$\mathbb{A}^1$-CYLINDERS OVER SMOOTH $\mathbb{A}^1$-FIBERED AFFINE SURFACES

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ABSTRACT. We give a general structure theorem for affine $\mathbb{A}^1$-fibrations on smooth quasi-projective surfaces. As an application, we show that every smooth $\mathbb{A}^1$-fibered affine surface non-isomorphic to the total space of a line bundle over a smooth affine curve fails the Zariski Cancellation Problem. The present note is an expanded version of a talk given at the Kinosaki Algebraic Geometry Symposium in October 2019.

INTRODUCTION

The Zariski Cancellation Problem asks under which circumstances the existence of a biregular isomorphism between the cartesian products $X \times \mathbb{A}^n$ and $Y \times \mathbb{A}^n$ of two algebraic varieties X and Y with the affine space $\mathbb{A}^n$, say over a an algebraically closed field of characteristic zero, implies that the varieties X and Y are isomorphic. Cancellation is known to hold for smooth curves [1] and for a large class of algebraic varieties characterized roughly by the property that they are not dominantly covered by images of the affine line $\mathbb{A}^1$ (see e.g. [18], [5]). A specific stronger characterization due to Makar-Limanov [2] asserts that if X is an affine variety which does not admit any effective algebraic action of the additive group $\mathbb{G}_a$ then every isomorphism $X \times \mathbb{A}^1 \simeq Y \times \mathbb{A}^1$ induces an isomorphism $X \simeq Y$ (this is no longer true for products with affine spaces $\mathbb{A}^n$ of higher dimension, see e.g. [9, 10]).

Among smooth affine surfaces with an effective action of the additive group $\mathbb{G}_a$, cancellation is known to hold for the affine plane $\mathbb{A}^2$ by [23, 25]. The first celebrated examples of smooth affine surfaces with effective $\mathbb{G}_a$-actions which fail cancellation were constructed by Danielewski [4]: he established that the smooth surfaces S_n in $\mathbb{A}^3$ defined by the equations $x^n z = y(y - 1)$, where $n \geq 1$, are pairwise non-isomorphic but that their $\mathbb{A}^1$-cylinders $S_n \times \mathbb{A}^1$ are all isomorphic. Since then, many other families of examples of smooth affine surfaces with effective $\mathbb{G}_a$-actions which fail cancellation have been constructed (see e.g. [26, 13] and the references therein for a survey). All these constructions are derived from variants of the nowadays called "Danielewski fiber product trick", which depends on the study of the structure of the algebraic quotient morphisms of $\mathbb{G}_a$-actions on affine surfaces. These quotient morphisms are surjections $\pi : S \to C$ onto smooth affine curves, with generic fiber isomorphic to the affine line $\mathbb{A}^1$ over the function field of C, called $\mathbb{A}^1$-*fibrations*. The local structure of these fibrations in a neighborhood of their degenerate fibers has been studied by many authors after the pioneering work of Miyanishi [24] and Fieseler [12]. The first result presented in this note is a general structure theorem for affine $\mathbb{A}^1$-fibrations on smooth quasi-projective surfaces which generalizes and encompasses, in a different language, all formerly known descriptions:

Theorem. *Let S be a smooth quasi-projective surface and let $\pi : S \to C$ be an affine $\mathbb{A}^1$-fibration over a smooth algebraic curve C. Then there exists a smooth algebraic space $\mathcal{C}$ of dimension 1 endowed with a surjective quasi-finite birational morphism of finite type $\alpha : \mathcal{C} \to C$ and a factorization*

$$\pi = \alpha \circ \rho : S \xrightarrow{\rho} \mathcal{C} \xrightarrow{\alpha} C$$

where $\rho : S \to \mathcal{C}$ is an étale locally trivial $\mathbb{A}^1$-bundle.

In the case of a smooth $\mathbb{A}^1$-fibration $\pi : S \to C$ on a smooth affine surface S, the above result was already established in [12] and [7] (see also [13]) with the additional observation that in this particular case, the algebraic space curve $\alpha : \mathcal{C} \to C$ is a smooth scheme, in general not separated. We will see below that for non-smooth $\mathbb{A}^1$-fibrations, the existence of multiple fibers forces to consider algebraic space curves $\alpha : \mathcal{C} \to C$ which are not schemes. This fact was already observed in [9, 10] where the existence of a factorization as above was established for some particular examples of $\mathbb{A}^1$-fibrations with multiple fibers.

Our second result is an application of the above structure theorem to the construction of smooth affine surfaces which fail cancellation. By applying a new variant of the Danielwski fiber product trick construction, we obtain

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the following characterization which basically fully settles the Zariski Cancellation Problem for smooth affine surfaces:

Theorem. *Let S be a smooth affine surface and let $\pi : S \to C$ be an $\mathbb{A}^1$-fibration over a smooth affine curve C. Then the following alternative holds:*

a) If $\pi : S \to C$ is isomorphic to the structure morphism of a line bundle over C then every smooth affine surface S' such that $S' \times \mathbb{A}^1 \simeq S \times \mathbb{A}^1$ is isomorphic to S.

b) Otherwise, there exists a smooth affine $\mathbb{A}^1$-fibered surface S' non-isomorphic to S such that $S \times \mathbb{A}^1$ is isomorphic to $S' \times \mathbb{A}^1$.

The characterization in the above theorem strongly overlaps similar results established in [13] for smooth surfaces which admit $\mathbb{A}^1$-fibrations with reduced fibers. We also refer the reader to a forthcoming article in collaboration S. Kaliman and M. Zaidenberg in which a similar result is established by different methods.

In this note, all schemes and algebraic spaces are assumed to be defined for simplicity over the field of complex numbers $\mathbb{C}$. We refer the reader to [20] for the basic properties of algebraic spaces which are used throughout the text. Some of the results in this note are given without complete and detailed proofs, these will appear elsewhere.

1. SMOOTH $\mathbb{A}^1$-FIBERED SURFACES AS ÉTALE LOCALLY TRIVIAL $\mathbb{A}^1$-BUNDLES

Definition 1. An $\mathbb{A}^1$-*fibration* on a smooth quasi-projective surface S is a surjective affine morphism $\pi : S \to C$ to a smooth algebraic curve C, whose fiber over the generic point of C is isomorphic to the affine line $\mathbb{A}^1_{K(C)}$ over the function field $K(C)$ of C.

An $\mathbb{A}^1$-fibration on a smooth quasi-projective surface S is said to be of *affine type* (resp. *complete type*) if the curve C is affine (resp. complete).

Example 2. Given a smooth algebraic curve C, a $\mathbb{P}^1$-bundle $\overline{\pi} : \mathbb{P}(E) \to C$ for some vector bundle E of rank 2 on C and a section $\sigma : C \to \mathbb{P}(E)$ of $\overline{\pi}$, the restriction $\pi : S = \mathbb{P}(E) \setminus \sigma(C) \to C$ of $\overline{\pi}$ to the complement of $\sigma(C)$ is an $\mathbb{A}^1$-fibration which is a *Zariski locally trivial $\mathbb{A}^1$-bundle* over C, that is, there exists a covering of C by Zariski open susbets C_i, $i \in I$, and isomorphisms $\pi^{-1}(C_i) \simeq C_i \times \mathbb{A}^1$ of schemes over C_i for every $i \in I$.

If C is affine, such Zariski locally trivial $\mathbb{A}^1$-fibrations $\pi : S \to C$ are simply line bundles. This is no longer the case in general when C is complete. For instance, let $S \subset \mathbb{A}^3$ be the smooth affine quadric surface with equation $xz = y^2 - 1$. Then the morphism

$$\pi : S \to \mathbb{P}^1, \quad (x, y, z) \mapsto [x : y - 1] = [y + 1 : z]$$

is a Zariski locally trivial $\mathbb{A}^1$-bundle which cannot be a line bundle. Indeed, otherwise the zero section of this line bundle would be a complete curve contained in S, which is impossible as S is affine.

Proposition 3. [19, Lemma 1.3] *Let S be a smooth quasi-projective surface and let $\pi : S \to C$ be an $\mathbb{A}^1$-fibration onto a smooth curve C. Assume that all scheme-theoretic fibers of $\pi : S \to C$ are irreducible and reduced. Then $\pi : S \to C$ is Zariski locally trivial $\mathbb{A}^1$-bundle.*

Definition 4. Let $\pi : S \to C$ be an $\mathbb{A}^1$-fibration on a smooth quasi-projective surface S. A scheme-theoretic fiber of π over a closed point c of C which is either reducible or non-reduced is called *degenerate*.

By [24, Lemma 1.4.2], every degenerate fiber of an $\mathbb{A}^1$-fibration $\pi : S \to C$ is a disjoint union of curves isomorphic to the complex affine line $\mathbb{A}^1$ when endowed with respective reduced structures.

Example 5. Let $n \geq 2$, let $P(y) = \prod_{i=1}^{r}(y - y_i)^{m_i} \in \mathbb{C}[y]$ be a non-constant monic polynomial with $r \geq 1$ distinct roots $y_i \in \mathbb{C}$ of respective multiplicities $m_i \geq 1$ and let $S \subset \mathbb{A}^3$ be the affine surface defined by the equation $x^n z = P(y) - x$. Then S is smooth by the Jacobian criterion and the projection pr_x induces an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1$ which restricts to the trivial $\mathbb{A}^1$-bundle $\mathbb{A}^1 \setminus \{0\} \times \mathrm{Spec}(\mathbb{C}[y])$ over $\mathbb{A}^1 \setminus \{0\} = \mathrm{Spec}(\mathbb{C}[x^{\pm 1}])$. On the other hand, the scheme-theoretic fiber $\pi^{-1}(\{0\})$ decomposes as the disjoint union of the schemes

$$F_i = \mathrm{Spec}(\mathbb{C}[y]/((y - y_i)^{m_i}[z]), \quad i = 1, \ldots, r$$

whose reductions are all isomorphic to the affine line $\mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[z])$.

1.1. **The smooth relatively connected quotient of an $\mathbb{A}^1$-fibration.** In this subsection, we show that every $\mathbb{A}^1$-fibration $\pi : S \to C$ on a smooth quasi-projective surface S factors through a smooth morphism with connected fibers $\rho : S \to \mathcal{C}$ over a suitably defined algebraic space curve $\mathcal{C}$ over C.

Definition 6. Let C be a smooth algebraic curve. A *smooth multifold algebraic space C-curve* is a smooth algebraic space $\mathcal{C}$ of dimension 1 endowed with a surjective quasi-finite birational morphism of finite type $\alpha : \mathcal{C} \to C$ such that $\alpha_* \mathcal{O}_\mathcal{C} = \mathcal{O}_C$.

By generic smoothness, there exists a non empty maximal Zariski open subset U of C over which $\alpha : \mathcal{C} \to C$ restricts to an étale morphism $\alpha : \alpha^{-1}(U) \to U$. Given any separated open subset $\mathcal{V} \subset \alpha^{-1}(U)$, the restriction $\alpha|_\mathcal{V} : \mathcal{V} \to C$ is a separated birational quasi-finite étale morphism. Since a quasi-finite morphism is quasi-affine, $\mathcal{V}$ is thus a quasi-projective scheme and $\alpha|_\mathcal{V} : \mathcal{V} \to C$ is an open immersion by Zariski main theorem [15, Théorème 8.12.6]. It follows in particular that there exists finitely many points $c_1, \ldots, c_s$ of C over which $\alpha : \mathcal{C} \to C$ is not an isomorphism. Furthermore, for every $i = 1, \ldots, s$, the fiber $\alpha^{-1}(c_i)$ consists of finitely many points $\check{c}_{i,1}, \ldots, \check{c}_{i,r_i}$, and if α is unramified at $\check{c}_{i,j}$ then there exists a separated Zariski open neighborhood $\check{U}_{i,j}$ of $\check{c}_{i,j}$ in $\mathcal{C}$ such that $\alpha|_{\check{U}_{i,j}} : \check{U}_{i,j} \to C$ is an isomorphism onto its image. So one can picture a smooth multifold algebraic space C-curve $\alpha : \mathcal{C} \to C$ as being obtained from the curve C by "replacing" finitely many points $c_1, \ldots, c_s$ of C by a collection of finitely many distinct algebraic space curve points $\check{c}_{i,1}, \ldots, \check{c}_{i,r_i}$.

Example 7. Let C be smooth affine curve, let $c_0 \in C$ be a closed point and let $\mathcal{C}$ be the curve obtained by gluing $r \geq 2$ copies $\alpha_i : \mathcal{C}_i \xrightarrow{\cong} C$, $i = 1, \ldots, r$, of C by the identity outside the points $c_i = \alpha_i^{-1}(c_0)$. The curve $\mathcal{C}$ is a non-separated scheme on which the morphisms α_i glue to a surjective quasi-finite birational morphism $\alpha : \mathcal{C} \to C$ which coincides with the canonical affinization morphism $\mathcal{C} \to \mathrm{Spec}(\Gamma(\mathcal{C}, \mathcal{O}_\mathcal{C}))$. The restriction of α over $C \setminus \{c_0\}$ is an isomorphism whereas $\alpha^{-1}(c_0)$ consists of r distinct point c_i, $i = 1, \ldots, r$.

Example 8. Let C be a smooth algebraic curve, let $c_0 \in C$ be a closed point and let $\varphi : \tilde{C} \to C$ be a quasi-finite morphism of degree $m \geq 2$, with branching index m at c and étale elsewhere. Let $\tilde{c}_0 = \varphi^{-1}(c_0)$ and let $\mathcal{C}$ be the smooth algebraic space obtained from $\tilde{C}$ by identifying two points $\tilde{c}, \tilde{c}' \in \tilde{C} \setminus \{\tilde{c}_0\}$ if $\varphi(\tilde{c}) = \varphi(\tilde{c}')$. More rigorously, letting $C_* = C \setminus \{c\}$ and $\tilde{C}_* = \tilde{C} \setminus \{\tilde{c}_0\} = \varphi^{-1}(C_*)$, $\mathcal{C}$ is the quotient of $\tilde{C}$ by the étale equivalence relation

$$R = \mathrm{Diag} \sqcup j : \tilde{C} \sqcup \tilde{C}_* \times_{C_*} \tilde{C}_* \longrightarrow \tilde{C} \times_C \tilde{C},$$

where $\mathrm{Diag} : \tilde{C} \to \tilde{C} \times_C \tilde{C}$ is the diagonal morphism and $j : \tilde{C}_* \times_{C_*} \tilde{C}_* \to \tilde{C} \times_C \tilde{C}$ is the natural open immersion. The R-invariant morphism $\varphi : \tilde{C} \to C$ descends through the quotient morphism $q : \tilde{C} \to \mathcal{C} = \tilde{C}/R$ to a bijective quasi-finite morphism $\alpha : \mathcal{C} \to C$ restricting to an isomorphism over C_*. On the other hand, the inverse image of c_0 by α consists of a unique point $\mathfrak{c}_0 = q(\tilde{c}_0)$, at which α ramification index m. The sheaf $\alpha_* \mathcal{O}_\mathcal{C}$ is equal to the $\mathcal{O}_C$-submodule of $\varphi_* \mathcal{O}_{\tilde{C}}$ consisting of germs of R-invariant regular functions on $\tilde{C}$, hence is equal to $\mathcal{O}_C$. Since R is not a locally closed immmersion in a neighborhood of the point $\tilde{c}_0 \in \tilde{C}$, it follows that the algebraic space $\mathcal{C}$ is not locally separated in a neighborhood of the point $\mathfrak{c}_0$, hence is not a scheme.

Theorem 9. *Let $\pi : S \to C$ be an $\mathbb{A}^1$-fibration on a smooth quasi-projective surface. Then there exists a smooth multifold algebraic space C-curve $\alpha : \mathcal{C} \to C$ unique up to C-isomorphism and a smooth affine morphism with connected fibers $\rho : S \to \mathcal{C}$ such that $\pi = \alpha \circ \rho$.*

Sketch of proof. A smooth multifold algebraic space C-curve $\alpha : \mathcal{C} \to C$ with the desired properties is obtained as follows. Let $c_1, \ldots, c_s$ be the points over which the fibers of $\pi : S \to C$ are degenerate and let $\check{\alpha} : \check{C} \to C$ be the scheme obtained from C as in Example 7 by replacing each point c_s by distinct scheme points $\check{c}_{i,1}, \ldots, \check{c}_{i,r_i}$, one for each connected component $F_{i,j}$ of the fiber $\pi^{-1}(c_i)$, $i = 1, \ldots, s$. The unique morphism $\check{\rho} : S \to \check{C}$ defined by

$$\check{\rho}(s) = \begin{cases} \check{\alpha}^{-1}(\pi(s)) & \text{if } \pi^{-1}(\pi(s)) \text{ is connected} \\ \check{c}_{i,j} & \text{if } \pi(s) = c_i \text{ and } s \in F_{i,j} \end{cases}$$

is affine with connected fibers and satisfies $\pi = \check{\alpha} \circ \check{\rho}$. Since S is smooth and every connected component of a fiber of π is irreducible and smooth when equipped with its reduced structure, we see that $\check{\rho} : S \to \check{C}$ is smooth over a point $\check{c}$ of $\check{C}$ if and only if $\check{\rho}^{-1}(\check{c})$ is a reduced irreducible component of $\pi^{-1}(\check{\alpha}(\check{c}))$.

Let $\check{c}_0 \in \check{C}$ be a point such that $\check{\rho}^{-1}(\check{c}_0) = mF$, where $F \simeq \mathbb{A}^1$, is multiple, of multiplicity $m \geq 2$ and let s be a closed point of S supported on F. Since S and F are smooth at s, there exists a germ of smooth curve $\tilde{C}_0 \hookrightarrow S$ intersecting F transversally at s. The restriction $\varphi = \check{\rho}|_{\tilde{C}_0} : \tilde{C}_0 \to \check{C}$ is quasi-finite onto its image. By shrinking $\tilde{C}_0$ if necessary we can assume without loss generality that the image of φ is an affine open neighborhood C_0 of $\check{c}_0$ in $\check{C}$ with the property that $\pi^{-1}(\check{\alpha}(\check{c}_0))$ is the unique degenerate fiber of π over $\check{\alpha}(C_0)$ and that $\varphi : \tilde{C}_0 \to C_0$ is a quasi-finite morphism of degree $m \geq 2$, with ramification index m at s and étale elsewhere. Let $\beta_{\check{c}_0} : \tilde{C}_0/R \to C_0$ be the algebraic space curve over C_0 determined by $\varphi : \tilde{C}_0 \to C_0$ as in Example 8 and let $\alpha_{\check{c}_0} : \mathcal{C}_{\check{c}_0} \to \check{C}$ be the algebraic space curve over $\check{C}$ obtained by gluing $\check{C} \setminus \{\check{c}_0\}$ with $\tilde{C}_0/R$ with by the identity along the open subsets $C_0 \setminus \{\check{c}_0\}$ and $\beta_{\check{c}_0}^{-1}(C_0 \setminus \{\check{c}_0\}) \simeq C_0 \setminus \{\check{c}_0\}$ of $\check{C} \setminus \{\check{c}_0\}$ and $\tilde{C}_0/R$

respectively. Letting $\mathfrak{c}_0 = \alpha_{\check{c}_0}^{-1}(\check{c}_0)$, one checks locally on an étale cover of S that $\check{\rho} : S \to \check{C}$ factors through an affine morphism $\rho_{\check{c}_0} : S \to \mathcal{C}_{\check{c}_0}$ smooth over a Zariski open neighborhood of $\mathfrak{c}_0$ and such that $\rho_{\check{c}_0}^{-1}(\mathfrak{c}_0) = F$. By repeating the above construction for each of the finitely many points of $\check{C}$ over which the fiber of $\check{\rho}$ is multiple, we obtain a smooth multifold algebraic space C-curve $\alpha : \mathcal{C} \to C$ and a smooth affine morphism with connected fibers $\rho : S \to \mathcal{C}$ factoring π.

By construction, for every closed point $c \in C$, the fibers of $\rho : S \to \mathcal{C}$ over the points in $\alpha^{-1}(c)$ are in one-to-one correspondence with the connected components of the fiber of $\pi : S \to C$ over c. Therefore, if $\alpha' : \mathcal{C}' \to C$ is another smooth multifold algebraic space C-curve with the same properties then its associated smooth morphism $\rho' : S \to \mathcal{C}'$ is locally constant on the fibers of $\pi : S \to C$ hence constant on the fibers of $\rho : S \to \mathcal{C}$. This implies in turn by faithfully flat descent that there exists a unique morphism $\varphi : \mathcal{C} \to \mathcal{C}'$ of algebraic spaces over C such that $\rho' = \varphi \circ \rho$. Reversing the roles of $\mathcal{C}$ and $\mathcal{C}'$, we conclude that φ is a C-isomorphism. $\qquad\square$

Definition 10. Let $\pi : S \to C$ be an $\mathbb{A}^1$-fibration on a smooth quasi-projective surface S. The smooth multifold algebraic space C-curve $\alpha : \mathcal{C} \to C$ such that π factors as $\pi = \alpha \circ \rho$ for some smooth affine morphism $\rho : S \to \mathcal{C}$ with connected fibers is called the *smooth relatively connected quotient* of $\pi : S \to C$.

It follows from the proof of Theorem 9 that the isomorphism type of $\alpha : \mathcal{C} \to C$ as a space over C depends only on the irreducible components, taken with their respective multiplicities of the scheme-theoretic degenerate fibers of $\pi : S \to C$.

1.2. **Illustration on a toy local model.** Let C be the spectrum of a discrete valuation ring $\mathcal{O}$ with maximal ideal $\mathfrak{m}$ and residue field $\mathbb{C}$ and let $t \in \mathfrak{m}$ be a uniformizing parameter. Given integers $n, m \geq 2$, let $S_{n,m}$ be the smooth affine surface in $C \times \mathbb{A}^2$ defined by the equation $t^n z = y^m - t$. The restriction to $S_{n,m}$ of the projection pr_C is an $\mathbb{A}^1$-fibration $\pi : S_{n,m} \to C$ whose fiber over the closed point c of C is irreducible of multiplicity m, isomorphic to $\mathrm{Spec}(\mathbb{C}[y]/(y^m)[z])$. The curve

$$\tilde{C} = \{z = 0\} \simeq \mathrm{Spec}(\mathcal{O}[y]/(y^m - t))$$

on S is smooth and the restriction of π to $\tilde{C}$ is a finite Galois cover $\varphi : \tilde{C} \to C$ totally ramified over the closed point of C, with Galois group equal to the group μ_m of complex m-th roots of unity. The normalization $\tilde{S}_{n,m}$ of the fiber product $S_{n,m} \times_C \tilde{C}$ is isomorphic to the smooth affine surface in $\tilde{C} \times \mathbb{A}^2$ defined by the equation $y^{(n-1)m} z = u^m - 1$. The Galois group μ_m acts on $\tilde{S}_{n,m}$ by $(y, z, u) \mapsto (\varepsilon y, \varepsilon^{-1} u, z)$, where ε is a primitive m-th root of unity, and the quotient morphism $\tilde{S}_{n,m} \to \tilde{S}_{n,m}/\mu_m \simeq S_{n,m}$ is étale. The restriction to $\tilde{S}_{n,m}$ of the projection $\mathrm{pr}_{\tilde{C}}$ is an $\mathbb{A}^1$-fibration $\tilde{\pi} : \tilde{S}_{n,m} \to \tilde{C}$ whose fiber over the closed point $\tilde{c}$ of $\tilde{C}$ is reduced, consisting of m disjoint copies $\tilde{F}_i = \{y = u - \varepsilon^i = 0\}$, $i = 0, \ldots, m-1$, of the affine line $\mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[z])$, which form a unique orbit of the action of μ_m on $\tilde{S}_{n,m}$.

Let $\tilde{\alpha} : \tilde{\mathcal{C}} \to \tilde{C}$ be the scheme obtained as in Example 7 by gluing m copies $\tilde{\alpha}_i : \tilde{\mathcal{C}}_i \overset{\simeq}{\Rightarrow} \tilde{C}$ of $\tilde{C}$ by the identity outside the points $\tilde{c}_i = \tilde{\alpha}_i^{-1}(\tilde{c})$ and let $\tilde{\rho} : \tilde{S}_{n,m} \to \tilde{\mathcal{C}}$ be the unique morphism lifting $\tilde{\pi} : \tilde{S}_{n,m} \to \tilde{C}$ and mapping $\tilde{F}_i$ to $\tilde{c}_i$. For every $i = 0, \ldots, m-1$, the open subset $\tilde{S}_{n,m}^i = \tilde{S}_{n,m} \setminus \bigcup_{j \neq i} \tilde{F}_j$ of $\tilde{S}_{n,m}$ is isomorphic to $\tilde{C} \times \mathrm{Spec}(\mathbb{C}[v_i])$ where v_i is the regular function on $\tilde{S}_{n,m}^i$ defined as the restriction of the rational function

$$v_i = \frac{(u - \varepsilon^i)}{y^{(n-1)m}} = \frac{z}{\prod_{j \neq i}(u - \varepsilon^j)}$$

on $\tilde{S}_{n,m}$. Via the so-defined isomorphism $\tilde{S}_{n,m}^i \simeq \tilde{C} \times \mathbb{A}^1$, the restriction of $\tilde{\rho} : \tilde{S}_{n,m} \to \tilde{\mathcal{C}}$ on $\tilde{S}_{n,m}^i$ coincides with the composition of the projection $\mathrm{pr}_{\tilde{C}}$ with the inclusion of $\tilde{C}$ as the open subset $\tilde{\mathcal{C}}_i$ of $\tilde{\mathcal{C}}$. It follows that $\tilde{\rho} : \tilde{S}_{n,m} \to \tilde{\mathcal{C}}$ is a Zariski locally trivial $\mathbb{A}^1$-bundle, in particular, a smooth morphism with connected fibers.

The action of the Galois group μ_m on $\tilde{S}_{n,m}$ descends to a free μ_m-action on $\tilde{\mathcal{C}}$ defined by

$$\tilde{\mathcal{C}}_i \ni y \mapsto \varepsilon y \in \tilde{\mathcal{C}}_{i+1 \bmod m}.$$

The geometric quotient of $\tilde{\mathcal{C}}$ by this μ_m-action is an algebraic space $\mathcal{C}$, isomorphic to the algebraic space $\tilde{C}/R$ associated to the finite cover $\varphi : \tilde{C} \to C$ as in Example 8, and the quotient morphism $q : \tilde{\mathcal{C}} \to \mathcal{C} = \tilde{\mathcal{C}}/\mu_m$ is an étale μ_m-torsor. Let

$$\rho : S_{n,m} \simeq \tilde{S}_{n,m}/\mu_m \to \tilde{\mathcal{C}}/\mu_m = \mathcal{C} \text{ and } \alpha : \mathcal{C} = \tilde{\mathcal{C}}/\mu_m \to \tilde{C}/\mu_m \simeq C$$

be the morphisms induced by the μ_m-equivariant morphisms $\tilde{\rho} : \tilde{S}_{n,m} \to \tilde{\mathcal{C}}$ and $\tilde{\alpha} : \tilde{\mathcal{C}} \to \tilde{C}$ respectively. The inverse image of c by α consists of a unique point $\mathfrak{c}$ which is the image by $q : \tilde{\mathcal{C}} \to \mathcal{C}$ of the μ_m-orbit formed by

the points $\tilde{c}_i$, $i = 0, \ldots, m-1$, and we have a commutative diagram

$$
\begin{array}{ccc}
\tilde{S}_{n,m} & \longrightarrow & S_{n,m} \simeq \tilde{S}_{n,m}/\mu_m \\
\downarrow{\scriptstyle\tilde{\rho}} & & \downarrow{\scriptstyle\rho} \\
\tilde{\mathcal{C}} & \xrightarrow{\ q\ } & \mathcal{C} = \tilde{\mathcal{C}}/\mu_m \\
\downarrow{\scriptstyle\tilde{\alpha}} & & \downarrow{\scriptstyle\alpha} \\
\tilde{C} & \longrightarrow & C \simeq \tilde{C}/\mu_m
\end{array}
$$

in which the top square is cartesian. Since $q : \tilde{\mathcal{C}} \to \mathcal{C}$ is étale and $\tilde{\rho} : \tilde{S}_{n,m} \to \tilde{\mathcal{C}}$ is smooth, the morphism $\rho : S_{n,m} \to \mathcal{C}$ is thus smooth, with fiber over $\mathfrak{c}$ equal to $\pi^{-1}(c)$ endowed with its reduced structure.

1.3. $\mathbb{A}^1$-fibrations as torsors under étale locally trivial line bundles.

Let $\pi : S \to C$ be an $\mathbb{A}^1$-fibration on a smooth quasi-projetive surface, let $\alpha : \mathcal{C} \to C$ be its smooth relatively connected quotient and let $\rho : S \to \mathcal{C}$ be the corresponding smooth affine morphism with connected fibers. For every étale morphism $f : B \to \mathcal{C}$ from a smooth algebraic curve B, the projection $\mathrm{pr}_B : S \times_{\mathcal{C}} B \to B$ is a smooth $\mathbb{A}^1$-fibration with connected fibers, hence is a Zariski locally trivial $\mathbb{A}^1$-bundle by virtue of Proposition 3. It follows that $\rho : S \to \mathcal{C}$ is an étale locally trivial $\mathbb{A}^1$-bundle over $\mathcal{C}$. Summing-up, we obtain:

Theorem 11. *Every $\mathbb{A}^1$-fibration $\pi : S \to C$ on a smooth quasi-projective surface S decomposes as an étale locally trivial $\mathbb{A}^1$-bundle $\rho : S \to \mathcal{C}$ over a smooth multifold algebraic C-space curve $\mathcal{C}$ followed by the structure morphism $\alpha : \mathcal{C} \to C$ of $\mathcal{C}$.*

Since the automorphism group of the affine line $\mathbb{A}^1$ is the affine group $\mathrm{Aff}_1 = \mathbb{G}_m \ltimes \mathbb{G}_a$, every étale locally trivial $\mathbb{A}^1$-bundle $\rho : S \to \mathcal{C}$ is an étale affine-linear bundle, isomorphic to the associated fiber bundle $V \times^{\mathrm{Aff}_1} \mathbb{A}^1 \to \mathcal{C}$ of a principal homogeneous Aff_1-bundle $V \to \mathcal{C}$ over $\mathcal{C}$. It follows that there exists a uniquely determined étale locally trivial line bundle $p : L_{S/\mathcal{C}} \to \mathcal{C}$, considered as a locally constant group scheme over $\mathcal{C}$ for the group law induced by the addition of germs of sections, such that $\rho : S \to \mathcal{C}$ can be further equipped with the structure of an étale $L_{S/\mathcal{C}}$-torsor, that is, a principal homogeneous bundle under the action of $L_{S/\mathcal{C}}$. Namely, the class of $L_{S/\mathcal{C}}$ in the Picard group $\mathrm{Pic}(\mathcal{C}) = H^1_{\text{ét}}(\mathcal{C}, \mathbb{G}_m)$ of $\mathcal{C}$ coincides with the image of the isomorphism class of $V \to \mathcal{C}$ in $H^1_{\text{ét}}(\mathcal{C}, \mathrm{Aff}_1)$ by the map $H^1_{\text{ét}}(\mathcal{C}, \mathrm{Aff}_1) \to H^1_{\text{ét}}(\mathcal{C}, \mathbb{G}_m)$ in the long exact sequence of non-abelian cohomology

$$
\cdots \to H^0(\mathcal{C}, \mathbb{G}_m) \to H^1_{\text{ét}}(\mathcal{C}, \mathbb{G}_a) \to H^1_{\text{ét}}(\mathcal{C}, \mathrm{Aff}_1) \to H^1_{\text{ét}}(\mathcal{C}, \mathbb{G}_m)
$$

associated to the short exact sequence $0 \to \mathbb{G}_a \to \mathrm{Aff}_1 \to \mathbb{G}_m \to 0$ of étale sheaves of groups on $\mathcal{C}$. Isomorphism classes of étale torsors under a given étale locally trivial line bundle $p : L \to \mathcal{C}$ are in turn classified by the cohomology group $H^1_{\text{ét}}(\mathcal{C}, L)$ (see e.g. [17, XI.4] or [14, III.2.4]).

The line bundle $p : L_{S/\mathcal{C}} \to \mathcal{C}$ associated to a given étale locally trivial $\mathbb{A}^1$-bundle $\rho : S \to \mathcal{C}$ can be alternatively described as follows: since $\rho : S \to \mathcal{C}$ is an étale locally trivial $\mathbb{A}^1$-bundle, the pull-back homomorphism

$$
\rho^* : H^1_{\text{ét}}(\mathcal{C}, \mathbb{G}_m) \to H^1_{\text{ét}}(S, \mathbb{G}_m) \simeq \mathrm{Pic}(S)
$$

is an isomorphism. The same local description as in [16, 16.4.7] then implies that $L_{S/\mathcal{C}}$ is the unique étale line bundle on $\mathcal{C}$ whose image by ρ^* is equal to the relative tangent line bundle $T_{S/\mathcal{C}} = \mathrm{Spec}(\mathrm{Sym}^{\cdot}\Omega_{S/\mathcal{C}}) \to S$ of $\rho : S \to \mathcal{C}$, where $\Omega_{S/\mathcal{C}}$ denotes the sheaf of relative Kähler differentials of S over $\mathcal{C}$.

Example 12. Let $Q \subset \mathbb{P}^2$ be a smooth plane conic. The pencil $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ generated by Q and twice its projective tangent line L_q at a given point $q \in Q$ restricts on $S_1 = \mathbb{P}^2 \setminus Q$ to an $\mathbb{A}^1$-fibration $\pi_1 : S_1 \to \mathbb{A}^1$ with irreducible fibers, having

$$
\pi_1^{-1}(0) = (2L_q \cap S_1) \simeq 2\mathbb{A}^1
$$

as a unique degenerate fiber of multiplicity 2. Similarly, given any interger $n \geq 2$, the smooth surface $S_{n,2} \subset \mathbb{A}^3$ defined by the equation $x^n z = y^2 - x$ has an $\mathbb{A}^1$-fibration $\pi_{n,2} : S_2 \to \mathbb{A}^1$ induced by the projection pr_x whose unique degenerate fiber $\pi_{n,2}^{-1}(0) \simeq \mathrm{Spec}(\mathbb{C}[y]/(y^2)[z])$ is irreducible of multiplicity 2.

By Theorem 11, π_1 and π_2 factors through étale locally trivial $\mathbb{A}^1$-bundles over the multifold algebraic space curve $\alpha : \mathcal{C} \to \mathbb{A}^1$ defined as the quotient of $\tilde{\mathbb{A}}^1 = \mathrm{Spec}(\mathbb{C}[u])$ by the equivalence relation

$$
\mathbb{A}^1 \setminus \{0\} \ni u \sim -u \in \mathbb{A}^1 \setminus \{0\}.
$$

Equivalently, $\mathcal{C}$ is the quotient of the affine line with a double origin $\tilde{\alpha} : \tilde{\mathcal{C}} \to \tilde{\mathbb{A}}^1$ obtained by gluing two copies $\tilde{\mathcal{C}}_{\pm}$ of $\tilde{\mathbb{A}}^1$ by the identity outside their respective origins by the free μ_2-action defined by $\tilde{\mathcal{C}}_{\pm} \ni u \mapsto -u \in \tilde{\mathcal{C}}_{\mp}$.

Étale descent along the quotient μ_2-torsor $q : \tilde{C} \to \tilde{C}/\mu_2 \simeq C$ induces a one-to-one correspondence between isomorphism classes of étale locally trivial line bundles on C and isomorphism classes of μ_2-linearized line bundles on $\tilde{C}$. Since the two origins of $\tilde{C}$ form a unique μ_2-orbit, every μ_2-linearized line bundle on $\tilde{C}$ is the pull-back by $\tilde{\alpha} : \tilde{C} \to \mathbb{A}^1$ of a μ_2-linearized line bundle on $\mathbb{A}^1$ for the μ_2-action $u \mapsto -u$. It follows that $\mathrm{Pic}(C)$ is isomorphic to $\mathbb{Z}_2$, generated by the class of the étale locally trivial line bundle $p : L \to C$ corresponding to the trivial line bundle $\tilde{C} \times \mathrm{Spec}(\mathbb{C}[\ell])$ endowed with the non-trivial μ_2-linearization given by $\ell \mapsto -\ell$ on the second factor. Noting that this line bundle on $\tilde{C}$ is isomorphic to the cotangent line bundle $T_{\tilde{C}}^\vee$ of $\tilde{C}$ endowed with its canonical μ_2-linearization, we see that L is isomorphic to the cotangent line bundle $T_C^\vee \to C$ of C.

Let $\rho_1 : S_1 \to C$ and $\rho_{n,2} : S_{n,2} \to C$ be the étale locally trivial $\mathbb{A}^1$-bundles factoring π_1 and $\pi_{n,2}$ respectively. The Picard group of $S_1 = \mathbb{P}^2 \setminus Q$ is isomorphic to $\mathbb{Z}_2$ generated by the restriction of $\mathcal{O}_{\mathbb{P}^1}(1)$, which is equal to canonical line bundle $\Lambda^2 T_{S_1}^\vee$ of S_1. The Picard group of $S_{n,2}$ is also isomorphic to $\mathbb{Z}_2$, generated for instance by the line bundle corresponding to the Cartier divisor $D = \{x = y = 0\}$ on $S_{n,2}$, but in contrast, since $S_{n,2}$ is a smooth hypersurface in $\mathbb{A}^3$, its canonical bundle $\Lambda^2 T_{S_{n,2}}^\vee$ is trivial by adjunction formula. Since the homomorphisms $\rho_1^* : \mathrm{Pic}(C) \to \mathrm{Pic}(S_1)$ and $\rho_{n,2}^* : \mathrm{Pic}(C) \to \mathrm{Pic}(S_{n,2})$ are isomorphisms, we then deduce from the cotangent exact sequences

$$0 \to \rho^* T_C^\vee \to T_S^\vee \to T_{S/C}^\vee \to 0$$

of the morphisms $\rho_1 : S_1 \to C$ and $\rho_{n,2} : S_{n,2} \to C$ respectively that

$$T_{S_1/C}^\vee \simeq (\rho_1^* T_C^\vee)^\vee \otimes \Lambda^2 T_{S_1}^\vee \simeq \rho_1^* T_C \otimes \rho_1^* T_C^\vee$$

is the trivial line bundle on S_1 and that the line bundle $T_{S_{n,2}/C}^\vee$ is isomorphic to $\rho_{n,2}^* T_C$.

This implies in turn that $\rho_1 : S_1 \to C$ is an étale locally trivial torsor under the trivial line bundle on C, in other words, a principal homogeneous $\mathbb{G}_{a,C}$-bundle, whereas $\rho_{n,2} : S_{n,2} \to C$ is an étale locally trivial torsor under the tangent line bundle T_C of C. One can check further on a suitable étale cover of C that the isomorphism classes in $H^1_{\text{ét}}(C, T_C)$ of the T_C-torsors $\rho_{n,2} : S_{n,2} \to C$, $n \geq 2$, are pairwise distinct.

2. $\mathbb{A}^1$-CYLINDERS OF SMOOTH $\mathbb{A}^1$-FIBERED SURFACES OF AFFINE TYPE

Given a smooth quasi-projective $\mathbb{A}^1$-fibered surface $\pi : S \to C$ with smooth relatively connected quotient $\alpha : C \to C$, the isomorphism class of S as a scheme over C is determined by the pair consisting of the isomorphism class in $\mathrm{Pic}(C)$ of the étale line bundle $p : L_{S/C} \to C$ under which the associated étale locally trivial $\mathbb{A}^1$-bundle $\rho : S \to C$ is an $L_{S/C}$-torsor and of the isomorphism class of this torsor in $H^1_{\text{ét}}(C, L_{S/C})$. In contrast, the following results shows in particular that when C is an affine curve, the isomorphism class as scheme over C of the cylinder $S \times \mathbb{A}^1$ over S is independent of the class of $\rho : S \to C$ in $H^1_{\text{ét}}(C, L_{S/C})$.

Theorem 13. *Let $\pi : S \to C$ and $\pi' : S' \to C$ be smooth quasi-projective $\mathbb{A}^1$-fibered surfaces over a same affine curve C. Let $\alpha : C \to C$ and $\alpha' : C' \to C$ be their respective smooth relatively connected quotients and let $\rho : S \to C'$ and $\rho' : S' \to C'$ be the associated étale $L_{S/C}$-torsor and $L_{S'/C'}$-torsor respectively.*

Then the threefolds $S \times \mathbb{A}^1$ and $S' \times \mathbb{A}^1$ are isomorphic as schemes over C if and only if there exists a C-isomorphism $\psi : C \to C'$ such that $\psi^* L_{S'/C'} \simeq L_{S/C}$.

Proof. A C-isomorphism $\Psi : S \times \mathbb{A}^1 \xrightarrow{\simeq} S' \times \mathbb{A}^1$ induces for each closed point $c \in C$ a multiplicity preserving one-to-one correspondence between the irreducible components of the fiber of $\pi \circ \mathrm{pr}_S$ over c and the irreducible components of the fiber of $\pi' \circ \mathrm{pr}_{S'}$ over c. It follows from the construction of the smooth relatively connected quotients $\alpha : C \to C$ and $\alpha' : C' \to C$ that Ψ induces a C-isomorphism $\psi : C \to C'$ such that $\psi \circ (\rho \circ \mathrm{pr}_S) = (\rho' \circ \mathrm{pr}_{S'}) \circ \Psi$. Since the morphisms $\rho \circ \mathrm{pr}_S : S \times \mathbb{A}^1 \to C$ and $\rho' \circ \mathrm{pr}_{S'} : S' \times \mathbb{A}^1 \to C'$ are étale locally trivial $\mathbb{A}^2$-bundles over C and C' respectively, the pull-back homomorphisms

$$(\rho \circ \mathrm{pr}_S)^* : \mathrm{Pic}(C) \to \mathrm{Pic}(S \times \mathbb{A}^1) \quad \text{and} \quad (\rho' \circ \mathrm{pr}_{S'})^* : \mathrm{Pic}(C') \to \mathrm{Pic}(S' \times \mathbb{A}^1)$$

are both isomorphisms. Let $T_{S \times \mathbb{A}^1/C}$ and $T_{S' \times \mathbb{A}^1/C'}$ be the relative tangent bundles of the morphisms $\rho \circ \mathrm{pr}_S$ and $\rho' \circ \mathrm{pr}_{S'}$ respectively. By definition of $L_{S/C}$ and $L_{S'/C'}$ (see subsection 1.3), we have

$$\Lambda^2 T_{S \times \mathbb{A}^1/C} \simeq \mathrm{pr}_S^* T_{S/C} \simeq (\rho \circ \mathrm{pr}_S)^* L_{S/C} \text{ and } \Lambda^2 T_{S' \times \mathbb{A}^1/C'} \simeq \mathrm{pr}_{S'}^* T_{S'/C'} \simeq (\rho' \circ \mathrm{pr}_{S'})^* L_{S'/C'}.$$

Since on the other hand Ψ is an isomorphism and $\psi \circ (\rho \circ \mathrm{pr}_S) = (\rho' \circ \mathrm{pr}_{S'}) \circ \Psi$, we have

$$
\begin{aligned}
(\rho \circ \mathrm{pr}_S)^* L_{S/C} \simeq \Lambda^2 T_{S \times \mathbb{A}^1/C} &\simeq \Psi^* \Lambda^2 T_{S' \times \mathbb{A}^1/C'} \\
&\simeq \Psi^*((\rho' \circ \mathrm{pr}_{S'})^* L_{S'/C'}) \\
&\simeq (\rho \circ \mathrm{pr}_S)^*(\psi^* L_{S'/C'})
\end{aligned}
$$

from which it follows that $L_{S/C} \simeq \psi^* L_{S'/C'}$.

Conversely, assume that there exists a C-isomorphism $\psi : \mathcal{C} \to \mathcal{C}'$ such that $\psi^* L_{S'/\mathcal{C}'} \simeq L_{S/\mathcal{C}}$. Letting $S'' = S' \times_{\mathcal{C}'} \mathcal{C}$ and $\pi'' = \alpha \circ \mathrm{pr}_{\mathcal{C}} : S'' \to C$, it is enough to constructs a $\mathcal{C}$-isomorphism $S \times \mathbb{A}^1 \simeq S'' \times \mathbb{A}^1$. We can thus assume without loss of generality that $\mathcal{C} = \mathcal{C}'$ and that $\psi = \mathrm{id}_{\mathcal{C}}$. We let $L = L_{S/\mathcal{C}} = L_{S'/\mathcal{C}}$. Note that since C is affine and the morphisms $\pi : S \to C$ and $\pi' : S' \to C'$ are affine by definition, the surfaces S and S' are both affine. Since C is affine, the étale line bundle $p : L_{S/\mathcal{C}} \to \mathcal{C}$ has a non-zero section. Indeed, given any rational section σ of L we can find a non-zero regular function f on C which vanishes sufficiently on the images by $\alpha : \mathcal{C} \to C$ of the poles of σ so that $s = (\alpha^* f)\sigma$ is a regular global section of L on $\mathcal{C}$. The cokernel $\mathcal{Q}$ of s viewed as an injective homomorphism $\mathcal{O}_{\mathcal{C}} \to \mathcal{L}$, where $\mathcal{L}$ denote the étale sheaf of germs of sections of L, is a torsion sheaf on $\mathcal{C}$. It follows that $H^1_{\text{ét}}(\mathcal{C}, \mathcal{Q}) = 0$. Considering the long exact sequence of étale cohomology associated to the short exact sequence

$$0 \to \mathcal{O}_{\mathcal{C}} \xrightarrow{s} \mathcal{L} \to \mathcal{Q} \to 0,$$

of étale sheaves on $\mathcal{C}$, we conclude that the homomorphism

$$H^1(s) : H^1_{\text{ét}}(\mathcal{C}, \mathbb{G}_a) = H^1_{\text{ét}}(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \to H^1_{\text{ét}}(\mathcal{C}, \mathcal{L}) = H^1_{\text{ét}}(\mathcal{C}, L)$$

is surjective. This implies the existence of an étale $\mathbb{G}_{a,\mathcal{C}}$-torsor $\rho_0 : S_0 \to \mathcal{C}$ and a $\mathcal{C}$-morphism $\xi : S_0 \to S$ of torsors which is equivariant for the homomorphism of group schemes

$$\mathrm{Spec}(\mathrm{Sym}^{\cdot}(^t s)) : \mathbb{G}_{a,\mathcal{C}} = \mathrm{Spec}(\mathrm{Sym}^{\cdot} \mathcal{O}_{\mathcal{C}}) \to L = \mathrm{Spec}(\mathrm{Sym}^{\cdot} \mathcal{L}^{\vee})$$

induced by s and which has the property that the isomorphism class of the L-torsor $\rho : S \to \mathcal{C}$ in $H^1_{\text{ét}}(\mathcal{C}, L)$ is the image by $H^1(s)$ of the isomorphism class of the étale $\mathbb{G}_{a,\mathcal{C}}$-torsor $\rho_0 : S_0 \to \mathcal{C}$ in $H^1_{\text{ét}}(\mathcal{C}, \mathbb{G}_a)$. Since S is affine and $\xi : S_0 \to S$ is an affine morphism, S_0 is an affine surface. Now consider the fiber products

$$W = S \times_{\mathcal{C}} S_0 \quad \text{and} \quad W' = S_0 \times_{\mathcal{C}} S'.$$

Since $\rho_0 : S_0 \to \mathcal{C}$ is an étale $\mathbb{G}_{a,\mathcal{C}}$-torsor, the projections $\mathrm{pr}_S : W \to S$ and $\mathrm{pr}_{S'} : W' \to S'$ are étale $\mathbb{G}_a$-torsors over S and S' respectively. Since S and S' are both affine, these torsors are thus trivial [16, Proposition 16.5.16], respectively isomorphic to $S \times \mathbb{A}^1$ and $S' \times \mathbb{A}^1$ on which $\mathbb{G}_a$ acts by translations on the second factor. On the other hand, the projections $\mathrm{pr}_{S_0} : W \to S_0$ and $\mathrm{pr}_{S_0} : W' \to S_0$ are both étale torsors under the étale line bundle $\rho_0^* L$ on S_0. Since S_0 is an affine scheme, $\rho_0^* L$ is a Zariski locally trivial line bundle and the vanishing of $H^1_{\text{ét}}(S_0, \rho_0^* L) = H^1(S_0, \rho_0^* L)$ implies that W and W' are both isomorphic to the trivial $\rho_0^* L$-torsor over S_0, that is, to the total space of the line bundle $\rho_0^* L$ on S_0. Summing up, we have constructed isomorphisms

$$S \times \mathbb{A}^1 \simeq W \simeq \rho^* L_0 \simeq W' \simeq S' \times \mathbb{A}^1$$

of schemes over $\mathcal{C}$. This completes the proof. $\qquad\square$

Remark 14. The proof of Theorem 13 depends in a crucial way on the existence of an étale $\mathbb{G}_{a,\mathcal{C}}$-torsor $\rho_0 : S_0 \to \mathcal{C}$ with affine total space S_0. Such a torsor does not exist in general if the curve $\mathcal{C}$ is not affine. For instance, if $\mathcal{C} = C = \mathbb{P}^1$, the vanishing of $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$ implies that the only $\mathbb{G}_a$-torsor over $\mathbb{P}^1$ is the trivial one $\mathbb{P}^1 \times \mathbb{A}^1$ on which $\mathbb{G}_a$ acts by translations on second factor, whose total space is not affine. Similarly, if $\mathcal{C} = C$ is a smooth elliptic curve, then the total space of the unique non-trivial $\mathbb{G}_a$-torsor $\rho_0 : S_0 \to C$ corresponding to the unique non-trivial extension

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{O}_C \to 0$$

via the isomorphism $\mathrm{Ext}^1(\mathcal{O}_C, \mathcal{O}_C) \simeq H^1(C, \mathcal{O}_C) \simeq \mathbb{C}$ is a quasi-projective surface which is not affine.

Example 15. Consider again the smooth affine surfaces $S_{n,2}$ in $\mathbb{A}^3$ defined by the equations $x^n z = y^2 - x$, $n \geq 2$. By Example 12, each of these surfaces is an étale torsor $\rho_{n,2} : S_{n,2} \to \mathcal{C}$ under the tangent line bundle $T_{\mathcal{C}}$ of the multifold algebraic space curve $\alpha : \mathcal{C} \to \mathbb{A}^1$ obtained as the quotient of the affine line with a double origin by a free μ_2-action. One can check that up to composition by automorphisms of $\mathbb{A}^1$, $\pi_{n,2} = \mathrm{pr}_x|_{S_{n,2}} : S_{n,2} \to \mathbb{A}^1$ is the unique $\mathbb{A}^1$-fibration of affine type on $S_{n,2}$. Combined with the fact that the étale $\mathbb{A}^1$-bundles $\rho_{n,2} : S_{n,2} \to \mathcal{C}$ are pairwise non-isomorphic, this implies that the surfaces $S_{n,2}$ are pairwise non-isomorphic as abstract varieties. On the other hand, it follows from Theorem 13 that the cylinders $S_{n,2} \times \mathbb{A}^1$, $n \geq 2$, are all isomorphic. We thus recover a particular case of non-cancellation for so-called *affine pseudo-planes* studied in [22].

We conclude with the following theorem which provides a complete answer to the cancellation problem for smooth affine surfaces admitting $\mathbb{A}^1$-fibrations of affine type.

Theorem 16. *Let S be a smooth affine surface and let $\pi : S \to C$ be an $\mathbb{A}^1$-fibration over a smooth affine curve C. Then the following alternative holds:*

a) If $\pi : S \to C$ is isomorphic to the structure morphism of a line bundle over C then every smooth affine surface S' such that $S' \times \mathbb{A}^1 \simeq S \times \mathbb{A}^1$ is isomorphic to S.

b) Otherwise, there exists a smooth affine $\mathbb{A}^1$-fibered surface S' non-isomorphic to S such that $S \times \mathbb{A}^1$ is isomorphic to $S' \times \mathbb{A}^1$.

Sketch of proof. Assume that $\pi : S \to C$ is a line bundle, let S' be a smooth affine surface and let $\Psi : S' \times \mathbb{A}^1 \overset{\simeq}{\to} S \times \mathbb{A}^1$ be an isomorphism of abstract algebraic varieties. If $C \simeq \mathbb{A}^1$ then $\pi : S \to \mathbb{A}^1$ is a trivial line bundle so that $S \simeq \mathbb{A}^2$. The assertion then follows from [25]. Otherwise, if $C \not\simeq \mathbb{A}^1$ then since every morphism $\mathbb{A}^1 \to C$ is constant, it follows that the composition of Ψ with the projection $\pi \circ \mathrm{pr}_S : S \times \mathbb{A}^1 \to C$ descends to a unique morphism $\pi' : S' \to C$ such that $(\pi \circ \mathrm{pr}_S) \circ \Psi = \pi' \circ \mathrm{pr}_{S'}$. Since $\pi \circ \mathrm{pr}_S$ is a Zariski locally trivial $\mathbb{A}^2$-bundle, it follows that the same holds for $\pi' \circ \mathrm{pr}_{S'}$. This implies in turn that $\pi' : S' \to C$ is an $\mathbb{A}^1$-fibration without degenerate fibers, hence is a line bundle as C is affine. Arguing as in the proof of Theorem 13, we conclude that $\pi : S \to C$ and $\pi' : S' \to C$ are isomorphic line bundles over C, hence that $S \simeq S'$.

Now assume that $\pi : S \to C$ is not isomorphic to the structure morphism of a line bundle over C. Since C is affine, it follows that $\pi : S \to C$ has at least a degenerate fiber. Let $\alpha : \mathcal{C} \to C$ be the smooth relatively connected quotient of $\pi : S \to C$ and let $\rho : S \to \mathcal{C}$ be the associated étale $L_{S/\mathcal{C}}$-torsor. Since $\pi : S \to C$ has a degenerate fiber, it follows from the construction of the smooth relatively connected quotient that $\mathcal{C}$ is a non-separated algebraic space or scheme and that α is not an isomorphism. Since S is affine, hence is a separated scheme, this implies in particular that $\rho : S \to \mathcal{C}$ is a non-trivial étale $L_{S/\mathcal{C}}$-torsor whose isomorphism class in $H^1_{\text{ét}}(\mathcal{C}, L_{S/\mathcal{C}})$ is thus a non-zero element.

Let $\mathcal{L}$ be the étale invertible sheaf of germs of section of $L_{S/\mathcal{C}}$ and let f be a non-zero regular function on C. The multiplication by $\alpha^* f$ defines an injective homomorphism $s : \mathcal{L} \to \mathcal{L}$ which determines in turn a non-zero homomorphism of group schemes $\zeta : L_{S/\mathcal{C}} \to L_{S/\mathcal{C}}$. Arguing as in the proof of Theorem 13, we can find an étale $L_{S/\mathcal{C}}$-torsor $\rho' : S' \to \mathcal{C}$ with affine total space and an s-equivariant $\mathcal{C}$-morphism $\xi : S' \to S$ of torsors with the property that the isomorphism class of $\rho : S \to \mathcal{C}$ in $H^1_{\text{ét}}(\mathcal{C}, L_{S/\mathcal{C}})$ is the image by $H^1(s)$ of the isomorphism class of $\rho' : S' \to \mathcal{C}$ in $H^1_{\text{ét}}(\mathcal{C}, L_{S/\mathcal{C}})$.

We claim that by choosing $f \in \Gamma(C, \mathcal{O}_C)$ appropriately, we can ensure simultaneously that on the one hand the morphism $\pi' = \alpha \circ \rho' : S' \to C$ is the unique $\mathbb{A}^1$-fibration of affine type on S' and that on the other hand, for every automorphism ψ of $\mathcal{C}$, the $\mathbb{A}^1$-bundles $\rho : S \to \mathcal{C}$ and $\mathrm{pr}_2 : S' \times_{\rho', \mathcal{C}, \psi} \mathcal{C} \to \mathcal{C}$ are not isomorphic. Indeed, the first condition is automatically satisfied unless C is isomorphic to the affine line $\mathbb{A}^1$. In the case where $C = \mathbb{A}^1$, it is enough to choose a regular function f which vanishes sufficiently at the points of C over which the fibers of $\pi : S \to C$ are degenerate to ensure the existence of an SNC-minimal projective completion $\overline{S}'$ of S' whose boundary divisor is not a chain, a property which implies that $\pi' : S' \to C$ is the unique $\mathbb{A}^1$-fibration of affine type on S' (see e.g. [3, Théorème 1.8] or [6, Theorem 2.16]). Similarly, choosing f so that it vanishes at the points of C over which the fibers of $\pi : S \to C$ are degenerate is enough to guarantee that the images of the isomorphism classes of $\rho : S \to \mathcal{C}$ and $\rho' : S' \to \mathcal{C}$ in $\mathbb{P} H^1_{\text{ét}}(\mathcal{C}, L_{S/\mathcal{C}})$ do not belong the same orbit of the action of the automorphism group of $\mathcal{C}$.

To conclude, one checks that for a regular function f satisfying the two properties above, the surfaces S and S' are not isomorphic as abstract algebraic varieties. On the other hand, since $\rho : S \to \mathcal{C}$ and $\rho' : S' \to \mathcal{C}$ are both étale $L_{S/\mathcal{C}}$-torsors, we deduce from Theorem 13 that $S \times \mathbb{A}^1$ is isomorphic to $S' \times \mathbb{A}^1$. $\square$

Example 17. (Danielewski counter-example [4] revisited). Let S_0 be the smooth surface in $\mathbb{A}^3$ defined by the equation $xz = y^2 - 1$. The restriction of the projection pr_x on S defines a smooth $\mathbb{A}^1$-fibration $\pi_0 : S_0 \to \mathbb{A}^1$ restricting to a trivial $\mathbb{A}^1$-bundle over $\mathbb{A}^1 \setminus \{0\}$ and whose fiber $\pi^{-1}(0)$ is reduced, consisting of two disjoint copies $\{x = y \pm 1 = 0\}$ of the affine line $\mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[z])$. The smooth relatively connected quotient of $\pi_0 : S_0 \to \mathbb{A}^1$ is thus isomorphic to the affine line with a double origin $\alpha : \tilde{\mathbb{A}}^1 \to \mathbb{A}^1$. On checks using local trivialization as in the Example in subsection 1.2 that the induced morphism $\rho_0 : S_0 \to \tilde{\mathbb{A}}^1$ is a $\mathbb{G}_{a, \mathbb{A}^1}$-torsor, whose isomorphism class in $H^1(\tilde{\mathbb{A}}^1, \mathbb{G}_a) = H^1(\tilde{\mathbb{A}}^1, \mathcal{O}_{\tilde{\mathbb{A}}^1})$ is equal to the class of the Čech cocycle $g_0 = 2x^{-1} \in C^1(\mathcal{U}, \mathcal{O}_{\tilde{\mathbb{A}}^1}) \simeq \mathbb{C}[x^{\pm 1}]$ for the natural open cover $\mathcal{U}$ of $\tilde{\mathbb{A}}^1$ by two copies of $\mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[x])$. Let $s_n : \mathcal{O}_{\tilde{\mathbb{A}}^1} \to \mathcal{O}_{\tilde{\mathbb{A}}^1}$ be the injective homomorphism given by the multiplication by x^n and let

$$\zeta : \mathbb{G}_{a, \tilde{\mathbb{A}}^1} = \mathrm{Spec}(\mathcal{O}_{\tilde{\mathbb{A}}^1}[t]) \to \mathrm{Spec}(\mathcal{O}_{\tilde{\mathbb{A}}^1}[t]) = \mathbb{G}_{a, \tilde{\mathbb{A}}^1}, \ t \mapsto x^n t$$

be the corresponding homomorphism of group schemes over $\tilde{\mathbb{A}}^1$. The induced homomorphism

$$H^1(s_n) : H^1(\tilde{\mathbb{A}}^1, \mathcal{O}_{\tilde{\mathbb{A}}^1}) \to H^1(\tilde{\mathbb{A}}^1, \mathcal{O}_{\tilde{\mathbb{A}}^1})$$

maps the class in $H^1(\tilde{\mathbb{A}}^1, \mathcal{O}_{\tilde{\mathbb{A}}^1})$ of the Čech cocycle $g_n = 2x^{-n-1} \in C^1(\mathcal{U}, \mathcal{O}_{\tilde{\mathbb{A}}^1})$ onto the isomorphism class of the $\mathbb{G}_{a, \tilde{\mathbb{A}}^1}$-torsor $\rho_0 : S_0 \to \tilde{\mathbb{A}}^1$. It follows that there exists a $\mathbb{G}_{a, \tilde{\mathbb{A}}^1}$-torsor $\rho_n : S_n \to \tilde{\mathbb{A}}^1$ whose isomorphism class in $H^1(\tilde{\mathbb{A}}^1, \mathcal{O}_{\tilde{\mathbb{A}}^1})$ is equal to the class of the cocyle g_n and a ζ-equivariant morphism of $\mathbb{G}_{a, \tilde{\mathbb{A}}^1}$-torsors $\xi : S_n \to S_0$.

The desired $\mathbb{G}_{a,\tilde{\mathbb{A}}^1}$-torsor $\rho_n : S_n \to \tilde{\mathbb{A}}^1$ is given for instance by the smooth affine surface S_n in $\mathbb{A}^3$ with equation $x^{n+1}z = y^2 - 1$ endowed with the factorization $\rho_n : S_n \to \tilde{\mathbb{A}}^1$ of the smooth $\mathbb{A}^1$-fibration $\pi_n : S_n \to \mathbb{A}^1$ induced by the restriction of the projection pr_x. A corresponding s_n-equivariant morphism is simply the birational morphism

$$\xi : S_n \to S_0, \quad (x, y, z) \mapsto (x, y, x^n z).$$

By Theorem 13, the cylinders $S_n \times \mathbb{A}^1$, $n \geq 0$, are all isomorphic. Clearly, the surface S_0 admits a second $\mathbb{A}^1$-fibration of affine type given by the restriction of the projection pr_z. On the other hand, for every $n \geq 2$, $\pi_n : S_n \to \mathbb{A}^1$ is the unique $\mathbb{A}^1$-fibration of affine type on S_n up to composition by automorphisms of $\mathbb{A}^1$ (see e.g. [21]). It follows that for every $n \geq 2$, S_n is not isomorphic to S_0. Actually, the surfaces S_n, $n \geq 0$, are even pairwise non isomorphic [4, 21].

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