On the geometry of singular K3 surfaces with discriminant 3, 4 and 7

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Introduction

We consider only algebraic K3 surfaces defined over $\mathbb{C}$.

Definition
A K3 surface X is said to be singular if its Picard rank is 20.

Shioda and Inose showed that any singular K3 surface admits infinitely many automorphisms. In spite of their significance, there are only eleven such K3 surfaces where the automorphism groups are actually calculated, as shown in the following table:

No.	[a,b,c]	
1	[2,1,2]	Vinberg
2	[2,0,2]	Vinberg
3	[2,1,4]	Ujikawa
4	[2,0,4]	Shimada
5	[2,1,6]	Shimada
6	[2,0,6]	Shimada
7	[4,2,4]	Keum and Kondo
8	[2,1,8]	Shimada
9	[4,1,4]	Shimada
10	[2,0,8]	Shimada
11	[4,0,4]	Keum and Kondo

Here, each of the K3 surfaces X is specified by a triple $[a, b, c]$, which indicates the Gram matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ of the transcendental lattice T_X. We denote by X_d three K3 surfaces with discriminant d (d=3, 4, 7) which correspond to No.1, 2 and 3, respectively.
In the case of No.1 and 2, Vinberg found "good" hyperbolic geometry in their Néron-Severi groups. and calculated the automorphism groups of X_3 and X_4. In the case of No.3, Ujikawa calculated automorphism group of X_7 using Borcherds method. In this study, we focus on these three K3 surfaces and aims at finding common properties among them, which should give us clue to understand the geometry of these surfaces more in depth.

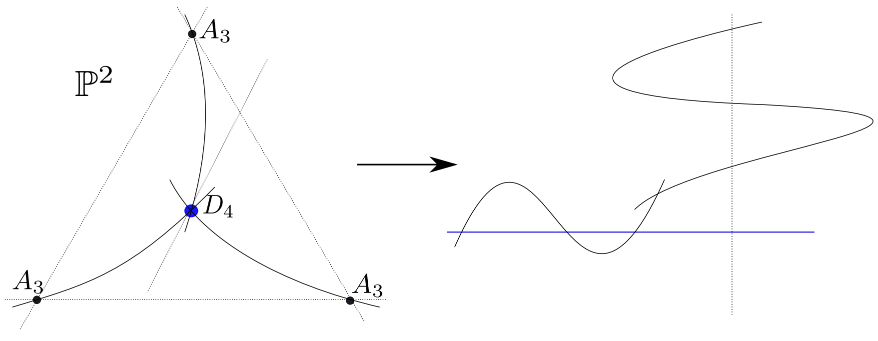

Figure: 1

Known Result

X_7 can be constructed by the double covering.
A branching locus of X_7 is given by the equation

$$(x_0^2x_1 + x_1^2x_2 + x_2^2x_0 - 3x_0x_1x_2)^2 - 4x_0x_1x_2(x_0 - x_1)(x_1 - x_2)(x_2 - x_0) = 0.$$

Proposition

X_d $(d = 3, 4)$ can be constructed by the double covering.
(i) A branching locus of X_3 is given by the equation

$$(x_0 - x_1)(x_1 - x_2)(x_2 - x_0)\{(x_0 + x_1 + x_2)^3 + (x_0 - x_1)(x_1 - x_2)(x_2 - x_0)\} = 0,$$

which has a D_4 singularity at $(1, 1, 1)$ and A_5 singularities at the intersections of lines and elliptic curve of components of C_3. In particular, $NS(X_3)$ has a sublattice of finite index isomorphic to $U \oplus E_6^{\oplus 3}$
(ii) A branching locus of X_4 is given by the equation

$$x_0x_1x_2(x_0 - x_1)(x_1 - x_2)(x_2 - x_0) = 0,$$

which has D_4 singularities at intersections of six lines. In particular, $NS(X_4)$ has a sublattice of finite index isomorphic to $U \oplus D_6^{\oplus 3}$.

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Notation

Let T be a lattice of sign (2,3) satisfying $q_T = -q_{U\oplus A_5^{\oplus 3}}$, and define a period domain $\mathcal{B}$, $\mathcal{H}_\delta$ and $\mathcal{H}$ as follows.

$$\mathcal{B} := \{\omega \in \mathbb{P}(T \otimes \mathbb{C}) |\ \langle \omega, \omega \rangle = 0, \langle \omega, \overline{\omega} \rangle > 0\},$$
$$\mathcal{H}_\delta := \{\omega \in \mathcal{B} |\ \langle \omega, \delta \rangle = 0\} \text{ for } \delta \in T,$$
$$\mathcal{H} := \cup \mathcal{H}_\delta.$$

Main Results

Theorem 1
Let (X, α_X) be a marked $K3$ surface and ω_X be a nowhere vanishing holomorphic 2-form of X. Then the following conditions are equivalent.

(i) There exists a sextic curve C such that X is isomorphic to X_C (the double covering branched over C) and the Mordell Weil group associated to C is trivial, and C satisfies the following three conditions in Figure 1.

1. C has a D_4 (resp. A_3) singularity at q (resp. p_i), and is smooth elsewhere.
2. $\{p_i, q |\ i = 1, 2, 3\}$ is in general position.
3. The multiplicity of C and the line p_iq at p_i (resp. q) is 2 (resp. 4).

(ii) There exists an elliptic fibration $f : X \to \mathbb{P}^1$ such that f has three singular fibres of types I_6, and others are I_1 or II, and $MW(f)$ is trivial.

(iii) $\alpha_X(\omega_X) \in \mathcal{B} \backslash \mathcal{H}$.

Theorem 2 Let ω_{X_d} $(d = 3, 4, 7)$ be a nowhere vanishing holomorphic 2-forms of X_d.
Then $\omega_{X_d} \in \mathcal{H} \subset \mathcal{B}$.