On the geometry of singular K3 surfaces with discriminant 3, 4 and 7

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Introduction

We consider only algebraic K3 surfaces defined over $\mathbb{C}$.

Definition

A K3 surface $X$ is said to be singular if its Picard rank is 20.

Shioda and Inose showed that any singular K3 surface admits infinitely many automorphisms. In spite of their significance, there are only eleven such K3 surfaces where the automorphism groups are actually calculated, as shown in the following table:

<table>
<thead>
<tr>
<th>No.</th>
<th>$[a,b,c]$</th>
<th>Authors</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$[2,1,2]$</td>
<td>Vinberg</td>
</tr>
<tr>
<td>2</td>
<td>$[2,0,2]$</td>
<td>Vinberg</td>
</tr>
<tr>
<td>3</td>
<td>$[2,1,4]$</td>
<td>Ujikawa</td>
</tr>
<tr>
<td>4</td>
<td>$[2,0,4]$</td>
<td>Shimada</td>
</tr>
<tr>
<td>5</td>
<td>$[2,1,6]$</td>
<td>Shimada</td>
</tr>
<tr>
<td>6</td>
<td>$[2,0,6]$</td>
<td>Shimada</td>
</tr>
<tr>
<td>7</td>
<td>$[4,2,4]$</td>
<td>Keum and Kondo</td>
</tr>
<tr>
<td>8</td>
<td>$[2,1,8]$</td>
<td>Shimada</td>
</tr>
<tr>
<td>9</td>
<td>$[4,1,4]$</td>
<td>Shimada</td>
</tr>
<tr>
<td>10</td>
<td>$[2,0,8]$</td>
<td>Shimada</td>
</tr>
<tr>
<td>11</td>
<td>$[4,0,4]$</td>
<td>Keum and Kondo</td>
</tr>
</tbody>
</table>

Here, each of the K3 surfaces $X$ is specified by a triple $[a, b, c]$, which indicates the Gram matrix

\[
\begin{bmatrix}
    a & b \\
    b & c
\end{bmatrix}
\]

of the transcendental lattice $T_X$. We denote by $X_d$ three K3 surfaces with discriminant $d$ ($d=3, 4, 7$) which correspond to No.1, 2 and 3, respectively.

In the case of No.1 and 2, Vinberg found "good" hyperbolic geometry in their Néron-Severi groups and calculated the automorphism groups of $X_3$ and $X_4$. In the case of No.3, Ujikawa calculated automorphism group of $X_7$ using Borcherds method. In this study, we focus on these three K3 surfaces and aim at finding common properties among them, which should give us clue to understand the geometry of these surfaces more in depth.

Known Result

$X_T$ can be constructed by the double covering.

A branching locus of $X_T$ is given by the equation

\[
(x_0^2x_1 + x_1^2x_2 + x_2^2x_0 - 3x_0x_1x_2)^2 - 4x_0x_1x_2(x_0 - x_1)(x_1 - x_2)(x_2 - x_0) = 0.
\]

Proposition

$X_5 (d = 3, 4)$ can be constructed by the double covering.

(i) A branching locus of $X_5$ is given by the equation

\[
(x_0 - x_1)(x_1 - x_2)(x_2 - x_0)(x_0 + x_1 + x_2)^2 + x_0(1-x_1)(1-x_2)(x_2 - x_0) = 0,
\]

which has a $D_4$ singularity at $(1,1,1)$ and $A_1$ singularities at the intersections of lines and elliptic curve of components of $C_4$. In particular, $NS(X_5)$ has a sublattice of finite index isomorphic to $U \oplus E_8^{(3)}$

(ii) A branching locus of $X_5$ is given by the equation

\[
x_0(x_1x_2(x_0 - x_1)(x_1 - x_2)(x_2 - x_0) = 0,
\]

which has $D_4$ singularities at intersections of six lines. In particular, $NS(X_5)$ has a sublattice of finite index isomorphic to $U \oplus D_6^{(3)}$.

Notation

Let $T$ be a lattice of sign $(2,3)$ satisfying $q_T = -q_U \oplus A_1^{(3)}$, and define a period domain $B$, $H_0$ and $H$ as follows.

\[
B := \{ \omega \in P(T \otimes \mathbb{C}) \mid (\omega, \omega) = 0, (\omega, \overline{\omega}) > 0 \}, \\
H_0 := \{ \omega \in B \mid (\omega, \delta) = 0 \text{ for } \delta \in T, \} \\
H := \cup H_0.
\]

Main Results

Theorem 1

Let $(X, \omega_X)$ be a marked K3 surface and $\omega_X$ be a nowhere vanishing holomorphic 2-form of $X$. Then the following conditions are equivalent.

(i) $C$ has a $D_4$ singularity at $q$ (resp. $p_i$), and is smooth elsewhere.

(ii) $\{p_i, q\}$ is in general position.

(iii) $\omega_X |_{\delta}$ is smooth elsewhere.

(iv) $C$ has an elliptic fibration $f : X \to \mathbb{P}^1$ such that $f$ has three singular fibres of types $I_0$, and others are $I_1, I_2, I_3, I_4$.

$\omega_X(t) \in B \backslash H$.

Theorem 2

Let $\omega_X (d = 3, 4, 7)$ be a nowhere vanishing holomorphic 2-forms of $X_5$.

Then $\omega_X \in H \subset B$. 

Figure 1

\[
\begin{array}{cccc}
A_3 & A_3 & D_4 & \mathbb{P}^2 \\
\end{array}
\]