The Sarkisov progaram on log surfaces

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1. Background

We will work over an algebraic closed field of any characteristic.

The goal of the minimal model theory is to find a minimal model or a Mori fiber space by a given projective variety.

We are interested in how uniquely minimal models and Mori fiber spaces, which are the outputs of the same minimal model program, are. We note that in surfaces minimal models are uniquely determined.

Known Results

- Any two birational minimal models of any dimensional $\mathbb{Q}$-factorial terminal pairs are connected by finitely many flops (cf. [K]).
- Any two birational Mori fiber spaces of any dimensional $\mathbb{Q}$-factorial kawamata log terminal pairs are connected by finitely many Sarkisov links (cf. [HM]).

Remark. We emphasize that the arguments above are carried out over the complex number field $\mathbb{C}$.

A *log surface* is a pair (X, Δ) consisting of a normal surface X and a boundary $\mathbb{R}$-divisor Δ on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. In addition, we say that (X, Δ) is a *$\mathbb{Q}$-factorial log surface* if X is $\mathbb{Q}$-factorial.

Fujino and Tanaka established the minimal model theory for $\mathbb{Q}$-factorial log surfaces and log canonical surfaces in full generality ([F1] and [T]).

2. Main Results

Theorem 1 (Main Theorem I)**.** Let (Z, Φ) be a projective $\mathbb{Q}$-factorial log surface. Let $\phi : X \to S$ and $\psi : Y \to T$ be two Mori fiber spaces which are outputs of the $(K_Z + \Phi)$-minimal model program.

Then the induced birational map $\sigma : X \dashrightarrow Y$ is a composition of Sarkisov links.

Theorem 2 (Main Theorem II)**.** Let (Z, Φ) be a projective log canonical surface. Let $\phi : X \to S$ and $\psi : Y \to T$ be two Mori fiber spaces which are outputs of the $(K_Z + \Phi)$-minimal model program.

Then the induced birational map $\sigma : X \dashrightarrow Y$ is a composition of Sarkisov links.

Remark. We emphasize that a $\mathbb{Q}$-factorial log surface (resp. log canonical surface) (X, Δ) is not necessarily log canonical (resp. $\mathbb{Q}$-factorial).

Definition 3 (Sarkisov link)**.** Let (Z, Φ) be a projective $\mathbb{Q}$-factorial log surface (or log canonical surface). Let $\phi : X \to S$ and $\psi : Y \to T$ be two Mori fiber spaces which are outputs of the $(K_Z + \Phi)$-minimal model program.

A *Sarkisov link* $\sigma : X \dashrightarrow Y$ between ϕ and ψ is one of four types:

Type (I)

$$\begin{tikzcd} X' \arrow[r, equal] \arrow[d] & Y \arrow[d, "\psi"] \\ X \arrow[d, "\phi"'] & T \arrow[dl] \\ \mathrm{pt.} & \end{tikzcd}$$

Type (II)

$$\begin{tikzcd} X' \arrow[r, equal] \arrow[d] & Y' \arrow[d] \\ X \arrow[d, "\phi"'] & Y \arrow[d, "\psi"] \\ S \arrow[r, equal] & T. \end{tikzcd}$$

Type (III)

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "\phi"'] & Y' \arrow[d] \\ S \arrow[dr] & Y \arrow[d, "\psi"] \\ & \mathrm{pt.} \end{tikzcd}$$

Type (IV)

$$\begin{tikzcd} X \arrow[rr, equal] \arrow[d, "\phi"'] & & Y \arrow[d, "\psi"] \\ S \arrow[dr] & & T \arrow[dl] \\ & \mathrm{pt.} & \end{tikzcd}$$

Every vertical arrow is an extremal contraction. Moreover if the target is X or Y, then it is a divisorial contraction. The space X' and Y' are realized as the ample model of (Z, Ψ) for some boundary $\mathbb{Q}$-divisor Ψ.

Remark. In fact, S and T appeared in links of Type (I), (III) and (IV) are isomorphic to $\mathbb{P}^1$.

3. Proof of Main Theorem I

Let $(Z.\Phi)$ be a $\mathbb{Q}$-factorial log surface and let V be a finite dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(Z)$ over $\mathbb{Q}$. Then we define

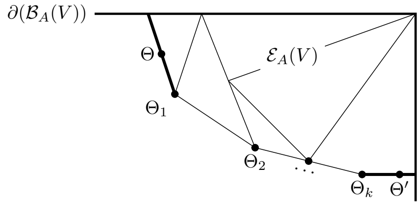
- $V_A = \{\Delta \mid \Delta = A + B, B \in V\}$,
- $\mathcal{B}_A(V) = \{\Delta = A + B \in V_A \mid \Delta \in [0, 1] \text{ and } B \geq 0\}$,
- $\mathcal{E}_A(V) = \{\Delta \in \mathcal{B}_A(V) \mid K_X + \Delta \text{ is pseudo-effective}\}$.

Remark. $\mathcal{B}_A(V)$ and $\mathcal{E}_A(V)$ are rational polytopes.

By the result of [BCHM] with some modifications, there are finitely many contraction morphisms $f_i : Z \to X_i$ $(1 \leq i \leq m)$ such that $\mathcal{A}_{A,f_i}(V) = \{\Delta \in \mathcal{E}_A(V) \mid f_i \text{ is the ample model of } K_X + \Delta \text{ over } U\}$ $(1 \leq i \leq m)$ are a partition of $\mathcal{E}_A(V)$.

Sketch of proof

- Take appropriate ample $\mathbb{Q}$-divisor A and two dimensional affine subspace $V \subset \mathrm{WDiv}_{\mathbb{R}}(Z)$. Then $\phi \circ f$ and $\psi \circ g$ are realized as the ample model for some boundary $\mathbb{Q}$-divisors Θ and Θ', respectively. Moreover $\overline{\mathcal{A}_{\phi \circ f}(V)}$ and $\overline{\mathcal{A}_{\psi \circ g}(V)}$ are one dimensional and contained in the boundary of $\mathcal{E}_A(V)$.
- If we trace the boundary of $\mathcal{E}_A(V)$ contained in the interior in $\mathcal{B}_A(V)$ from Θ to Θ', we obtain finitely many points Θ_i $(1 \leq i \leq k)$.
- By comparing the Picard number of a contraction morphism between ample models, we see that each of these points Θ_i corresponds to a Sarkisov link.



4. Toward log canonical surfaces

The essential difference between $\mathbb{Q}$-factorial log surfaces and log canonical surfaces is that in log canonical surfaces we do not necessarily define the pull-back of divisors by a morphism. However, in our arguments it is sufficitent to be able to define the pull-back of the strict transform of $\mathbb{Q}$-Cartier divisors by a divisorial contraction. This problem is solved by the following definition and theorem and so we can treat $\mathbb{Q}$-factorial log surfaces and log canonical surfaces simultaneously.

Definition 4. Let $f : X \to Y$ be a birational contraction between normal surfaces.

Then we say that f is *special* if there are two Zariski open sets $U_X \subset X$ and $U_Y \subset Y$ such that f induces $U_X \simeq U_Y$ and, U_X and U_Y respectively contain all non-$\mathbb{Q}$-factorial singularities. In particular, the exceptional locus $\mathrm{Exc}(f)$ does not pass through any non-$\mathbb{Q}$-factorial singularities.

Theorem 5 (cf. [F2])**.** Let $f : X \to Y$ be a projective birational morphism between normal surfaces. Let (X, Δ) be log canonical. If $-(K_X + \Delta)$ is f-ample, then every f-exceptional curve is $\mathbb{R}$-Cartier. In particular, all divisorial contractions are special.

References

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