Radiance obstructions of tropical Kummer surfaces and their quotients

Yuki Tsutsui (University of Tokyo), 城崎代数幾何学シンポジウム 2019

1. Classical results from complex geometry

$T_{\mathbb{C}}$: a complex 2-torus, $-1_{T_{\mathbb{C}}} : T_{\mathbb{C}} \to T_{\mathbb{C}}; x \mapsto -x$.
$X = \mathrm{Km}(T_{\mathbb{C}}) :=$ the minimal resolution of $T_{\mathbb{C}}/-1_{T_{\mathbb{C}}}$.
ω: a holomorphic 2-form on X.
$\mathrm{NS}(X) = \{x \in H^2(X, \mathbb{Z}) | \omega \cup x = 0\}$.
$U(n)$: the lattice with a Gram matrix $\begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix}$.

Theorem (Classical result for Kummer surfaces $\mathrm{Km}(T_{\mathbb{C}})$)

$\exists q_* : H^2(T_{\mathbb{C}}, \mathbb{Z}) \to H^2(\mathrm{Km}(T_{\mathbb{C}}), \mathbb{Z})$, $q_*(D) \cup q_*(D) = 2D \cup D$, $q_*(H^2(T_{\mathbb{C}}, \mathbb{Z})) \simeq U(2)^{\oplus 3}$ *and* $q_*(\mathrm{NS}(T_{\mathbb{C}})^{\perp}) \simeq \mathrm{NS}(\mathrm{Km}(T_{\mathbb{C}}))^{\perp}$.

The main result is about a tropical analog of the above theorem for tropical Kummer surfaces.

complex Kummer surfaces	tropical Kummer surfaces
$\wedge_{i=1}^2 \Omega_{\mathrm{Km}(T_{\mathbb{C}})} \simeq \mathcal{O}_{\mathrm{Km}(T_{\mathbb{C}})}$	$\iota_*(\wedge_{i=1}^2 \mathcal{T}^{\vee}_{\mathrm{Km}(T)_0}) \simeq \mathbb{R}_{\mathrm{Km}(T)}$
$h^i(\mathrm{Km}(T_{\mathbb{C}}), \mathcal{O}_{\mathrm{Km}(T_{\mathbb{C}})})$	$h^i(\mathrm{Km}(T), \mathbb{R}_{\mathrm{Km}(T)})$
$\mathcal{O}^{\times}_{\mathrm{Km}(T_{\mathbb{C}})}$	$\iota_* \mathit{Aff}_{\mathbb{Z},\mathrm{Km}(T)_0}$
ω (holomorphic 2-form)	$c_{\mathrm{Km}(T)}$ (radiance obstruction)
$\mathrm{NS}(\mathrm{Km}(T_{\mathbb{C}})) = \{\omega \cup d = 0\}$	$\mathrm{Im}\, c_1 = \{c_{\mathrm{Km}(T)} \cup d = 0\}$
$\dim \Gamma(\check{X}, L) = \int_{\check{X}} \frac{c_1(L)^2}{2} + 2$	$\lvert \mathrm{Km}(T)(\mathbb{Z}) \rvert = \frac{c_{\mathrm{Km}(T)} \cup c_{\mathrm{Km}(T)}}{2} + 2$
$q_*(H^2(T_{\mathbb{C}}, \mathbb{Z})) \simeq U(2)^{\oplus 3}$	$q_*(H^1(T, \mathcal{T}_{\mathbb{Z},T})) \simeq U(2)^{\oplus 2}$
$q_*(\mathrm{NS}(T_{\mathbb{C}})^{\perp}) \simeq \mathrm{NS}(\mathrm{Km}(T_{\mathbb{C}}))^{\perp}$	$q_*((\mathrm{Im}\, c_{1,T})^{\perp}) = (\mathrm{Im}\, c_{1,\mathrm{Km}(T)})^{\perp}$

Table: Comparison table for complex and tropical Kummer surfaces

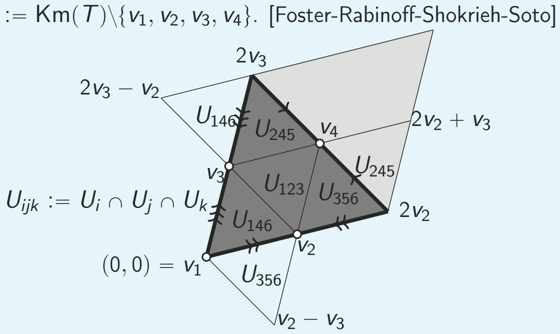
2. TASS and tropical Kummer surfaces

Definition (cf. Gross-Siebert'03, Kontsevich-Soibelman'04)

(1) B_0 is an n-dim'l *tropical (resp. integral) affine manifold* if B_0 is a manifold equipped with an atlas $\{(U_i, \psi_i)\}$ s.t. $\psi_i \circ \psi_j^{-1}|_{U_i \cap U_j} \in \mathrm{GL}_n(\mathbb{Z}) \ltimes \mathbb{R}^n$ (resp. $\mathrm{GL}_n(\mathbb{Z}) \ltimes \mathbb{Z}^n$).
(2) An n-dim'l topological mfd B equipped with an inclusion $\iota : B_0 \hookrightarrow B$ of tropical (resp. integral) affine manifold is a *tropical (resp. integral) affine manifold with singularieties* if $\mathrm{codim}\, B \backslash B_0 \geqslant 2$.

Example

(1) $B = B_0 = \mathbb{R}^n/\Lambda$: tropical tori [Mikhalkin-Zharkov'06].
(2) $T = \mathbb{R}^2/\Lambda$: a tropical 2-torus, $-1_T : T \to T; x \mapsto -x$,
$\mathrm{Km}(T) := T/-1_T (\simeq S^2)$: the **tropical Kummer surface**.
$\mathrm{Km}(T)_0 := \mathrm{Km}(T) \backslash \{v_1, v_2, v_3, v_4\}$. [Foster-Rabinoff-Shokrieh-Soto]



The light gray region is a fundamental domain of T.
The dark gray region is that of $\mathrm{Km}(T)$.

A *tropical affine surface with singularities* (**TASS** for short) is a 2-dim'l tropical affine manifold with singularities satisfying the following condition;

Condition (cf. Kontsevich-Soibelman'04)

$\forall x \in B \backslash B_0$, $\exists \{V_i\}_{i \in I}$ s.t. (i) $|I| < \infty$, (ii) $\bigcap_{i \in I} V_i = \{x\}$
(iii) $\bigcup_{i \in I} V_i \ni x$: open, (iv) $\exists u_i, v_i \in \mathbb{R}^2$,
$U_i \backslash \{x\} \simeq \{r u_i + s v_i \in \mathbb{R}^2 | 0 \leqslant r, s \text{ and } r + s < 1\} \backslash \{(0,0)\}$,
(v) $\forall i \neq j \in I$, $(V_i \cap V_j) \backslash \{x\} \simeq (0, r)$ or $\varnothing$.

Example

$B_0 = \mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathbb{R} \hookrightarrow \mathbb{C}P^1 = B$ does not satisfy the above Condition.

3. Main result

B: a TASS, $\{(U_i, \psi_i)\}$: an affine atlas of B_0.
Aff_{B_0}: the sheaf of locally affine functions.
$\mathit{Aff}_{\mathbb{Z},B_0}$: the sheaf of locally affine functions with integer slopes.
$\mathcal{T}_{B_0}$: the locally constant sheaf associated with the local trivialization $\theta_i : TU_i \to U_i \times \mathbb{R}^2; (x, v) \mapsto (x, d\psi_{i,x}(v))$.
$\mathcal{T}_{\mathbb{Z},B_0}$: the subsheaf of $\mathcal{T}_{B_0}$ consisting of integer valued vectors.
$\mathcal{T}^{\vee}_{\mathbb{Z},B_0} := \mathcal{H}om_{\mathbb{Z}_{B_0}}(\mathcal{T}_{\mathbb{Z},B_0}, \mathbb{Z}_{B_0})$, $\mathcal{T}^{\vee}_{B_0} := \mathcal{H}om_{\mathbb{R}_{B_0}}(\mathcal{T}_{B_0}, \mathbb{R}_{B_0})$.

$$0 \to \mathbb{R}_B \to \iota_* \mathit{Aff}_{\mathbb{Z},B_0} \to \iota_* \mathcal{T}^{\vee}_{\mathbb{Z},B_0} \to 0 : \text{ exact}$$

$c_B \in H^1(B, \mathcal{H}om_{\mathbb{Z}_B}(\iota_* \mathcal{T}^{\vee}_{\mathbb{Z},B_0}, \mathbb{R}_B)) \subset \mathrm{Ext}^1(\iota_* \mathcal{T}^{\vee}_{\mathbb{Z},B_0}, \mathbb{R}_B)$
: the extension class of the above exact sequence.
$\mathcal{H}om_{\mathbb{Z}_B}(\iota_* \mathcal{T}^{\vee}_{\mathbb{Z},B_0}, \mathbb{R}_B) \simeq \iota_* \mathcal{T}_{B_0}$,
c_B is called the **radiance obstruction** of B.
$c_1 : H^1(B, \iota_* \mathit{Aff}_{\mathbb{Z},B_0}) \to H^1(B, \iota_* \mathcal{T}^{\vee}_{\mathbb{Z},B_0})$, $\mathrm{Im}\, c_1 = \{c_B \cup d = 0\}$.
B is a *special TASS* if $\forall \psi_i \circ \psi_j^{-1} \in \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{R}^2$.
$\cup : H^1(B, \iota_* \mathcal{T}_{\mathbb{Z},B_0})^{\otimes 2} \to H^2(B, \iota_*(\wedge_{i=1}^2 \mathcal{T}_{\mathbb{Z},B_0})) \simeq \mathbb{Z}$.

Lemma (cf. Degtyarev'12)

If B is a compact special TASS, then $H^1(B, \iota_* \mathcal{T}_{\mathbb{Z},B_0})_{\mathrm{free}}$ is an even lattice.

$B(\mathbb{Z})$: the set of integer points of an integral TASS B.

Theorem (T.)

$q : T \to T/-1_T = \mathrm{Km}(T)$
(1) $\exists q_* : H^1(T, \mathcal{T}_{\mathbb{Z},T}) \xrightarrow{\sim} H^1(\mathrm{Km}(T), \iota_* \mathcal{T}_{\mathbb{Z},\mathrm{Km}(T)_0})$,
$q_*(D) \cup q_*(D) = 2D \cup D$, $(q_* \otimes_{\mathbb{Z}} \mathbb{R})(c_T) = 2c_{\mathrm{Km}(T)}$,
$q_*((\mathrm{Im}\, c_{1,T})^{\perp}) = (\mathrm{Im}\, c_{1,\mathrm{Km}(T)})^{\perp}$.
In particular, $H^1(\mathrm{Km}(T), \iota_* \mathcal{T}_{\mathbb{Z},\mathrm{Km}(T)_0}) \simeq U(2)^{\oplus 2}$.
(2) If $c_{\mathrm{Km}(T)} \in H^1(\mathrm{Km}(T), \iota_* \mathcal{T}_{\mathbb{Z},\mathrm{Km}(T)_0})$, *then*
$\lvert \mathrm{Km}(T)(\mathbb{Z}) \rvert = \frac{c_{\mathrm{Km}(T)} \cup c_{\mathrm{Km}(T)}}{2} + 2 = \mathrm{vol}(\mathrm{Km}(T)) + \chi_{\mathrm{top}}(\mathrm{Km}(T))$.

Remark

(i) (2) is true for every compact special TASS admitting an integer affine triangulation of B (cf. Kontsevich-Soibelman).
(ii) L: an ample line bundle on an complex K3 surface X.
$\dim \Gamma(X, L) = \int_X \frac{c_1(L)^2}{2} + 2$.
(iii) X_{Σ}: a projective toric variety, D: an ample torus-invariant divisor on X_{Σ}, P_D: the Newton polytope of D.
$|P_D(\mathbb{Z})| = \dim \Gamma(X_{\Sigma}, \mathcal{O}_{X_{\Sigma}}(D))$ (e.g. Pick's formula)

4. TASS with boundary and wallpaper groups

Theorem (Classical result for Abelian surfaces)

*$T_{\mathbb{C}}$: an Abelian surface, G: a finite subgroup of $\mathrm{Aut}(T_{\mathbb{C}})$.
The minimal model of $T_{\mathbb{C}}/G$ is either Abelian, bielliptic, K3, Enriques, ruled, or rational.*

There exists its tropical analog as follows;
$T = \mathbb{R}^2/\Lambda$: a tropical 2-torus
G: a finite subgroup of the affine automorphisms group of T. T/G has the canonical structure of TASS with boundary.

Example (Known facts)

17 flat (parabolic) 2-orbifolds $\mathbb{R}^2/\mathfrak{C}$ $\Longleftrightarrow$ 17 wallpaper groups $\mathfrak{C}$.
flat tori $\Longleftrightarrow$ principally polarized tropical abelian varieties

flat 2-orbifolds	tropical —
T (2-torus)	Abelian surface
K (Klein bottle)	bielliptic surface
$S^2(2,2,2,2) = T/-1_T = \mathrm{Km}(T)$	Kummer surface
$S^2(3,3,3)$, $S^2(2,4,4)$, $S^2(2,3,6)$	Kummer type K3 surface
$\mathbb{R}P^2(2,2)$	Enriques surface
$[0,1] \times S^1$ (Cylinder)	$\mathbb{T}P^1 \times S^1$
$(S^1)^2/\mathfrak{S}_2$ (Möbius band)	$\mathbb{T}P^1$-bundle
$D^2(a,b,c;p,q,r)$ (closed disk)	rational surface

Table: flat 2-orbifolds v.s. TASS with boundary