The moduli space of polynomial maps and their fixed-point multipliers

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1 Setting of the Problem

$$\mathrm{Poly}_d := \{f(z) \in \mathbb{C}[z] \mid \deg f = d\} \quad \text{for} \quad d \geq 2$$
$$\mathrm{Fix}(f) := \{\zeta \in \mathbb{C} \mid f(\zeta) = \zeta\} \quad \text{for} \quad f \in \mathrm{Poly}_d$$

Then $\#\mathrm{Fix}(f) = \deg f$ holds counted with multiplicity. For $\zeta \in \mathrm{Fix}(f)$, we call $f'(\zeta)$ the multiplier of f at ζ.

Putting $\mathrm{MP}_d := \mathrm{Poly}_d/(\text{Affine conj.})$, one can define the map

$$\mathrm{MP}_d \to \mathbb{C}^d/\mathfrak{S}_d \quad \text{by} \quad [f] \mapsto (f'(\zeta))_{\zeta\in\mathrm{Fix}(f)},$$

where $\mathbb{C}^d/\mathfrak{S}_d$ is the set of unordered collections of d complex numbers.

Proposition (fixed point theorem). *Let $f \in \mathrm{Poly}_d$ and suppose $f'(\zeta) \neq 1$ for every $\zeta \in \mathrm{Fix}(f)$, then*

$$\sum_{\zeta\in\mathrm{Fix}(f)} \frac{1}{1 - f'(\zeta)} = 0.$$

Hence putting

$$\Lambda_d := \left\{ (\lambda_1, \dots, \lambda_d) \in \mathbb{C}^d \;\middle|\; \sum_{i=1}^{d} \prod_{j\neq i} (1 - \lambda_j) = 0 \right\},$$

one can define the map

$$\Phi_d : \mathrm{MP}_d \to \Lambda_d/\mathfrak{S}_d =: \widetilde{\Lambda}_d \quad \text{by} \quad [f] \mapsto (f'(\zeta))_{\zeta\in\mathrm{Fix}(f)}.$$

Our aim is to analyze the map Φ_d as precisely as possible.

2 Results

We consider the map $\Phi_d : \mathrm{MP}_d \to \widetilde{\Lambda}_d$ on the domain where $[f] \in \mathrm{MP}_d$ has no multiple fixed point. Put

- $V_d := \{(\lambda_1, \dots, \lambda_d) \in \Lambda_d \mid \lambda_i \neq 1 \text{ for every } i\}$
- $\widetilde{V}_d := V_d/\mathfrak{S}_d \subset \widetilde{\Lambda}_d$.

Theorem 1 (S. [1]). *$d \geq 4$ and $\lambda = (\lambda_1, \dots, \lambda_d) \in V_d$, $\bar{\lambda} = \{\lambda_1, \dots, \lambda_d\} \in \widetilde{V}_d$. Then*

(1) $0 \leq \#\Phi_d^{-1}(\bar{\lambda}) \leq (d-2)!$

(2) $\#\Phi_d^{-1}(\bar{\lambda})$ *is a function of*

$$\mathfrak{I}(\lambda) := \left\{ \{I_1, \dots, I_\ell\} \;\middle|\; \begin{array}{c} \ell \geq 2, \quad \emptyset \neq I_u \subset \{1, \dots, d\}, \\ I_1 \amalg \cdots \amalg I_\ell = \{1, \dots, d\}, \\ \sum_{i\in I_u} 1/(1 - \lambda_i) = 0 \textit{ for every } u \end{array} \right\}$$

and

$$\mathcal{K}(\lambda) := \left\{ K \subset \{1, \dots, d\} \;\middle|\; \begin{array}{c} K \neq \emptyset \\ i, j \in K \Longrightarrow \lambda_i = \lambda_j \\ i \in K,\ j \notin K \Longrightarrow \lambda_i \neq \lambda_j \end{array} \right\}.$$

(3) $\mathfrak{I}(\lambda) \subseteq \mathfrak{I}(\lambda')$ *and "$\mathcal{K}(\lambda)$ is a refinement of $\mathcal{K}(\lambda')$" for* $\lambda, \lambda' \in V_d \implies \#\Phi_d^{-1}(\bar{\lambda}) \geq \#\Phi_d^{-1}(\bar{\lambda}')$

(4) $\#\Phi_d^{-1}(\bar{\lambda}) = (d-2)!$
$\iff \mathfrak{I}(\lambda) = \emptyset,\ \mathcal{K}(\lambda) = \{\{1\}, \dots, \{d\}\}$

(5) $\exists c_1, \dots, c_d \in \mathbb{Z}$ *with* $c_1 : \cdots : c_d = \frac{1}{1-\lambda_1} : \cdots : \frac{1}{1-\lambda_d}$ *and* $\sum |c_i| \leq 2(d-2) \implies \Phi_d^{-1}(\bar{\lambda}) = \emptyset$

(6) $d \leq 7 \implies$ *converse of (5) holds.*

(7) *For $\lambda \in V_d$, if $\exists c_1, \dots, c_d \in \mathbb{Z}$ such that*
- $c_1 : \cdots : c_d = \frac{1}{1-\lambda_1} : \cdots : \frac{1}{1-\lambda_d}$
- $\sum |c_i| = 2(d-1)$
- $|c_i| \leq 2$ *for every* $i \in \{1, \dots, d\}$,

then $\#\Phi_d^{-1}(\bar{\lambda}) = 1$.

Theorem 2 (S. [1], (roughly)). *The local fiber structure of $\Phi_d : \Phi_d^{-1}(\widetilde{V}_d) \to \widetilde{V}_d$ is also determined by $\mathfrak{I}(\lambda)$ and $\mathcal{K}(\lambda)$. Moreover*

- *$\mathfrak{I}(\lambda)$ stands for the configuration of 'holes' of Φ_d.*
- *$\mathcal{K}(\lambda)$ stands for the configuration of 'branches' of Φ_d.*

Theorem 3 (S. [1]). *For $\lambda \in V_d$, $\#\Phi_d^{-1}(\bar{\lambda})$ is calculated by the following steps:*

- *we define the non-negative integers $s_d(\lambda)$ for each $\lambda \in V_d$ and $e_{\mathbb{I}}(\lambda)$ for each $\mathbb{I} \in \mathfrak{I}(\lambda)$ satisfying the equalities*
  - ▶ $s_d(\lambda) = (d-2)! - \sum_{\mathbb{I}\in\mathfrak{I}(\lambda)} \left\{ e_{\mathbb{I}}(\lambda) \cdot \prod_{k=d-\#\mathbb{I}+1}^{d-2} k \right\}$,
  - ▶ $e_{\mathbb{I}}(\lambda) = \prod_{I\in\mathbb{I}} (\#I - 1) \cdot s_{\#I}(\lambda_I)$,
    where $\lambda_I := (\lambda_i)_{i\in I} \in V_{\#I}$ *for* $I \in \mathbb{I} \in \mathfrak{I}(\lambda)$.
- *Then in most cases (omit precise description),*
$$\#\Phi_d^{-1}(\bar{\lambda}) = s_d(\lambda) / \prod_{K\in\mathcal{K}(\lambda)} (\#K)!.$$

Theorem 4 (S. [2]). *$s_d(\lambda)$ defined in Theorem 3 is also obtained from the equality*

$$s_d(\lambda) = (d-2)! - \sum_{\mathbb{I}\in\mathfrak{I}(\lambda)} \left[\{-(d-1)\}^{\#\mathbb{I}-2} \cdot \prod_{I\in\mathbb{I}} (\#I - 1)! \right].$$

Put $\mathrm{MC}_d := \left\{ f(z) = z^d + \sum_{k=0}^{d-2} a_k z^k \;\middle|\; a_k \in \mathbb{C} \right\}$. The natural map $p : \mathrm{MC}_d \to \mathrm{MP}_d$ is generically $(d-1)$-to-one. $\widehat{\Phi}_d := \Phi_d \circ p : \mathrm{MC}_d \to \widetilde{\Lambda}_d$ is defined by $f \mapsto (f'(\zeta))_{\zeta\in\mathrm{Fix}(f)}$.

Theorem 5 (S. [2]). *For $d \geq 2$ and $\lambda \in V_d$, we always have*

$$\#\widehat{\Phi}_d^{-1}(\bar{\lambda}) = \frac{(d-1)s_d(\lambda)}{\prod_{K\in\mathcal{K}(\lambda)} (\#K)!}.$$

References

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