Kawaguchi-Silverman conjecture for endomorphisms on rationally connected varieties admitting an int-amplified endomorphism

Shou Yoshikawa (j.w.w. Yohsuke Matsuzawa)

Graduate School of Mathematical Sciences, The University of Tokyo

Conjecture and Main Result

Let X be a smooth projective variety and $f\colon X \to X$ a surjective morphism, both defined over $\overline{\mathbb{Q}}$.

Arithmetic degree & Dynamical degree

1. Let H be an ample divisor on X. Fix a Weil height function h_H associated with H. The *arithmetic degree* $\alpha_f(x)$ of f at $x \in X(\overline{\mathbb{Q}})$ is defined by
$$\alpha_f(x) = \lim_{n\to\infty} \max\{1, h_H(f^n(x))\}^{1/n}.$$
2. The *dynamical degree* is defined by
$$\delta_f = \lim_{n\to\infty} ((f^n)^*H \cdot H^{\dim X-1})^{1/n},$$
where H is any nef and big Cartier divisor on X.

The above definitions are independent on the choice of H.

The Arithmetic degree measures the arithmetic complexity of f-orbits by means of Weil height functions and the dynamical degree measures the geometric complexity of the dynamical system. Kawaguchi and Silverman proposed the following conjecture.

Conjecture 1 ([KS1])

Let $x \in X(\overline{\mathbb{Q}})$. If the orbit $O_f(x) = \{f^n(x) \mid n = 0, 1, 2, \dots\}$ is Zariski dense in X, then $\alpha_f(x) = \delta_f$.

This conjecture is called Kawaguchi-Silverman conjecture. It holds if X is a surface, X is an abelian variety or f is polarized.

An endomorphism $\Phi\colon Y \to Y$ on a normal projective variety is int-amplified if there exists an ample divisor H on Y such that $\Phi^*H - H$ is ample. For example, polarized endomorphisms are int-amplified, and every toric variety has an int-amplified endomorphism.

Main Results (Matsuzawa-Y)

Conjecture 1 holds if X is rationally connected and has an int-amplified endomorphism.

Strategy

One strategy to prove the conjecture for higher dimensional algebraic varieties is to use minimal model program (MMP, for short) and reduce the problem to the problem on relatively easier varieties. In order to attack Conjecture 1, we use the following theorem.

Equivariant MMP ([MZ1])

If X has an int-amplified endomorphism, then every MMP is f-equivarinat up to replacing f with some iterate.

Let $X \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_r \to Y$ be an f^n-equivariant MMP for X for some $n \in \mathbb{Z}_{\geq 0}$, where for every i, $X_i \dashrightarrow X_{i+1}$ is a flip or a divisorial contraction and $X_r \to Y$ is a Mori fiber space. One can easily check that Conjecture 1 for f is reduced to it for $f^n|_{X_r}$. A main difficulty of the proof of the main theorem is to deal with the case where Conjecture 1 for $f^n|_{X_r}$ is not reduced to it for $f|_Y$. We actually prove that such a Mori fiber space do not appear if X is rationally connected (see, Key Lemma 1,2) by using the following condition.

Condition $(*)$

Let Y be a normal projective variety and $\Phi\colon Y \to Y$ a surjective endomorphism. We say Φ satisfies $(*)$ if the following holds. There exits a following equivariant commutative diagram

$$\begin{array}{ccc} \tilde{\Phi} \circlearrowright \tilde{Y} & \xrightarrow{\tilde{\pi}} & A \circlearrowleft \Psi_A \\ \mu_Y \downarrow & & \downarrow \mu_Z \\ \Phi \circlearrowright Y & \xrightarrow[\pi]{} & Z \circlearrowleft \Psi, \end{array}$$

where

1. Z is a normal projective variety of positive dimension, A is an abelian variety,
2. π is an algebraic fiber space, μ_Z is a finite surjective morphism,
3. $\tilde{Y}$ is the normalization of the main component of $Y \times_Z A$,
4. μ_Y is a finite surjective étale in codimension one morphism, $\tilde{\pi}$ is an algebraic fiber space,
5. Ψ, Ψ_A and $\tilde{\Phi}$ are surjective endomorphisms.

Proof of the main theorem

We prove Conjecture 1 for endomorphisms on varieties with mild singularities admitting an int-amplified endomorphism not satisfying $(*)$ by the induction on the sum of the dimension and the Picard rank. We remark that every endomorphism on X does not satisfy $(*)$ if X is smooth rationally connected. By the following Lemma, every int-amplified endomorphism does not satisfy $(*)$ induced by a MMP for every variety admitting an int-amplified endomorphism not satisfying $(*)$.

Key Lemma 1 (Matsuzawa-Y)

Let $\pi\colon Y \to Z$ be a Φ-equivariant flip, divisorial contraction or Mori fiber space, where Φ is an int-amplified endomorphism on Y. If Φ does not satisfy $(*)$, then $\Phi|_Z$ does not satisfy $(*)$.

Let $g, \Phi\colon Y \to Y$ be endomoprhisms, where Φ is int-amplified and $\pi\colon Y \to Z$ a Φ and g-equivariant Mori fiber space. By [MZ2, Proposition 9.2], if the ramification divisor R_Φ of Φ contains some vertical component of π, then Conjecture 1 for g follows from it for $g|_{Z'}$, where Z' is an output of some MMP for Y and the Picard rank or the dimension of Z' is smaller than it of Y. Hence, by the induction hypothesis, it is enough to show that R_Φ contains some vertical component if Φ does not satisfy $(*)$ of π. It follows from the next covering lemma.

Key Lemma 2 (Y)

If R_Φ does not contain any vertical component of π, then Φ satisfies $(*)$.

[KS1] Kawaguchi, S., Silverman, J. H., *On the dynamical and arithmetic degrees of rational self-maps of algebraic varieties*, J. Reine Angew. Math. **713** (2016), 21–48.

[MZ1] Meng, S. and Zhang, D.-Q., *Semi-group structure of all endomorphisms of a projective variety admitting a polarized endomorphism*, arXiv:1806.05828.

[MZ2] S. Meng, D.-Q. Zhang, *Kawaguchi-Silverman conjecture for surjective endomorphisms*, arXiv:1908.01605