Bogomolov–Sommese vanishing theorem on globally F-regular threefolds

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Introductuion and Main Result

Differential sheaves are vector bundles naturally attached to smooth algebraic varieties and it is important to study their positivity properties. The following theorem states the positivity of line bundles contained in the differential sheaves.

Bogomolov–Sommese vanishing

Let X be a smooth projective variety over $\mathbb{C}$. Then for any invertible subsheaf $\mathcal{L} \subset \Omega^i_X$, we have $\kappa(\mathcal{L}) \leq i$, where $\kappa(\mathcal{L})$ denotes the Iitaka dimension of $\mathcal{L}$.

In general, Bogomolov–Sommese vanishing fails in positive characteristic. There exists a smooth projective surface of general type which is liftable to $W(k)$, but violates Bogomolov–Sommese vanishing (See [3, Example 1]). We consider the following question.

Quension

Does the Bogomolov–Sommese vanishing hold for smooth projective F-split varieties?

We can see that Bogomolov–Sommese vanishing holds for smooth F-split surfaces, but the proof heavily depends on the classification result of surfaces and does not work for higher-dimensional varieties. In higher-dimensional cases, we consider globally F-regular varieties, a special class of F-split varieties. We give a partial affirmative answer to Question when X is a smooth globally F-regular threefold.

Main Result[2, Theorem 4.10]

Let X be a smooth projective globally F-regular threefold over a perfect field k of characteristic $p > 5$. Then for any invertible subsheaf $\mathcal{L} \subset \Omega_X$, we have $\kappa(\mathcal{L}) \leq 1$. Furthermore, if $p > 7$, then we have $\kappa(\mathcal{L}) \leq 0$.

We need the assumption "$p > 5$" only for running an MMP.

Key Theorem

Let X be a normal variety over a perfect field k of characteristic $p > 0$. X is said to be *globally F-regular* if for every effective Weil divisor D on X, there exists an integer $e \geq 1$ such that the composite map $\mathrm{O}_X \to F^e_*\mathcal{O}_X \hookrightarrow F^e_*\mathcal{O}_X(D)$ splits as an $\mathcal{O}_X$-module homomorphism. In order to prove Main Result, we show that $H^0(X, \Omega^1_X \otimes \mathcal{O}_X(-D)) = 0$ for every Cartier divisor with $\kappa(D) \geq 2$. We first consider the case where D is nef and big.

Theorem 1 [2, Theorem 4.5]

Let X be a projective globally F-regular variety over a perfect field k of characteristic $p > 0$ and Δ be a Weil divisor on X. Assume that $\dim X \geq 2$ and non-simple normal crossing locus has codimension at least 3. Then $H^0(X, (\Omega^{[1]}_X(\log \Delta) \otimes \mathcal{O}_X(-D))^{**}) = 0$ for every nef and big $\mathbb{Q}$-Cartier Weil divisor D on X.

Here, $\Omega^{[i]}_X(\log \Delta)$ denotes the reflexive differential form $\Omega^{[i]}_X(\log \Delta) := j_*(\Omega^i_U(\log \Delta))$, where $j : U := (X, \Delta)_{\mathrm{snc}} \hookrightarrow X$ is a canonical inclusion map and $\Omega^j_U(\log \Delta)$ is the sheaf of logarithmic Kähler differentials. In Theorem 1, we use the global F-regularity of X to singular varieties. Therefore, if X is smooth in Theorem 1, then we can prove a similar assertion by only assuming F-splitness. In the proof of Main result, we run an MMP to make D satisfy the assumption of Theorem 1. In general, even if we start from a smooth variety, the output of the MMP is not necessarily smooth. This is the reason why we have to consider singular varieties in Theorem 1.
By using Theorem 1, we can show Akizuki–Nakano type vanishing on globally F-regular surface.

Corollary 2 [2, Corollary 4.8]

Let X be a globally F-regular projective surface over a perfect field k of characteristic $p > 0$. Then $H^0(X, \Omega^{[1]}_X \otimes \mathcal{O}_X(-D)) = 0$ for every nef and big Cartier divisor D on X.

Sketch of the proof of Main Result

By running a K_X-MMP, we have a birational map $f\colon X \dashrightarrow X'$ and X' has a Mori fiber space structure $g\colon X' \to Y$. X' is $\mathbb{Q}$-factorial projective globally F-regular variety with isolated singularities. Let D be a Cartier divisor on X and $D' := f_*D$ be a push-forward D by f. Then we can check that $\kappa(D) \leq \kappa(D')$ and to show

$$H^0(X, \Omega_X \otimes \mathcal{O}_X(-D)) = 0 \tag{1}$$

it is enough to show

$$H^0(X', (\Omega^{[1]}_{X'} \otimes \mathcal{O}_X(-D'))^{**}) = 0. \tag{2}$$

- $\dim Y = 0$. If $\kappa(D') \geq \kappa(D) > 0$, then D' is ample by $\rho(X) = 1$ and (2) follows from Theorem 1.
- $\dim Y = 1$. Let F be a general fiber of g. Then F is a globally F-regular surface. By considering a restriction to F, it is enough to show $H^0(F, \Omega^{[i]}_F \otimes \mathcal{O}_F(-A)) = 0$ for any i and for any ample Cartier divisor A on F. This follows from Corollary 2. If $p > 7$, then X' is separably rationally connected by [1, Theorem 4.1] and thus so is X and we get (1) directly for any D with $\kappa(D) \geq 0$ by using the separably rationally connectedness of X.
- $\dim Y = 2$. In this case, X is separably rationally connected by [1, Theorem 4.1] and we can get (1) for any D with $\kappa(D) \geq 0$.

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References

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