Curves whose automorphism groups are $\mathfrak{A}_6$

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Introduction

We study automorphism groups of projective plane curves over $\mathbb{C}$.
Recently, Harui gave a classification of automorphism groups of smooth curves $C \subset \mathbb{P}^2$ of degree ≥ 4 ([1]). Roughly, the classification is as follows:

Harui's classification of automorphism groups

Let $d \geq 4$.
C : smooth projective plane curve $/\mathbb{C}$ of degree d
Then C satisfies one of the following:
(a) $\mathrm{Aut}\, C$ fixes a point on $\mathbb{P}^2$,
(b) $(C, \mathrm{Aut}\, C)$ is a "descendant" of a Fermat curve or a Klein curve,
(c) $\mathrm{Aut}\, C$ is a finite "primitive" subgroup of $\mathrm{PGL}(3, \mathbb{C})$.

Note that an automorphism of such a curve C extends uniquely to an automorphism of $\mathbb{P}^2$.
If $G < \mathrm{PGL}(3, \mathbb{C})$ is **primitive**, then $G \cong \mathfrak{A}_6, \mathrm{PSL}(2, \mathbb{F}_7), \mathfrak{A}_5, H_{216}, H_{72}$ or H_{36}.
We consider the case $G \cong \mathfrak{A}_6$ in this poster.

Question

For which degree does there exist a smooth curve C such that $\mathrm{Aut}\, C \cong \mathfrak{A}_6$?

Valentiner Group

We give a definition to the "Valentiner group" $\mathcal{V}$ according to [2].
Let $\mathcal{V}$ be the subgroup of $\mathrm{PGL}(3, \mathbb{C})$ generated by the equivalence classes of

$$Z := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, T := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, Q := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \rho \\ 0 & -\rho^2 & 0 \end{pmatrix}, P := \frac{1}{2}\begin{pmatrix} 1 & \tau^{-1} & -\tau \\ \tau^{-1} & \tau & 1 \\ \tau & -1 & \tau^{-1} \end{pmatrix}$$

where $\rho = e^{\frac{2}{3}\pi i} = \frac{-1+\sqrt{3}i}{2}$ and $\tau = \frac{1+\sqrt{5}}{2}$.
$\mathcal{V}$ is called the **Valentiner group** and is isomorphic to $\mathfrak{A}_6$.
There is an extension $\overline{\mathcal{V}} < \mathrm{SU}(3, \mathbb{C})$ of $\mathcal{V}$ by $\mathbb{Z}/3\mathbb{Z}$.
We have the following facts:

- The Poincaré series of $\overline{\mathcal{V}}$ is $P(t) = \dfrac{1 - t^{90}}{(1-t^6)(1-t^{12})(1-t^{30})(1-t^{45})}$, and the invariant ring of $\overline{\mathcal{V}}$ is generated by homogeneous polynomials of degrees $6, 12, 30, 45$.
- Since $\mathfrak{A}_6$ is simple, a $\mathcal{V}$-invariant curves is defined by a homogeneous element of $\mathbb{C}[x, y, z]^{\overline{\mathcal{V}}}$.

Invariant polynomials

The icosahedral group $\mathcal{I} := \langle [Z], [T], [P] \rangle (\cong \mathfrak{A}_5)$ fixes the conic defined by

$$c_1(x, y, z) := x^2 + y^2 + z^2.$$

We define five conics invariant under $\mathcal{V}$:

$$c_2(x, y, z) := c_1(Q^{-1}\boldsymbol{x}),\ c_3(x, y, z) := c_2(P^{-4}\boldsymbol{x}),\ c_4(x, y, z) := c_2(P^{-3}\boldsymbol{x}),$$
$$c_5(x, y, z) := c_1(P^{-2}\boldsymbol{x}),\ c_6(x, y, z) := c_2(P^{-1}\boldsymbol{x})$$

where $\boldsymbol{x} = {}^t(x, y, z)$.
We define $\mathcal{V}$-invariant homogeneous polynomials F, Φ, Ψ, X by

$$F(x, y, z) := \sum_{i=1}^{6} c_i(x, y, z)^3,$$

$$\Phi(x, y, z) := \det H(F)(x, y, z) := \begin{vmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} & \frac{\partial^2 F}{\partial x \partial z} \\ \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} & \frac{\partial^2 F}{\partial y \partial z} \\ \frac{\partial^2 F}{\partial z \partial x} & \frac{\partial^2 F}{\partial z \partial y} & \frac{\partial^2 F}{\partial z^2} \end{vmatrix},$$

$$\Psi(x, y, z) := \det BH(F, \Phi) := \begin{vmatrix} & H(\Phi) & & \frac{\partial F}{\partial x} \\ & & & \frac{\partial F}{\partial y} \\ & & & \frac{\partial F}{\partial z} \\ \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} & 0 \end{vmatrix},$$

$$X(x, y, z) := \det J(F, \Phi, \Psi) := \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial \Phi}{\partial x} & \frac{\partial \Phi}{\partial y} & \frac{\partial \Phi}{\partial z} \\ \frac{\partial \Psi}{\partial x} & \frac{\partial \Psi}{\partial y} & \frac{\partial \Psi}{\partial z} \end{vmatrix}.$$

Invariant ring of the Valentiner group

The invariant ring of $\overline{\mathcal{V}}$ is $\mathbb{C}[x, y, z]^{\overline{\mathcal{V}}} = \mathbb{C}[F, \Phi, \Psi, X]$.

We may drop X by the following remark.

Remark on X

- $V(X)$ consists of 45-lines.
- $X^2 \in \mathbb{C}[F, \Phi, \Psi]$.

Hence, irreducible homogeneous invariant polynomials belong to $\mathbb{C}[F, \Phi, \Psi]$.

Smooth invariant curves

Let $\mathfrak{d}_d := \mathbb{C}[F, \Phi, \Psi]_d = \langle F^i \Phi^j \Psi^k \mid 6i + 12j + 30k = d \rangle$.
By Bertini's theorem, we have only to care about singularities on the base points of $\mathfrak{d}_d$ to find nonsingular curves of degree d.
The base points of $\mathfrak{d}_d$ are on one of the six sets

$V(F)$, $V(\Phi)$, $V(\Psi)$, $V(F) \cap V(\Phi)$, $V(F) \cap V(\Psi)$ and $V(\Phi) \cap V(\Psi)$.

We obtain the following facts about F, Φ and Ψ by explicit calculation.

Curves defined by F, Φ, Ψ

- $V(F, \Phi, \Psi) = \emptyset$,
- $V(F)$ is nonsingular,
- $V(\Phi)$ is nonsingular,
- $V(\Psi)$ is nonsingular at least on $V(\Psi) \cap (V(F) \cup V(\Phi))$.

Smoothness at each of the six sets can be translated to a condition on degrees.

Conditions on degrees for nonsingularity

Assume $d \geq 30$.
A general element of $\mathfrak{d}_d$ is nonsingular if and only if d satisfies the following:

(1) $\exists (j, k)$ s.t. $d = 12j + 30k$, ($\longleftrightarrow$ (base locus) $\not\supseteq V(F)$)
(2) $\exists (i, k)$ s.t. $d = 6i + 30k$, ($\longleftrightarrow$ (base locus) $\not\supseteq V(\Phi)$)
(3) $\exists (i, j)$ s.t. $d = 6i + 12j$, ($\longleftrightarrow$ (base locus) $\not\supseteq V(\Psi)$)
(4) $d \equiv 0, 6, 12 \mod 30$, ($\longleftrightarrow$ smoothness at $V(F) \cap V(\Phi)$)
(5) $d \equiv 0, 6, 30 \mod 12$, ($\longleftrightarrow$ smoothness at $V(F) \cap V(\Psi)$)
(6) $d \equiv 0, 12, 30 \mod 6$. ($\longleftrightarrow$ smoothness at $V(\Phi) \cap V(\Psi)$)

Simplifying the conditions (1)-(6), we obtain the following.

Main result on smooth curves

There exists a smooth curve C of degree d such that $\mathrm{Aut}\, C \cong \mathfrak{A}_6$

$$\Updownarrow$$

$$d \equiv 0, 6, 12 \mod 30.$$

Note that the automorphism group of a smooth curve which is invariant under $\mathfrak{A}_6$ is isomorphic to $\mathfrak{A}_6$ by Harui's theorem since it is primitive.

Irreducible invariant curves

In addition to nonsingular curves, we also study irreducible curves in $\mathfrak{d}_d$.
First, from conditions on nonsingularity, if a general element of $\mathfrak{d}_d$ is singular, then it is singular only on $V(F) \cap V(\Phi)$.
Furthermore, we know the following fact about $V(F)$ and $V(\Phi)$.

Intersection of $V(F)$ and $V(\Phi)$

$V(F)$ and $V(\Phi)$ meet transversally at 72-points.

Thus F and Φ can be considered as two independent variables in a neighborhood of each point of $V(F) \cap V(\Phi)$. Note that Ψ is nonzero on $V(F) \cap V(\Phi)$.

Singularity on $V(F) \cap V(\Phi)$

Assume $d \equiv 18, 24 \mod 30$.
A general element $C \in \mathfrak{d}_d$ is singular only at $V(F) \cap V(\Phi)$.
Moreover,

- $d \equiv 18 \mod 30 \Rightarrow$ the singularities of a general element $C \in \mathfrak{d}_d$ are nodes.
- $d \equiv 24 \mod 30 \Rightarrow$ the singularities of a general element $C \in \mathfrak{d}_d$ are tacnodes.

We assume that a general element C of $\mathfrak{d}_d$ is reducible.
Let $C_1, \cdots, C_n$ be the irreducible component of C.
For example, if $d = 78$, then their degrees satisfy

$$\sum_{i<j} \deg C_i \deg C_j \geq \deg C_1 \sum_{j=2}^{n} \deg C_j = \deg C_1 (78 - \deg C_1).$$

However, the left hand side is $72 (= \#(V(F) \cap V(\Phi)))$.
There is no such positive integer $\deg C_1$, a contradiction.

Main result on irreducible curves

there exists an irreducible curve of degree d invariant under $\mathfrak{A}_6$

$$\Updownarrow$$

$6 | d$ and $d \neq 18$ and $d \neq 24$.

We also determined the degrees of smooth (or irreducible) curves invariant under $\mathrm{PSL}(2, \mathbb{F}_7)$ and $\mathfrak{A}_5$ in a similar way.

[1] Harui, Takeshi, *Automorphism groups of smooth plane curves*. Kodai Math. J. 42(2019), no. 2, 308-331, 2019.
[2] Crass, Scott, *Solving the sextic by iteration: a study in complex geometry and dynamics*. Experiment. Math. 8 (1999), no. 3, 209–240, 1999