Seshadri constants of non-principal abelian varieties

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1 Seshadri constant

- $X/\mathbb{C}$: smooth projective variety
- L : ample line bundle on X.
- For an integral curve $C \subset X$ and $x \in X$,

$$\varepsilon_{C,x}(L) := \frac{L.C}{\mathrm{mult}_x(C)}.$$

Definition 1.1 (The Seshadri constant).
The Seshadri constant of L at $x \in X$ is defined as

$$\varepsilon(X, L; x) := \inf_{x \in C}\{\varepsilon_{C,x}(L)\},$$

where the infimum is taken over all curves passing through x.

The Seshadri constant measures the local positivity of the line bundle.

Example 1.2.
- $\varepsilon(X, L; x) \geq 1$ for a very ample line bundle L and any $x \in X$.
- $\varepsilon(\mathbb{P}^n, \mathcal{O}(1); x) = 1$ for any $x \in \mathbb{P}^n$.
- $\varepsilon(J(C), \theta; x) = 4/3$ for the Jacobian $J(C)$ of a genus two curve.

2 Preceding results on abelian varieties

- (A, L): Polarized abelian variety of dimension n.
- $\varepsilon_C(L) := \varepsilon_{C,0}(L)$

Nakamaye proves the following result.

Lemma 2.1 (=[2, Lemma 3.3]).
Assume

$$\varepsilon_{C,0}(L) < \frac{\sqrt[n]{L^n}}{n}. \tag{2.1}$$

Then C is a degenerate curve. Namely, C is contained in a proper abelian subvariety of A.

For abelian surfaces, Bauer shows the following.

Lemma 2.2 (=[1, Theorem A.1.(b)]).
Let (S, L) be a polarized abelian surface. Then one has a lower bound

$$\varepsilon(S, L) \geq \min\left\{\varepsilon_0, \frac{\sqrt{14L^2}}{4}\right\}, \tag{2.2}$$

where ε_0 is the minimal degree of the elliptic curves in S with respect to L.

3 Main theorems

The following Theorem 3.1 is inspired by Lemma 2.1.

Theorem 3.1 (=[3, Theorem 1.3]).
Assume that

$$\varepsilon(A, L) < \frac{\sqrt[n]{L^n}}{n}. \tag{3.1}$$

Then there exists a proper abelian subvariety B of A such that $\varepsilon(A, L) = \varepsilon(B, L|_B)$.

Key idea of the proof of Theorem 3.2

For the proof, we construct a sequence $\{a_i\}$ satisfying

$$\varepsilon(A, L) \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq a_{k+1} = \frac{\sqrt[n]{L^n}}{n}, \tag{3.2}$$

such that the following condition (1^i) implies either (1^{i-1}) or (2^{i-1}) by using Lemma 2.1 and finiteness theorems.
(1^i) $\varepsilon(A, L) < a_i$, and if a curve C satisfies $\varepsilon(A, L) \leq \varepsilon_C(L) < a_i$, then C generates an abelian subvariety of dimension at most $(i-1)$.
(2^i) $\varepsilon(A, L) = \varepsilon(B, L|_B)$ for some i-dimensional abelian subvariety B.

Theorem 3.2 (=[3, Proposition 3.7]).
Let D be an abelian divisor on A. Suppose that

$$\sqrt[n]{L^n} > \sqrt[n-1]{a(L|_D)^{n-1}} \tag{3.3}$$

for a positive real number a. Then any curve C satisfying the inequality

$$\varepsilon_C(L) \leq \frac{a\sqrt[n]{L^n}}{n} \tag{3.4}$$

is contained in D.

Rough sketch proof of Theorem 3.2

Consider the nef threshold of D.

$$\sigma(L, D) := \sup\{t \in \mathbb{R} \mid L - tD \text{ is ample}\} \in \mathbb{R}_{>0} \cup \{\infty\}.$$

Then we obtain the equation $\sigma(L, D)(L|_D)^{n-1} = L^n/n$. This implies $L - \frac{(a\sqrt[n]{L^n})}{n}D$ is ample.
Considering the intersection number of a curve satisfying (3.4) and this ample line bundle, we obtain the theorem.

4 Applications

The following Proposition 4.1 is obtained from Lemma 2.1 and Theorem 3.2.

Proposition 4.1.
Let (S, L) be a polarized abelian surface. Assume that

$$\varepsilon(S, L) < \frac{\sqrt{L^2}}{2}. \tag{4.1}$$

Then there exists exactly one submaximal curve with respect to L and it is elliptic.

By our theorem, we can use results for abelian surfaces to study the Seshadri constants of abelian threefolds in some cases.

Theorem 4.2.
Let (A, L) be a polarized abelian threefold. Assume that $\varepsilon(A, L) < \sqrt[3]{L^3}/3$.

1. *If there exists an abelian surface S which satisfy*

$$\sqrt[3]{L^3} > \frac{3\sqrt{14(L|_S)^2}}{4}, \tag{4.2}$$

then $\varepsilon(A, L) = \varepsilon(S, L|_S)$.

2. *Otherwise, $\varepsilon(A, L)$ is computed by an elliptic curve.*

References

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