Duality for Witt sheaves

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Vanishing in positive characteristic

Motivation

Kodaira vanishing can fail if $\mathrm{char}(k) = p > 0$. Is there a similar result which holds when $\mathrm{char}(k) = p > 0$?

Tanaka proved Kodaira-like vanishing of ample invertible sheaves using the Witt–canonical sheaf $W\Omega_X^N$:

Theorem (Tanaka [Tan18]).

1. - $H^j(X, \mathcal{A}^{-s}) = 0$ for any $s \gg 0, j < N$,
   - $H^j(X, \mathcal{A}^{-1})_{\mathbb{Q}} = 0$ for any $j < N$,
2. - $H^i(X, W\Omega_X^N \otimes_{W\mathcal{O}_X} \mathcal{A}) = H^i(X, W\Omega_X^N \otimes_{W\mathcal{O}_X} \mathcal{A}^s)$ for any $s > 0$,
   - $H^i(X, W\Omega_X^N \otimes_{W\mathcal{O}_X} \mathcal{A}) = 0$ for any $i > 0$.

- $\mathcal{A}$: ample invertible sheaf on X
- X: N–dimensional smooth projective variety / $k = k^p, \mathrm{char}(k) = p > 0$
- $\mathcal{A}$: Teichmueller–lift of $\mathcal{A}$, invertible sheaf of $W\mathcal{O}_X$–modules (cf. [Tan18])
- $W\mathcal{O}_X$: $W\mathcal{O}_X(U) := W(\mathcal{O}_X(U))$
- $W(A)$: for a ring A, $W(A)$ is the ring of Witt–vectors over A
- $W\Omega_X^N$: top term of de Rham–Witt complex

Remark.
Actually, for applications in MMP we need vanishing for nef and big instead of ample invertible sheaves.

Remark.
The theorem suggests some asymmetric duality property.
The proof of (1) is trivial compared to the proof of (2).
Thus duality may help us prove the nef and big case.

[Tan18] Hiromu Tanaka, *Vanishing theorems of Kodaira type for Witt Canonical sheaves*, arXiv:1707.04036v2 [math.AG] (2018).

Duality

Duality Theorem

$$R\Gamma(W\Omega_X^N \otimes_{W\mathcal{O}_X} \mathcal{F}) \cong R\,\mathcal{H}om_\omega(R\Gamma(\mathcal{F}^\vee), \breve{\omega}[-N])$$

- $\mathcal{F}$: invertible sheaf on X
- ω: non-commutative W–algebra gen. by V s.t. $aV = VF(a)$, for any $a \in W$, emulating the relation $V(F(a)b) = aV(b)$ of the maps V, F on $W := W(k)$
- F: Frobenius map on W, induced by Frobenius on k
- V: shift map on W
- $\breve{\omega}$: V–adic completion of ω
- $[-N]$: shift in complex degree

Both results rest on the definition of ω.

Key steps of proof
Let $X \xrightarrow{\phi} S = \mathrm{Spec}\, k$.
Step 1. Truncated case (using results from Ekedahl [Eke84]):

$$H^i(W_n\Omega_X^N \otimes_{W_n\mathcal{O}_X} \mathcal{F}_{\leq n}) \cong \mathrm{Hom}_{W_n\mathcal{O}_X}(H^{N-i}(\mathcal{F}_{\leq n}^\vee), W_n\Omega_X^N)$$

for any $i \geq 0, n > 0$.

Step 2. Key proposition:

$$\omega_n \overset{L}{\underset{\omega}{\otimes}} R\Gamma(\mathcal{F}) \cong R\Gamma(\mathcal{F}_{\leq n})$$

Step 3. Apply derived pushforward and pass to the limit:

$$\begin{aligned} & R\phi_*(W\Omega_X^N \otimes_{W\mathcal{O}_X} \mathcal{F}) \\ & \cong R\lim_n R\,\mathcal{H}om_{W_n\mathcal{O}_S}(R\phi_*\mathcal{F}_{\leq n}^\vee, W_n[-N]) \\ & \cong R\lim_n R\,\mathrm{Hom}_{W_n}(\omega_n \overset{L}{\underset{\omega}{\otimes}} R\Gamma(\mathcal{F}^\vee), W_n[-N]) \\ & \cong R\lim_n R\,\mathrm{Hom}_\omega(R\Gamma(\mathcal{F}^\vee), R\,\mathrm{Hom}_{W_n}(\omega_n, W_n[-N])) \\ & \cong R\,\mathrm{Hom}_\omega(R\Gamma(\mathcal{F}^\vee), R\lim_n \mathrm{Hom}_{W_n}(\omega_n, W_n[-N])) \end{aligned}$$

Step 4.

$$R\lim_n \mathrm{Hom}_{W_n}(\omega_n, W_n) \cong \breve{\omega}$$

[Eke84] Torsten Ekedahl, *On the multiplicative properties of the de Rham–Witt complex. I*, Arkiv för Matematik 22 (1984), no. 2, 185–239.

Duality applied to Tanaka's vanishing

Theorem: ω–duality eliminates torsion

$$\begin{aligned} R^i\,\mathrm{Hom}_\omega(H^\bullet, \breve{\omega}) &\cong \bigoplus_{-i=p+q} R^p\,\mathrm{Hom}_\omega(H^q, \breve{\omega}) \\ &\cong \mathrm{Hom}_\omega(H^{-i}, \breve{\omega}) \oplus \mathrm{Ext}^1_\omega(H^{-i-1}, \breve{\omega}) \end{aligned}$$

If H^i is p–torsion, then

$$\mathrm{Ext}^1_\omega(H^i, \breve{\omega}) = 0.$$

- $H^\bullet$ = bounded complex of left–ω–modules for which $p = VF$ (or at least p–torsion = V–torsion), with $H^i = 0$ for any $i < 0$.

Key steps of proof
Write $\breve{\omega}_{\mathbb{Q}} := \Pi_{0 \leq i} W_{\mathbb{Q}} V^i$.
Step 1. Take SES:

$$0 \longrightarrow \breve{\omega} \longrightarrow \breve{\omega}_{\mathbb{Q}}[V^{-1}] \longrightarrow \breve{\omega}_{\mathbb{Q}}[V^{-1}]/\breve{\omega} \longrightarrow 0,$$

Step 2. Apply $\mathrm{Hom}_\omega(H, \bullet)$ to obtain LES
Step 3. $\breve{\omega}_{\mathbb{Q}}[V^{-1}]$ and $\breve{\omega}_{\mathbb{Q}}[V^{-1}]/\breve{\omega}$ are injective left–ω–modules
Step 4. Suppose H^i is p–torsion. Since **in $\breve{\omega}_{\mathbb{Q}}[V^{-1}]/\breve{\omega}$ p–torsion and V–torsion are disjoint, but in H^i they are identical,**

$$\mathrm{Ext}^1_\omega(H^i, \breve{\omega}) = \mathrm{im}(\mathrm{Hom}_\omega(H^i, J)) = 0$$

Corollary

If (1) of Tanaka's vanishing holds for nef and big $\mathcal{F}$, then so does (2).

Open questions

The most obvious next step is to attempt proving (1) of Tanaka's vanishing for nef and big invertible sheaves.
Further generalizing the result (say to $X \xrightarrow{\phi} Y$ = finite type over k, as in [Tan18]) may also be of interest.

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