A note on retracts of polynomial rings in three variables

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Key words and phrases

Retracts of polynomial rings, Zariski's cancellation problem

Abstract

For retracts of the polynomial ring, in [Cos77], Costa asks us whether every retract of $k[x_1 \dots, x_n]$ is also the polynomial ring or not, where k is a field. We call it the *polynomial retraction problem* (PRP).

In this paper, we give an affirmative answer to PRP in the case where k is a field of characteristic zero and $n = 3$ ([Nag19]). Also, we state relations between PRP and Zariski's cancellation problem.

Definition (retracts of a commutative ring)

B: commutative ring,
$A \subset B$: subring of B.
We say A is a **retract** of B if
$\exists$ an ideal $I \subset B$ such that $B \cong A \oplus I$ as A-modules,
$\Leftrightarrow \exists\ \varphi : B \to A$ such that the following splits:

$$0 \to \ker \varphi \to B \xrightarrow{\varphi} A \to 0,$$

$\Leftrightarrow \exists\ \varphi : B \to A$ such $\varphi|_A = \mathrm{id}_A$.

Example

$B = k[x, y, z]$: polynomial ring in three variables.
Then,

- k, $k[x]$, $k[x, y]$ and $k[x, y, z]$ are retracts of B.
- $k[xz, yz]$ is a retract of B.
  $\because$ Define $\varphi : B \to k[xz, yz]$ by $x \mapsto xz,\ \ y \mapsto yz,\ \ z \mapsto 1$.
  Then $\varphi|_{k[xz,yz]} = \mathrm{id}_{k[xz,yz]}$.
- $k[x, xz + y^2]$ is NOT a retract of B.

Polynomial Retraction Problem (PRP)

Is every retract of $k[x_1, \dots, x_n]$ the polynomial ring?

dimension n	char $k = 0$	char $k > 0$
$n = 1$	YES	YES
$n = 2$	YES ([Cos77])	YES ([Cos77])
$n = 3$	**YES (Main Theorem)**	???
$n \geq 4$	???	NO ([Gup14a], [Gup14b])

Zariski's Cancellation Problem (ZCP)

$X \times \mathbb{A}^1_k \cong_k \mathbb{A}^{n+1}_k \Longrightarrow X \cong_k \mathbb{A}^n_k$?

dimension n	char $k = 0$	char $k > 0$
$n = 1$	YES	YES
$n = 2$	YES ([Fuj79], [MS80])	YES ([Rus81])
$n = 3$	???	NO ([Gup14a])
$n \geq 4$	???	NO ([Gup14b])

Proposition (PRP vs ZCP)

Let $n \geq 1$. Then the affirmative answer to PRP for n implies the affirmative answer to ZCP for $n - 1$.

Proof of Proposition

Suppose that PRP holds true for $n \geq 1$.
Let $X = \mathrm{Spec}\,(A)$ such that $X \times \mathbb{A}^1_k \cong_k \mathbb{A}^n_k$.
Then $A[t] = k[x_1, \dots, x_n]$.
Define $\varphi : A[t] \to A$ by $\varphi(f(t)) = f(0)$.
Then A is a retract of $k[x_1, \dots, x_n]$.
Therefore $A = k[y_1, \dots, y_{n-1}]$, hence $X \cong_k \mathbb{A}^{n-1}_k$. $\square$

Main theorem (N. 2019)

k: field of characteristic zero.
$k[x_1, \dots, x_n]$: polynomial ring in $n \geq 3$ variables.
$A \subset k[x_1, \dots, x_n]$: sub k-algebra.
Assume that A is a retract of $k[x_1, \dots, x_n]$ of dimension d.
If $0 \leq d \leq 2$ or $d = n$, then $A = k[y_1, \dots, y_d]$.

Corollary (the answer to PRP)

k: field of characteristic zero.
Every retract of $k[x, y, z]$ is the polynomial ring.

Outline of the proof

k: field of characteristic zero.
$B = k[x_1, \dots, x_n]$: polynomial ring in n variables.
$A \subset B$: retract of B.

- $\mathrm{tr.deg}_k\, A = 0, n \Rightarrow$ easy to show that A is the polynomial ring.
- $\mathrm{tr.deg}_k\, A = 1 \Rightarrow A = k[t]$ (follows from [Cos77]).

Suppose that $\mathrm{tr.deg}_k\, A = 2$.
Due to [Kam75], we may assume that k is algebraically closed.

By combing results in [Eak72], [Cos77] and [Iit77], we have:

- A is a UFD, finitely generated over k, and $A^* = k^*$,
- $X = \mathrm{Spec}\,(A)$ is a smooth affine surface over k,
- the logarithmic Kodaira dimension of X is $-\infty$.

By combing results in [Miy75], [Fuj79] and [MS80], we have $X \cong_k \mathbb{A}^2_k$.
This implies that $A = k[s, t]$. $\square$

[Cos77] D. Costa, J. Algebra, 1977.
[Eak72] P. Eakin, Proc. Amer. Math. Soc., 1972.
[Fuj79] T. Fujita, Proc. Japan Acad., Ser. A, 1979.
[Gup14a] N. Gupta, Invent. Math., 2014.
[Gup14b] N. Gupta, Adv. Math., 2014.
[Iit77] S. Iitaka, Complex Analysis and Algebraic Geometry, 1977.
[Kam75] T. Kambayashi, J. Algebra, 1975.
[Miy75] M. Miyanishi, J. Math. Kyoto Univ., 1975.
[MS80] M. Miyanishi and T. Sugie, J. Math. Kyoto Univ., 1980.
[Nag19] T. Nagamine, J. Algebra, 2019.
[Rus81] P. Russell, Math. Ann., 1981.

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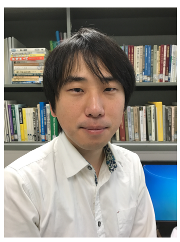

Research of the author was partially supported by Grant-in-Aid for JSPS Fellows (No. 18J10420) from Japan Society for the Promotion of Science