The universal cover of hypertoric varieties and Bogomolov's decomposition

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§1 Background and Motivation

Definition ((conical) symplectic variety)

(Y,ω) is a *(conical) symplectic variety* if Y is normal and satisfies
(1) $\omega \in \Omega^2_{Y_{\mathrm{reg}}}$ is symplectic and extends to a 2-form on any resolution.
((2) Y is affine, and $\exists \mathbb{C}^* \curvearrowright Y$ s.t. positive weight and $s^*\omega = s^\ell \omega$.)

Example
(1) ADE type surface singularity (3) (affine) Nakajima quiver variety
(2) Nilpotent orbit closures $\overline{\mathcal{O}} \subset \mathcal{N} \subset \mathfrak{g}$ (4) (affine) Hypertoric variety

Consider $\pi_1(Y_{\mathrm{reg}})$ of conical symplectic varieties (Y,ω).

Theorem (Namikawa)

For a conical symplectic variety (Y,ω), $|\pi_1^{\mathrm{alg}}(Y_{\mathrm{reg}})| < \infty$.

It is expected but not known that $|\pi_1(Y_{\mathrm{reg}})| < \infty$.
Example
(1) $Y = \mathbb{C}^n/\Gamma$: isolated quotient singularity, $\pi_1(Y_{\mathrm{reg}}) = \Gamma$.
(2) (Beauville) $Y = \overline{\mathcal{O}}^{\min}_{A_n} \subset \mathfrak{sl}_{n+1}$: min nilpotent orbit closure, $\pi_1(Y_{\mathrm{reg}}) = 0$.

Proposition (The universal coverings of conical symplectic varieties)

Assume $|\pi_1(Y_{\mathrm{reg}})| < \infty$. Then, $\exists!$ a conical symplectic variety $(\overline{Y},\overline{\omega})$ s.t.

$$\begin{array}{ccc} (\overline{Y},\overline{\omega}) & \supset & \varphi^{-1}(Y_{\mathrm{reg}}) \\ \downarrow{\scriptstyle \exists!\varphi\ :\ \text{finite}} & & \downarrow{\scriptstyle \varphi_0 := \varphi|\ :\ \text{universal cover}} \\ (Y,\omega) & \supset & Y_{\mathrm{reg}} \end{array},$$

Problem (Namikawa, Analogue of Bogomolov's decomposition)

Assume $|\pi_1(Y_{\mathrm{reg}})| < \infty$. Is $\overline{Y}$ (uniquely) decomposed into *irreducible* ones as

$$(\overline{Y},\overline{\omega}) \cong (Y_1,\omega_1) \times \cdots \times (Y_r,\omega_r)?$$

(That is, any conical symplectic form Y_m is ω_m up to scalar.)

Goal

- Describe $\pi_1(Y_A(0)_{\mathrm{reg}})$ and $\overline{Y_A(0)}$ for affine hypertoric varieties $Y_A(0)$.
- Establish Bogomolov's decomposition of $Y_A(0)$.

§2 Hypertoric variety $Y_A(\alpha)$

Principle in (hyper)toric geometry : Combinatorics $\leftrightarrow$ Geometry

Polytope (eg. $\triangle$) $\leftrightarrow$ (projective) Toric variety (eg. $\mathbb{P}^2$)

Hyperplane arrangement $\mathcal{H}^\alpha_B$ $\leftrightarrow$ Hypertoric variety $Y_A(\alpha)$

Ingredients of hypertoric variety $Y_A(\alpha)$

$$0 \to \mathbb{Z}^{n-d} \xrightarrow{B} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^d \to 0, \qquad \tilde{\alpha} \longmapsto \alpha \quad (\tilde{\alpha} \in \mathbb{Z}^n,\ \alpha \in \mathbb{Z}^d)$$

$$B^T = \left(\boldsymbol{b_1}\ \boldsymbol{b_2} \cdots \boldsymbol{b_n}\right)\}\, n-d \qquad A = \begin{pmatrix} \boldsymbol{a_1}\ \boldsymbol{a_2} \cdots \boldsymbol{a_n} \end{pmatrix}\Big\}\, d$$

Construction of $Y_A(\alpha) = \mu_A^{-1}(0)^{\alpha-ss}/\!/\mathbb{T}^d_{\mathbb{C}}$

$$\mathbb{T}^d_{\mathbb{C}} \overset{A^T}{\hookrightarrow} \mathbb{T}^n_{\mathbb{C}} \curvearrowright (\mathbb{C}^{2n},\omega_{\mathbb{C}}) \qquad \boldsymbol{t}\cdot(z_i,w_i) := (\boldsymbol{t}^{\boldsymbol{a_i}} z_i, \boldsymbol{t}^{-\boldsymbol{a_i}} w_i)$$

$$\downarrow \mu_A : \text{moment map}$$

$$\mathbb{C}^d = \mathrm{Lie}(\mathbb{T}^d_{\mathbb{C}})^*$$

Definition (Hypertoric variety)

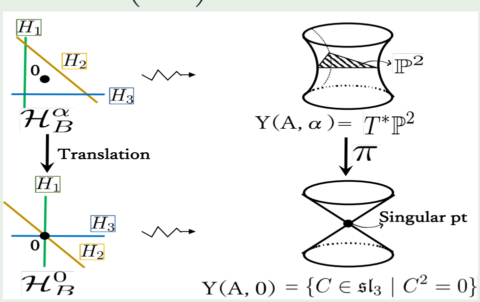
Hypertoric variety is defined as the GIT quotient $\boldsymbol{Y_A(\alpha)} := \mu_A^{-1}(0)^{\alpha-ss}/\!/\mathbb{T}^d_{\mathbb{C}}$.
Define the ass. hyperplane arrangement $\mathcal{H}^\alpha_B := \{H_i : \langle \boldsymbol{b_i}, - \rangle = \tilde{\alpha}_i\} \subset \mathbb{R}^{n-d}$.

By definition, $\mu_A^{-1}(0)^{\alpha-ss} \subset \mu_A^{-1}(0)$ induces a projective morphism

$$\pi_\alpha : Y_A(\alpha) \to Y_A(0).$$

Example (Springer resolution $T^*\mathbb{P}^2 \to \overline{\mathcal{O}}^{\min}_{A_2}$)

Let $A = (1\,1\,1)$, $B^T = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} = (\boldsymbol{b_1}\,\boldsymbol{b_2}\,\boldsymbol{b_3})$. $\tilde{\alpha} = (1,1,1)$



Basic Properties of $Y_A(\alpha)$

Let α be generic. Then, $Y_A(0)$ and $Y_A(\alpha)$ are sympl. var. of dim $2(n-d)$, and
(1) $\pi_\alpha : Y_A(\alpha) \to Y_A(0)$ gives a crepant (symplectic) resolution.
(2) $\mathrm{Sing}(Y_A(0))$ corresponds to the "**non-general**" locus of $\mathcal{H}^0_B$.

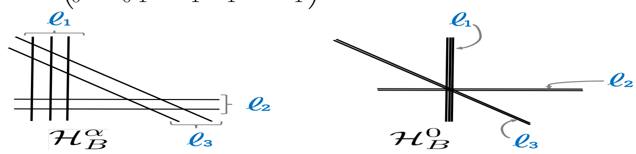
translation of the arrangement $\mathcal{H}^0_B$ $\leftrightarrow$ taking the crepant resolution

Theorem (Classification of 4-dimensional $Y_A(0)$)

Every 4-dim $Y_A(0)$ is isomorphic to one of the following:
(1) $S_{A_{\ell_1-1}} \times S_{A_{\ell_2-1}}$: the product of A type surface singularities.
(2)

$$\overline{\mathcal{O}^{\min}}(\ell_1,\ell_2,\ell_3) := \left\{ \begin{pmatrix} u_1 & x_1 & x_3 \\ y_1 & u_2 & x_2 \\ y_3 & y_2 & u_3 \end{pmatrix} \in \mathfrak{sl}_3 \;\middle|\; \text{All } 2\times 2\text{-minors of } \begin{pmatrix} u_1^{\ell_1} & x_1 & x_3 \\ y_1 & u_2^{\ell_2} & x_2 \\ y_3 & y_2 & u_3^{\ell_3} \end{pmatrix} = 0 \right\}.$$

Case (2) $B^T = \begin{pmatrix} \overbrace{1 \cdots 1}^{\ell_1} & \overbrace{0 \cdots 0}^{\ell_2} & \overbrace{-1 \cdots -1}^{\ell_3} \\ 0 \cdots 0 & 1 \cdots 1 & -1 \cdots -1 \end{pmatrix}$



Remark

In general, isomorphism classes of $Y_A(0)$ as conical symplectic varieties are classified by the combinatorial str. of $\mathcal{H}^0_B$, i.e., "matroid".

§3 Main result 1 : $\pi_1(Y_A(0)_{\mathrm{reg}})$ and the universal cover

In general, we can assume as $B^T = (\overbrace{\boldsymbol{b^{(1)}} \cdots \boldsymbol{b^{(1)}}}^{\ell_1} \cdots \overbrace{\boldsymbol{b^{(s)}} \cdots \boldsymbol{b^{(s)}}}^{\ell_s})$
It is natural to ask whether the simplification $Y_{\underline{A}}(0)$ gives univ. cover of $Y_A(0)$.

$$\begin{array}{ccc} B^T = (\boldsymbol{b^{(1)}} \cdots \boldsymbol{b^{(1)}} \cdots \boldsymbol{b^{(s)}} \cdots \boldsymbol{b^{(s)}}) & \dashrightarrow & Y_A(0) \\ \downarrow{\scriptstyle \text{simplification}} & & \uparrow{\scriptstyle \exists ?\ \text{finite}} \\ \underline{B}^T = (\boldsymbol{b^{(1)}} \cdots \boldsymbol{b^{(s)}}) & \dashrightarrow & Y_{\underline{A}}(0) \end{array},$$

$$0 \longrightarrow \mathbb{Z}^{n-d} \xrightarrow{\underline{B}} \mathbb{Z}^s \xrightarrow{\underline{A}} \mathbb{Z}^{s-(n-d)} \longrightarrow 0$$

Theorem (N.)

$\exists$ finite $\varphi : Y_{\underline{A}}(0) \twoheadrightarrow Y_A(0) \cong Y_{\underline{A}}(0)/G$: universal cover, and we have

$$G \cong \pi_1(Y_A(0)_{\mathrm{reg}}) \cong \Gamma/\Gamma \cap \mathbb{T}^{s-(n-d)}_{\mathbb{C}} = \Gamma/(\mathrm{Ker}\,\widetilde{B}^T)\Gamma,$$

where $\Gamma := \prod_{k=1}^s \mathbb{Z}/\ell_k\mathbb{Z} \hookrightarrow \mathbb{T}^s_{\mathbb{C}} \overset{\underline{A}^T}{\hookleftarrow} \mathbb{T}^{d-(n-s)}_{\mathbb{C}}$ and $\widetilde{\boldsymbol{B}}^T := (m_1\boldsymbol{b^{(1)}} \cdots m_s\boldsymbol{b^{(s)}})$ $(m_k := \ell_1 \cdots \ell_s/\ell_k)$.

simplification of the arrangement $\mathcal{H}^0_B$ $\leftrightarrow$ taking the universal covering

Corollary

$$\pi_1(Y_A(0)_{\mathrm{reg}}) = 0 \Leftrightarrow \ell_1 = \cdots = \ell_s = 1, \text{i.e., } \mathcal{H}^0_B \text{ is simple.}$$

Example ($\overline{\mathcal{O}^{\min}}(\ell_1,\ell_2,\ell_3)$)

$\widetilde{B}^T = \begin{pmatrix} \ell_2\ell_3 & 0 & -\ell_1\ell_2 \\ 0 & \ell_1\ell_3 & -\ell_1\ell_2 \end{pmatrix}$, and $g := \gcd(\ell_1,\ell_2,\ell_3)$. Then,

$$\pi_1(\overline{\mathcal{O}^{\min}}(\ell_1,\ell_2,\ell_3)_{\mathrm{reg}}) \cong \frac{\prod_{k=1}^3 \mathbb{Z}/\ell_k\mathbb{Z}}{\mathbb{Z}\langle\left(\frac{\ell_1}{g},\frac{\ell_2}{g},\frac{\ell_3}{g}\right)\rangle}$$

§4 Main result 2 : Bogomolov's decomposition for $Y_A(0)$

If $A = A_1 \oplus \cdots \oplus A_r$, then $Y_A(0) \cong \prod_{m=1}^r Y_{A_m}(0)$. So, we can expect as follows.

Theorem (N.)

$$Y_A(0) \text{ is irreducible} \Leftrightarrow A \text{ is indecomposable as a matrix.}$$

【Key of the proof】
(1) Reduce to the simple and smooth case $Y_{\underline{A}}(\alpha)$.
(2) $\mu_{\underline{A}}^{-1}(0)$ is C.I. & codim 3 smooth. $\rightsquigarrow$ $\Omega^2_{\mu_{\underline{A}}^{-1}(0)}$ is reflexive.
(3) Consider which $\sigma \in \Omega^2_{\mu_{\underline{A}}^{-1}(0)}$ descends to a $\overline{\sigma} \in \Omega^2_{Y_{\underline{A}}(\alpha)}$.

Corollary

For any two smooth hypertoric varieties $Y_A(\alpha)$ and $Y_{A'}(\alpha')$, TFAE:
(1) $Y_A(\alpha) \cong Y_{A'}(\alpha')$: $\mathbb{C}^*$-equivariant isomorphic.
(2) $Y_A(\alpha) \cong Y_{A'}(\alpha')$: $\mathbb{C}^* \times \mathbb{T}^{n-d}_{\mathbb{C}}$-equivariant isomorphic as symplectic varieties.
(3) $\mathcal{H}^\alpha_B \cong \mathcal{H}^{\alpha'}_{B'}$.
The same statement will hold for any two affine hypertoric varieties.

【Reference】

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