Multidimensional continued fractions for Gorenstein cyclic quotient singularities

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Background and Definition

Question

Let G be a finite subgroup of $SL(n,\mathbb{C})$, then the quotient $\mathbb{C}^n/G$ has a Gorenstein canonical singularity. When does $\mathbb{C}^n/G$ have a crepant resolution?

- In the case $n = 2, 3$, it is known that $\mathbb{C}^n/G$ has crepant resolutions.
- However, in higher dimension, $\mathbb{C}^n/G$ does not always have crepant resolutions.
  In this poster, we show a sufficient condition of existence of crepant resolution in all dimensions by using Ashikaga's continuous fractions.(This is joint work with Kohei Sato.)

Definition

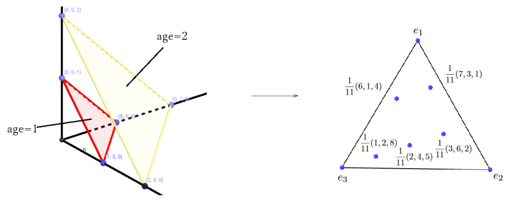
A resolution $f : Y \to X$ is called a crepant resolution if the adjunction formula $K_Y = f^*K_X + \sum_{i=1}^n a_iD_i$ satisfies $a_i = 0$ for all i

Crepant resolution as toric varieties

G:finite abelian subgroups of $SL(n,\mathbb{C})$. Any $g \in G$ is of the form $g = \mathrm{diag}(\varepsilon_r^{a_1}, \dots, \varepsilon_r^{a_n})$, where $\varepsilon_r^r = 1$ primitive. Then we can represent it as $g = \frac{1}{r}(a_1, \dots, a_n)$. Also, we define $\bar{g} = \frac{1}{r}(a_1, \dots, a_n) \in \mathbb{R}^n$. Let $N := \mathbb{Z}^n + \mathbb{Z}\bar{g}$, and σ be the region of $\mathbb{R}^n$ whose all entries are non-negative.
Then the toric variety determined by σ and N is isomorphic to $\mathbb{C}^n/G$

Remark

Let Σ is subdivision of σ using by lattice points of age($\bar{g}$)=1. If the toric variety U_Σ is smooth, then U_Σ is a crepant resolution of $\mathbb{C}^n/G$ where we define $age(g) = \frac{1}{r}\sum_{i=1}^n a_i$.



This figure shows the triangle of $\mathrm{age} = 1$(For $\frac{1}{11}(1,2,8)$-type singularity).

Let $n \in \mathbb{N}$. Let $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$ and $d \in \mathbb{N}$ which satisfies $0 \le a_i \le d-1$ for $1 \le i \le n$. We call the symbol

$$\frac{\mathbf{a}}{d} = \frac{(a_1, \dots, a_n)}{d}$$

an *n-dimensional proper fraction.*

Ashikaga's continued fraction

Let $\mathbb{Q}_n^{prop}$ be the set of n-dimensional proper fractions, and $\overline{\mathbb{Q}_n^{prop}} = \mathbb{Q}_n^{prop} \cup \{\infty\}$.

(i) The *i-th remainder map*:$\mathbb{Q}_n^{prop} \to \overline{\mathbb{Q}_n^{prop}}$ is defined by

$$R_i\left(\frac{(a_1,\dots,a_n)}{d}\right) = \begin{cases} \left(\frac{\overline{a_1},\dots,\overline{a_{i-1}},\overline{-d},\overline{a_{i+1}},\dots,\overline{a_n}}{a_i}\right) & \text{if } a_i \neq 0 \\ \infty & \text{if } a_i = 0 \end{cases}$$

and $R_i(\infty) = \infty$.

(ii) Let $\frac{\mathbf{a}}{d}$ be n-dimensional proper fraction, *the remainder polynomial* $\mathcal{R}_*(\frac{\mathbf{a}}{d}) \in \overline{\mathbb{Q}_n^{prop}}[x_1, \dots, x_n]$ is defined by

$$\mathcal{R}_*\left(\frac{\mathbf{a}}{d}\right) = \frac{\mathbf{a}}{d} + \sum_{(i_1,i_2,\dots,i_l)\in \mathbf{I}^l \ l\ge 1} (R_{i_l}\cdots R_{i_2}R_{i_1})\left(\frac{\mathbf{a}}{d}\right) \cdot x_{i_1}x_{i_2}\cdots x_{i_l}$$

excluding the term with coefficient ∞ or $\frac{(0,0,\dots,0)}{1}$.

Ashikaga's continued fraction summarizes informations of Fujiki-Oka resolution for $\frac{1}{r}(1, a_2, \dots, a_n)$-type cyclic quotient singularities.

Lemma(Ashikaga) Let $G=\frac{1}{r}(1, a_2, \dots, a_n) \subset GL(n,\mathbb{C})$ and the cone $\sigma_i = CONE(e_1, \dots, \hat{e}_i, \dots, e_n, \frac{1}{r}(1, a_2, \dots, a_n))$. Then the affine toric variety U_{σ_i} is isomorphic to $R_i(\frac{1}{r}(1, a_2, \dots, a_n))$-type quotient singularity.

Main Result

Theorem 1 (K.Sato, S)

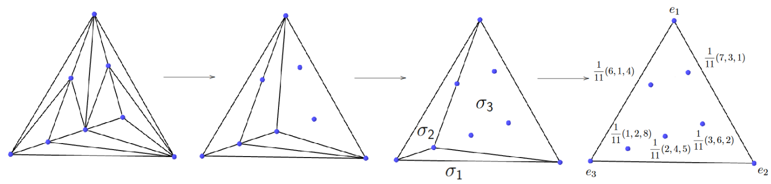
Let $G = \frac{1}{r}(1, a_2, \dots, a_n) \subseteq SL(n,\mathbb{C})$. Suppose that all coefficients of $\mathcal{R}_*(G)$ satisfy $\mathrm{age} = 1$, then $\mathbb{C}^n/G$ has a crepant resolution.

- Example

Let $G = \frac{1}{11}(1,2,8)$. Then, the remainder polynomial is

$$\mathcal{R}_*\left(\frac{(1,2,8)}{11}\right) = \frac{1}{11}(1,2,8) + \frac{1}{8}(1,2,5)x_3 + \{\frac{1}{2}(1,1,0)x_2 + \frac{1}{2}(1,0,1)x_3x_2\} + \frac{1}{5}(1,2,2)x_3x_3 + \{\frac{1}{2}(1,1,0)x_3x_3x_2 + \frac{1}{2}(1,0,1)x_3x_3x_3\}$$

Step 1:We subdivide the cone σ at the point $\frac{1}{11}(1,2,8)$. Then we can obtain three 3-dimensional cone σ_1 , σ_2, and σ_3.By above lemma, σ_1 is smooth, σ_2 is $\frac{1}{2}(1,0,1)$-type quotient singularity and σ_3 is $\frac{1}{8}(1,2,5)$-type.
Step 2: Since the coefficient $\frac{1}{8}(1,2,5)$ corresponds to the point $\frac{1}{11}(2,4,5)$, we consider star subdivison at $\frac{1}{11}(2,4,5)$.



By repeating this operation, we get the smooth fan Δ corresponding to the crepant resolution.

Theorem 2 (K.Sato, S)

If there is a repdigit proper point satisfy $\mathrm{age} \ge 2$. Then $\mathbb{C}^n/G$ has not any crepant resolutions.

- The term of the remainder polynomial $R_i \cdots R_i\left(\frac{\mathbf{a}}{r}\right) x_ix_i\cdots x_i$ is called *repdigit term*, and its coefficient is called *repdigit coefficient.*
- *A repdigit point* is a point of N corresponding to repdigit coefficient $\mathbf{a}/r$.
- Example
  Let $G = \frac{1}{15}(1,6,4,4)$, then $R_2(\frac{1}{15}(1,6,4,4)) = \frac{1}{6}(1,3,4,4)$
  The coefficient $\frac{1}{6}(1,3,4,4)$ corresponds to repdigit point $\frac{1}{15}(3,3,12,12)$, and $\mathrm{age}\left(\frac{1}{15}(3,3,12,12)\right) = 2$.
  Therefore, $\mathbb{C}^4/G$ has not any crepant resolutions.

Reference

[1] T. Ashikaga, *Multidimensional continued fractions for cyclic quotient singularities and Dedekind sums*, To appear in Kyoto J. Math. Advance publication (2019).