Generalized Zariski cancellation problem and principal $\mathbb{G}_a$-bundles over prevarieties

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Introduction

Generalized Zariski cancellation problem

V, W: varieties over k ($k = \overline{k}$, $\mathrm{char}\, k = 0$).
Then $V \times \mathbb{A}^1 \simeq W \times \mathbb{A}^1 \Rightarrow V \simeq W$?

- $\exists$ counterexamples constructed as principal $\mathbb{G}_a$-bundles $\longrightarrow$

Fact (Danielewski [1])

X: a k-scheme, V, W: affine k-schemes equiped with principal $\mathbb{G}_a$-bundle structures over X. Then $V \times_k \mathbb{A}^1 \simeq W \times_k \mathbb{A}^1$.

Main Problem

X: a prevariety
V: an affine variety with a principal $\mathbb{G}_a$-bundle structure over X
W: an affine variety

$V \times \mathbb{A}^1 \simeq W \times \mathbb{A}^1 \Longleftrightarrow W$: a principal $\mathbb{G}_a$-bundle over X?

Counterexamples for GZCP have been constructed by Danielewski [1], Fieseler [4], Dubouloz [3] and Dryło [2] as principal $\mathbb{G}_a$-bundles over prevarieties $X'_{+}rZ$.

Definition

X': a variety, Z: a closed subvariety of X', $r \in \mathbb{N}$. Then

$$X'_{+}rZ := X' \sqcup_{X'\setminus Z} \underbrace{X' \sqcup_{X'\setminus Z} \cdots \sqcup_{X'\setminus Z} X'}_{r}$$

Especially Dryło's counterexamples were constructed in the case X' is a non $\mathbb{A}^1$-uniruled affine variety.

Counterexample (Dryło [2])

X': a non $\mathbb{A}^1$-uniruled affine variety, Z: a principal hypersurface. Then $\exists$counter examples for GZCP as principal $\mathbb{G}_a$-bundles over $X = X'_{+}Z$.

Main Result

Main Theorem

X: a prevariety with $\exists f : X \to X'$: quasi-finite dominant morphism
X': either ① a non $\mathbb{A}^1$-uniruled affine variety or
② a variety with $\overline{\kappa}(X') \geq 0$
V: an affine variety with a principal $\mathbb{G}_a$-bundle structure over X
W: an affine variety

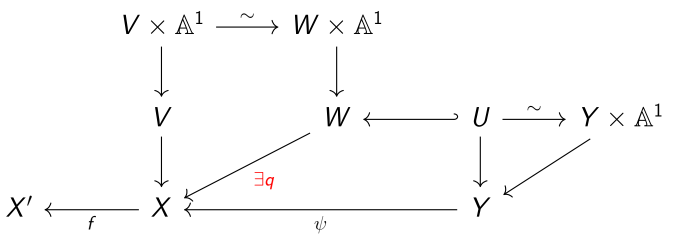
$V \times \mathbb{A}^1 \simeq W \times \mathbb{A}^1 \Longleftrightarrow$ **W: a principal $\mathbb{G}_a$-bundle over X.**

Remark

- prevariety = integral scheme of finite type over k
- X': $\mathbb{A}^1$-uniruled $\Leftrightarrow \exists Y \times \mathbb{A}^1 \to X'$: a dominant morphism of same dimensional varieties
- $X = X'_{+}rZ$ satisfies the assumption of X in Theorem if X' is non $\mathbb{A}^1$-uniruled affine variety or $\overline{\kappa}(X') \geq 0$.
  $\to$ Theorem can be applied to Dryło's counterexamples

Proof of Theorem

Step 1: W: an $\mathbb{A}^1$-bundle over X



it is enouch to show the existence of $q : W \to X$.

Case ①
1-1: $\exists q' : W \to X'$ ($\leftarrow X'$: non $\mathbb{A}^1$-uniruled affine)
1-2: q' lifts to $q : W \to X$ ($\leftarrow f$: quasi-finite)

Case ②
2-1: $\exists Y \times \mathbb{A}^1 \simeq U \subseteq W$:open subset
2-2: $\exists \psi : Y \to X$ ($\leftarrow \overline{\kappa}(X') \geq 0$)
2-3: $U \simeq Y \times \mathbb{A}^1 \to Y \to X \to X'$ extends to $q' : W \to X'$
2-4: q' lifts to $q : W \to X$ ($\leftarrow f$: quasi-finite)

Step 2: W: a principal $\mathbb{G}_a$-bundle over X

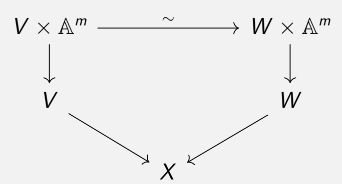
L_V, L_W: the line bundles associated to $\mathbb{A}^1$-bundles V, W

Lemma

W: a principal $\mathbb{G}_a$-bundle over $X \Longleftrightarrow L_W$: trivial.

Lemma

Suppose $V \times \mathbb{A}^m \simeq W \times \mathbb{A}^m$ and the following diagram is commutative.



Then $L_V \simeq L_W$ as line bundles over X.

Remark: The case V and W: line bundles $\longrightarrow$ proved by Dryło [2].

- Two lemmas $\longrightarrow$ W: a principal $\mathbb{G}_a$-bundle over X

References

[1] W. Danielewski, *On a cancellation problem and automorphism groups of affine algebraic varieties*, preprint, Warsaw (1989).
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[3] A Dubouloz, *Additive group actions on Danielewski varieties and the cancellation problem*, Math. Z. **255** (2007), no. 1, 77–93.
[4] K. H. Fieseler, *On complex affine surfaces with $\mathbf{C}^+$-action*, Comment. Math. Helv. **69** (1994), no. 1, 5–27.