Arakelov geometry over a trivially valued field

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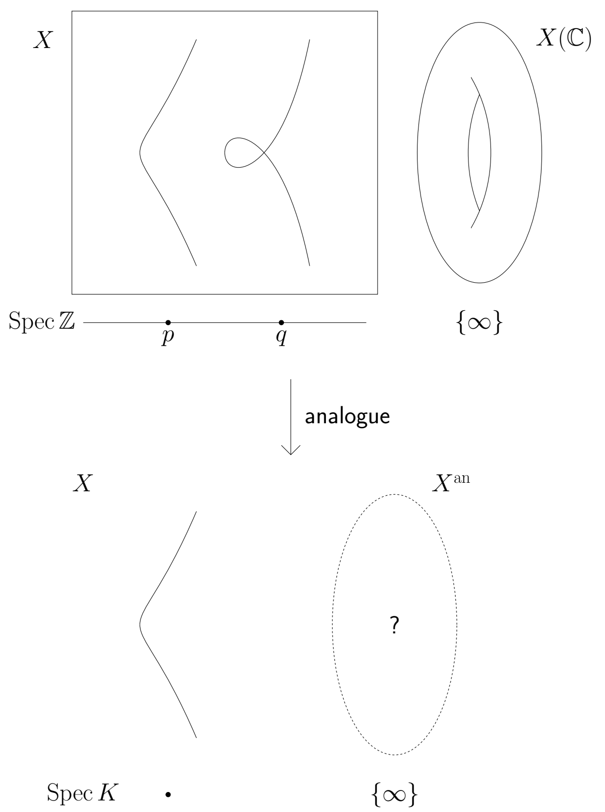
Abstract

We introduce Arakelov theory over a trivially valued field, mainly treat the volume function of adelic Cartier divisors. We show several properties of the volume function, for example, the integral formula, continuity, concavity and so on.

Arakelov geometry

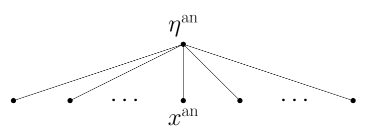
Arakelov geometry is a kind of arithmetic geometry. Beyond scheme theory, it has been developed to study a system of equations with integer coefficients. In some sense, it is an extension of Diophantine geometry.
An arithmetic variety X is a scheme over $\operatorname{Spec}\mathbb{Z}$ which is equipped with a (complex) analytic space $X(\mathbb{C})$ as a fiber of the infinite point.
Hence X consists of two parts, geometric part and analytic part:



A trivially valued field is a field K equipped with the trivial absolute value.
In Arakelov geometry over a trivially valued field, an "arithmetic variety" X consists of a scheme over $\operatorname{Spec}K$ and an analytic space X^{an} associated with the trivial absolute value. This analytic space X^{an} is given by Berkovich space.

Berkovich space

Let X be a scheme over a field K and $|\cdot|$ be an absolute value on K. We assume that $(K, |\cdot|)$ is complete.
Roughly speaking, the Berkovich space associated with X is the set of pairs $(x, |\cdot|_x)$ where $x \in X$ and $|\cdot|_x$ is an absolute value on $\kappa(x)$ over $|\cdot|$.
Let K be a trivially valued field. If X is a regular projective curve over $\operatorname{Spec}K$, we can illustrate the Berkovich space X^{an} by an infinite tree as follows:



where η is the generic point of X, x is a closed point of X and $x^{\mathrm{an}} = (x, |\cdot|_0)$.

Volume function

In algebraic geometry, to determine the set of global sections $H^0(X, D)$ of a Cartier divisor D is difficult in general:

$$H^0(X, D) := \{f \in K(X)^\times \mid D + (f) \geq 0\} \cup \{0\}.$$

However, to calculate the asymptotic behavior of $H^0(X, nD)$ is the easier problem. Hence the volume function of D is well studied:

$$\operatorname{vol}(D) := \limsup_{n\to+\infty} \frac{\dim H^0(X, nD)}{n^d/d!},$$

where $d = \dim X$.

On the other hand, in Arakelov geometry, we equip a Cartier divisor D with an analytic object g, which is called a Green function, like an arithmetic variety:

$$\overline{D} := (D, g).$$

This pair is called an adelic Cartier divisor.
In this setting, we also want to consider the set of "global sections" of $\overline{D}$:

$$H^0(X, \overline{D}) := \left\{ f \in K(X)^\times \,\middle|\, \overline{D} + \widehat{(f)} \geq 0 \right\} \cup \{0\}.$$

What is the relationship between $H^0(X, D)$ and $H^0(X, \overline{D})$?

By using a Green function g, we can introduce an ultrametric norm $\|\cdot\|_g$ on $H^0(X, D)$. Then we have the following statement:

$$H^0(X, \overline{D}) = \left\{ s \in H^0(X, D) \,\middle|\, \|s\|_g \leq 1 \right\}.$$

Hence an element of $H^0(X, \overline{D})$ is called a small section.

By analogy with $\operatorname{vol}(D)$, the volume function $\widehat{\operatorname{vol}}(\overline{D})$ estimates the asymptotic behavior of amounts of small sections:

$$\widehat{\operatorname{vol}}(\overline{D}) := \limsup_{n\to+\infty} \frac{\widehat{\deg}_+(n\overline{D})}{n^{d+1}/(d+1)!},$$

where $d = \dim X$.
Here is the definition of $\widehat{\deg}_+(\overline{D})$:

$$\widehat{\deg}_+(\overline{D}) := \int_0^{+\infty} \dim\left\{ s \in H^0(X, D) \,\middle|\, \|s\|_g \leq e^{-t} \right\} dt.$$

Main theorem (O.)

1. (integral formula). $\widehat{\operatorname{vol}}(\overline{D}) = (d+1)\displaystyle\int_0^{\lambda_{\max}^{\mathrm{asy}}(\overline{D})} \operatorname{vol}(D_{\mu(g-t)})\,dt$,
   where $D_{\mu(g-t)}$ is an $\mathbb{R}$-Weil divisor associated with $(D, g-t)$.
2. (limit existence). $\widehat{\operatorname{vol}}(\overline{D}) = \displaystyle\lim_{n\to+\infty} \frac{\widehat{\deg}_+(n\overline{D})}{n^{d+1}/(d+1)!}$.
3. (continuity). $\displaystyle\lim_{\epsilon\to 0} \widehat{\operatorname{vol}}(\overline{D} + \epsilon\overline{E}) = \widehat{\operatorname{vol}}(\overline{D})$.
4. (homogeneity). $\forall a \in \mathbb{R}_{>0}$, $\widehat{\operatorname{vol}}(a\overline{D}) = a^{d+1}\widehat{\operatorname{vol}}(\overline{D})$.
5. (log concavity). $\widehat{\operatorname{vol}}(\overline{D} + \overline{E})^{\frac{1}{d+1}} \geq \widehat{\operatorname{vol}}(\overline{D})^{\frac{1}{d+1}} + \widehat{\operatorname{vol}}(\overline{E})^{\frac{1}{d+1}}$.

Bibliography

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