 + \widehat{\Omega}\big)\Big)_Q = a_1 + \Big(E \cdot \widehat{\Omega}\Big)_Q = a_1 + m,$$

which contradicts our assumptions. This shows that $Q = \widehat{C}_2 \cap E$. Thus, the log pair

$$\boxed{\Big(\widehat{S}, a_2\widehat{C}_2 + (a_1 + a_2 + m - 1)E + \widehat{\Omega}\Big)}$$

is not log canonical at the point Q. Applying Theorem 6 to this log pair, we obtain

$$1 < \Big(E\cdot\big(a_2\widehat{C}_2+\widehat{\Omega}\big)\Big)_Q = a_2+\Big(E\cdot\widehat{\Omega}\Big)_Q = a_2+m.$$

Then, by our assumption, we have

$$\boxed{\Big(C_2\cdot\Omega\Big)_P \leqslant \frac{m}{a_2+m-1}(1-a_1)-1+a_2}$$

since we just proved that $a_2+m>1$.

Set $\widehat{a}_1 = a_1+a_2+m-1$. Now we are going to apply our theorem to the log pair

$$\Big(\widehat{S}, (a_1+a_2+m-1)E+a_2\widehat{C}_2+\widehat{\Omega}\Big).$$

Here, we use induction on the number of blow ups used in the definition of log canonicity. Of course, we have to check that the new log pair satisfies all the condition of our theorem.

We know that $\widehat{a}_1\leqslant 1$. Set $\widehat{m}=\mathrm{mult}_Q(\widehat{\Omega})$. Then $\widehat{m}\leqslant m\leqslant 1$. We have to check that

$$\boxed{\Big(E\cdot\widehat{\Omega}\Big)_Q \leqslant 2-\widehat{a}_1-a_2}$$

in the case when $\widehat{a}_1+\widehat{m}>1$, and we have to check that

$$\boxed{\Big(\widehat{C}_2\cdot\widehat{\Omega}\Big)_Q \leqslant \frac{\widehat{m}}{a_2+\widehat{m}-1}(1-\widehat{a})-1+a_2}$$

in the case when $a_2+\widehat{m}>1$.

Suppose that $\widehat{a}_1+\widehat{m}>1$ and $(E\cdot\widehat{\Omega})_Q > 2-\widehat{a}_1-a_2$. Let us seek for a contradiction. We have

$$a_1+2(a_2+m-1)+1 = a_1+2a_2+2m-1=\widehat{a}_1+a_2+m\geqslant \widehat{a}_1+a_2+(E\cdot\widehat{\Omega})_Q>2.$$

so that

$$\boxed{2>\frac{1-a_1}{a_2+m-1}}$$

since we proved earlier that $a_2+m>1$. Similarly, we have

$$\widehat{a}_1-m+\Big(C_2\cdot\Omega\Big)_P\geqslant \widehat{a}_1+\widehat{m}>1.$$

because $(C_2\cdot\Omega)_P\geqslant m+\widehat{m}$. Since $\widehat{a}_1=a_1+a_2+m-1$, this gives

$$a_1+a_2+\Big(C_2\cdot\Omega\Big)_P>2,$$

which can be rewritten as

$$2(1-a_2)=2-2a_2<\Big(C_2\cdot\Omega\Big)_P+a_1-a_2=\Big(C_2\cdot\Omega\Big)_P-(1-a_1)+1-a_2.$$

Thus, we have

$$\boxed{2<\frac{\Big(C_2\cdot\Omega\Big)_P-(1-a_1)+1-a_2}{1-a_2}}$$

which immediately gives

$$2<\frac{\frac{m}{a_2+m-1}(1-a_1)-1+a_2-(1-a_1)+1-a_2}{1-a_2}=\frac{1-a_1}{a_2+m-1},$$

because $(C_2 \cdot \Omega)_P \leqslant \frac{m}{a_2+m-1}(1-a_1)-1+a_2$. Hence, summarizing, we get

$$\frac{1-a_1}{a_2+m-1} > 2 > \frac{1-a_1}{a_2+m-1},$$

which is absurd. This shows that $(E \cdot \widehat{\Omega})_Q \leqslant 2-\widehat{a}_1-a_2$ if $\widehat{a}_1+\widehat{m}>1$.

Now we suppose that $a_2+\widehat{m}>1$ and we suppose that $(\widehat{C}_2 \cdot \widehat{\Omega})_Q > \frac{\widehat{m}}{a_2+\widehat{m}-1}(1-\widehat{a})+a_2-1$. Let us seek for a contradiction. We have

$$\frac{m}{a_2+m-1}(1-a_1)-1+a_2 \geqslant \Big(C_2 \cdot \Omega\Big)_P > \Big(\widehat{C}_2 \cdot \widehat{\Omega}\Big)_Q + m >$$
$$> \frac{\widehat{m}}{a_2+\widehat{m}-1}(1-\widehat{a})+a_2-1+m \geqslant \frac{m}{a_2+m-1}(1-\widehat{a})+a_2-1+m,$$

since $m \geqslant \widehat{m}$. This gives

$$m = \frac{m}{a_2+m-1}(a_2+m-1) = \frac{m}{a_2+m-1}(\widehat{a}-a_1) > m,$$

which is absurd. This shows that $(\widehat{C}_2 \cdot \widehat{\Omega})_Q \leqslant \frac{\widehat{m}}{a_2+\widehat{m}-1}(1-\widehat{a})+a_2-1$ if $a_2+\widehat{m}>1$.

Thus, we can apply our theorem to the pair $(\widehat{S}, (a_1+a_2+m-1)E+a_2\widehat{C}_2+\widehat{\Omega})$ at Q, which gives us a contradiction, since this pair is not log canonical at Q. □

Corollary 27 ([15, Corollary 3.5])**.** Suppose that $P \in C_1 \cap C_2$, the curves C_1 and C_2 are smooth at the point P, and they intersect transversally at the point P. Let $m = \mathrm{mult}_P(\Omega)$. Suppose that $a_1 \leqslant 1$, $a_2 \leqslant 1$ and $m \leqslant 1$. If

$$\boxed{\Big(C_1 \cdot \Omega\Big)_P \leqslant 1-a_2}$$

or

$$\boxed{\Big(C_2 \cdot \Omega\Big)_Q \leqslant \frac{\Big(C_1 \cdot \Omega\Big)_P}{\Big(C_1 \cdot \Omega\Big)_P - 1 + a_2}(1-a_1)-1+a_2}$$

then the log pair (S, D) is log canonical at the point P.

Proof. Suppose that (S, D) is not log canonical at P. Then $(C_1 \cdot \Omega)_P > 1-a_2$ by Theorem 6. Thus, to complete the proof, we may assume that

$$\Big(C_2 \cdot \Omega\Big)_Q \leqslant \frac{\Big(C_1 \cdot \Omega\Big)_P}{\Big(C_1 \cdot \Omega\Big)_P - 1 + a_2}(1-a_1)-1+a_2.$$

Let us seek for a contradiction. If $m > 1-a_2$, then

$$\Big(C_2 \cdot \Omega\Big)_Q \leqslant \frac{m}{m-1+a_2}(1-a_1)-1+a_2,$$

since $m \leqslant (C_1 \cdot \Omega)_Q$. Then $(C_1 \cdot \Omega)_P > 2-a_1-a_2$ and $a_1+m>1$ by Theorem 26. But

$$m \leqslant \Big(C_2 \cdot \Omega\Big)_P \leqslant \frac{\Big(C_1 \cdot \Omega\Big)_P}{\Big(C_1 \cdot \Omega\Big)_P - 1 + a_2}(1-a_1)-1+a_2 = a_2-a_1+\frac{(1-a_1)(1-a_2)}{\Big(C_1 \cdot \Omega\Big)_P - 1 + a_2},$$

which implies that

$$\boxed{1 < a_1 + m \leqslant 1 - (1 - a_2)\frac{a_1 + a_2 + \Big(C_1 \cdot \Omega\Big)_P}{\Big(C \cdot \Omega\Big)_P - 1 + a_2} \leqslant 1}$$

which is absurd. □

Exercise 28 ([15])**.** Let us use assumptions and notation of Example 25. Suppose that

$$\boxed{a_1 \leqslant \frac{1}{2} - \frac{\beta(r-4)}{8}}$$

and $r \geqslant 7$. Let $\lambda = 1 + \frac{\beta}{100}$. Prove that $(S, a_1C_1 + \lambda a_2 C_2 + \lambda\Omega)$ is log canonical at P.

Exercise 29 ([14, Proposition 7.6])**.** Suppose that S is a smooth cubic surface in $\mathbb{P}^3$, and P is its Eckardt point. Suppose also that C_1 is a line in S that contains P, and

$$D = a_1C_1 + \Delta \sim_{\mathbb{Q}} -K_S.$$

Let $\lambda = \frac{2-a_1}{3-a_1}$. Suppose that $a_1 \leqslant 1$. Prove that $(S, a_1C_1 + \lambda\Delta)$ is log canonical at P.

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47/3 Warrender Park Road
Edinburgh EH9 1EU, Scotland

I.Cheltsov@ed.ac.uk