TITLE:
Periods of tropical Calabi-Yau hypersurfaces

AUTHOR(S):
Yamamoto, Yuto

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1. Background

- \( K := \mathbb{C}(t) \): the convergent Laurent series field
- \( f = \sum_{\alpha} k_{\alpha} t^{\alpha} \in K[x_1, \ldots, x_n] \)

**Definition**

- The **tropicalization** of \( f \) is the piecewise affine function \( \text{trop}(f) : \mathbb{R}^{d+1} \to \mathbb{R} \) defined by \( \text{trop}(f)(X) := \max_{\alpha} \{ \text{val}(k_{\alpha}) + m \cdot X \} \).
- The **tropical hypersurface** \( \mathcal{V}(\text{trop}(f)) \subset \mathbb{R}^{d+1} \) is the corner locus of \( \text{trop}(f) \).

**Conjecture** (Gross–Wilson, Kontsevich–Soibelman)

Maximally degenerating families of Calabi–Yau manifolds with Ricci-flat Kähler metrics converge to \( \mathcal{A} \)-spheres with integral affine structures with singularities in the Gromov–Hausdorff topology.

In Gross–Siebert program, an integral affine manifold with singularities \( B \) is constructed as the dual intersection complex of a toric degeneration \([2]\). In the case of hypersurfaces, it coincides with the central part of a tropical Calabi–Yau hypersurface \( \mathcal{V}(\text{trop}(f)) \) \([1]\).

2. Goal

To describe the asymptotics of Hodge structure of a degenerating family of Calabi–Yau hypersurfaces by use of the tropical Calabi–Yau hypersurface \( \mathcal{V}(\text{trop}(f)) \equiv B \).

3. Radiance obstructions

- \( B \): an integral affine manifold with singularities
- \( t : B_0 \hookrightarrow B \): the smooth part
- \( T_\mathbb{E} \): the sheaf of integral tangent vectors on \( B_0 \)
- \( T_0 \equiv T_{\mathbb{E}} \otimes \mathbb{Q} \) for \( \mathbb{Q} = \mathbb{R}, \mathbb{C} \)
- \( \{U_i\} \): a sufficiently fine open covering of \( B \)
- \( \{ s_i \in \Gamma(U_i \cap B_0, T_{\mathbb{E}} B_0) \} \)

**Definition** (Goldman–Hirsch ’84)

The **radiance obstruction** \( c_B \in H^1(B, t^* T_\mathbb{E}) \) is defined by

\[
    c_B(U_i, U_j) := s_j - s_i
\]

for each 1-simplex \((U_i, U_j)\) of \( \{U_i\} \).

4. Main results

- \( \Delta \subset M_\mathbb{R}, \Lambda \subset N_\mathbb{R} \): reflexive polytopes dual to each other
- \( f = \sum_{\alpha} k_{\alpha} t^{\alpha} \in K[x_1, \ldots, x_n] \)
- \( B \): an integral affine \( d \)-sphere obtained by contracting the tropical hypersurface \( \mathcal{V}(\text{trop}(f)) \)
- \( H^* (B, t^* T_\mathbb{E}) := \bigoplus_{m=0}^d H^m(B, t^* T_\mathbb{E}) \)

**Definition**

- \( \Sigma \subset M_{\mathbb{R}} \): a subdivision of the normal fan of \( \Lambda \) that gives rise to a crepant resolution
- \( X_\mathbb{E} \): the complex toric variety associated with \( \Sigma \)
- \( D_{\rho} \): the toric divisor on \( X_\mathbb{E} \) corresponding to \( \rho \in \Sigma \)
- \( Y \subset X_\mathbb{E} \): an anti-canonical hypersurface
- \( H^0_{\text{amb}} (Y, \mathbb{Z}) := \text{Im} \left[ \mathcal{V} : H^0(X_\mathbb{E}, \mathbb{Z}) \to H^0(Y, \mathbb{Z}) \right] \)
- \( H^*_{\text{amb}} (Y, \mathbb{Z}) := \bigoplus_{m=0}^d H^m_{\text{amb}} (Y, \mathbb{Z}) \)

**Theorem 1** (Y.)

1. There is an injective graded ring homomorphism

\[
    \psi : H^*_{\text{amb}} (Y, \mathbb{Z}) \hookrightarrow H^* (B, t^* \Lambda^* T_\mathbb{E}).
\]

2. The radiance obstruction \( c_B \) is given by

\[
    c_B = \sum_{\rho \in \Sigma} \left[ h(\rho) - h(0) \right] \psi(D_{\rho}).
\]

**Theorem 2** (Y.)

The **radiance obstruction** \( c_B \) is isomorphic to \( H^* (B, t^* T_\mathbb{E}) \) and the monodromy is given by the cup product with \( \exp(-2\pi \sqrt{-1} c_1) \).

**Definition**

The **tropical period** of \( B \) is the following polarized logarithmic Hodge structure \((H_\mathbb{E}, Q, F)\) on the standard log point \( \{0\} \):

- the locally constant sheaf \( H_\mathbb{E} \) on \( \{0\} \) isomorphic to \( \mathbb{S} \) whose stalk is
- the \((-1)^d\) symmetric pairing
- the decreasing filtration \((F^p)_{p=1}^d\) of

\[
    F^p := \bigoplus_{q=p+1}^d H^q (B, t^* \Lambda^* T_\mathbb{E}).
\]

**Theorem 2** (Y.)

The restriction of this logarithmic VPH to \( \{0\} \) is isomorphic to the tropical period of \( B \).

**References**


