Computable Bialynicki-Birula decomposition of the Hilbert scheme

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Hilbert scheme

$$\mathrm{Hilb}_n^P = \left\{ Y \subset \mathbb{P}_k^n \;\middle|\; \begin{array}{l} Y \text{ is a closed subscheme in } \mathbb{P}_k^n \\ \text{with Hilbert polynomial } P \end{array} \right\}$$

How do we study the Hilbert scheme?

- **Geometries** in the projective space.
- **Deformation theory** in the projective space.

Strongly depends on **degree**, **codimension** and so on.

Find another tool for studying that:

- Does not depend on **specific data**.

Find:

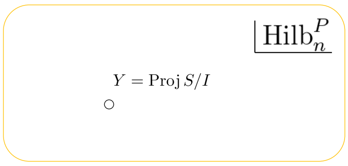
- **Algorithms** on Hilbert schemes.

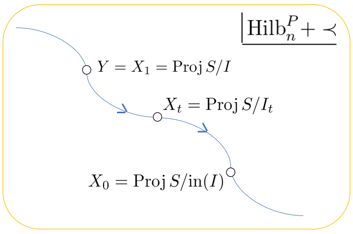
Gröbner basis theory

Gröbner degeneration

$\forall I \subset S$: homogeneous ideal,
$\exists X \subset \mathbb{P}_k^n \times_k \mathbb{A}_k^1$: closed subscheme,
$\exists \mathbb{G}_m \curvearrowright S$ such that

- X is flat over $\mathbb{A}_k^1$.
- $X_t = S/(t \cdot I)$ for $t \in \mathbb{G}_m \subset \mathbb{A}_k^1$.
- $X_0 = S/(\mathrm{in}_\prec I)$.





Gröbner scheme

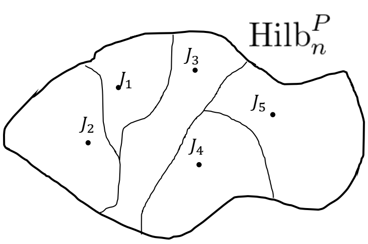
$$\mathrm{Gröb}_\prec^J = \left\{ I \subset S \;\middle|\; \begin{array}{l} I \text{ is a homogeneous ideal} \\ \text{and } \mathrm{in}_\prec I = J \end{array} \right\}$$

$$\mathrm{Gröb}_\prec^J \to \mathrm{Hilb}_n^P \;;\; I \mapsto \mathrm{Proj} S/J$$

Computable decomposition

$\exists \mathcal{M}_{P,n}$: A set of monomial ideals

- $\forall J \in \mathcal{M}_{P,n}$, $\mathrm{Gröb}_\prec^J \to \mathrm{Hilb}_n^P$ is a locally closed immersion
- $\mathrm{Hilb}_n^P(k) = \coprod_{J \in \mathcal{M}_{P,n}} \mathrm{Gröb}_\prec^J(k)$.



Where $\mathcal{M}_{P,n} = \{J_1, J_2, J_3, J_4, J_5\}$

$d \setminus m$	0	1	2	3	4	5	6	7	8	9	10
1	1	1	1								
2	1	2	3	2	1						
3	1	2	5	6	5	2	1				
4	1	2	6	10	13	10	6	2	1		
5	1	2	6	12	21	24	21	12	6	2	1

Table: the **number** of $J \in \mathcal{M}_{d,2}$ s. t. $\mathrm{Gröb}_\prec^J \cong \mathbb{A}_k^m$

Consider

Hilb_2^d : d points in $\mathbb{P}^2$

In fact, left numbers are the **Betti numbers** of the Hilbert schemes.

Why?

Bialynicki-Birula decomposition

Bialynicki-Birula introduced a **cell decomposition** of a smooth projective variety that equips an action of $\mathbb{G}_m = \mathrm{spec} k[t, t^{-1}]$ on that.

Thanks to [Dri13] and [JS18], we can define a generalized **BB decomposition** and **BB cells** on Hilbert schemes.

Main theorem

$\exists \mathbb{G}_m \curvearrowright \mathrm{Hilb}_n^P$ such that:

- $\{\mathrm{Gröb}_\prec^J \mid J \in \mathcal{M}_{P,n}\}$ is the **BB cells** with respect to the $\mathbb{G}_m$ action.
- If Hilb_n^P is smooth, then
  $$\mathrm{Hilb}_n^P(k) = \coprod_{J \in \mathcal{M}_{P,n}} \mathrm{Gröb}_\prec^J(k)$$
  gives a cell decomposition of Hilb_n^P.
- If $\mathrm{Proj}\, S/J$ is a **non-singular** point for $J \in \mathcal{M}_{P,n}$, then $\mathrm{Gröb}_\prec^J$ is isomorphic to an **affine space**.

Example

Singular points on Hilb_3^{2t+2}

Sernesi(2006) showed that the monomial ideal

$$J = \langle x_1x_2, x_1x_3, x_2x_3, x_3^2 \rangle_{\geq 3}$$

defines a singular point on Hilb_3^{2t+2}.

Let us see the Gröbner schemes in Hilb_3^{2t+2}.

$\prec$: Reverse lexicographic order

- $\#(M_{2t+2,3}) = 159$.
- **144** monomial ideals define **non-singular** Gröbner schemes, those include $J = \langle x_1x_2, x_1x_3, x_2x_3, x_3^2 \rangle_{\geq 3}$.
- **15** monomial ideals $J_1, \ldots, J_{15}$ define **singular** Gröbner schemes.

$\prec$: Lexicographic order

- $\#(M_{2t+2,3}) = 159$.
- **143** monomial ideals define **non-singular** Gröbner schemes, those include $J = \langle x_1x_2, x_1x_3, x_2x_3, x_3^2 \rangle_{\geq 3}$.
- **13** monomial ideals among $J_1, \ldots, J_{15}$ and new **3** monomial ideals J_{16}, J_{17}, J_{18} define **singular** Gröbner schemes.

Therefore Hilb_3^{2t+2} has **18** singular points corresponding to monomial schemes which are **different** from the Sernesi's example.

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