Periods of tropical Calabi–Yau hypersurfaces

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1. Background

- $K := \mathbb{C}\{t\}$: the convergent Laurent series field
- $f = \sum_m k_m x^m \in K[x_1^{\pm}, \cdots, x_{d+1}^{\pm}]$

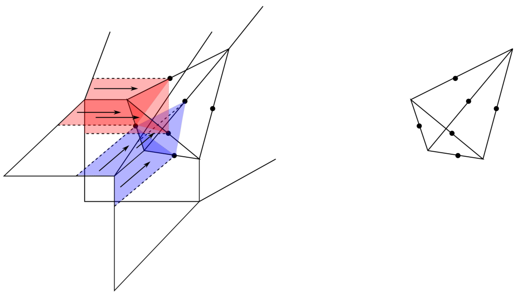
Definition

- The tropicalization of f is the piecewise affine function $\mathrm{trop}(f) \colon \mathbb{R}^{d+1} \to \mathbb{R}$ defined by
$$\mathrm{trop}(f)(X) := \max_m \{\mathrm{val}(k_m) + m \cdot X\}.$$
- The tropical hypersurface $V(\mathrm{trop}(f)) \subset \mathbb{R}^{d+1}$ is the corner locus of $\mathrm{trop}(f)$.

Conjecture (Gross–Wilson, Kontsevich–Soibelman)

Maximally degenerating families of Calabi–Yau manifolds with Ricci-flat Kähler metrics converge to d-spheres with integral affine structures with singularities in the Gromov–Hausdorff topology.

In Gross–Siebert program, an integral affine manifold with singularities B is constructed as the dual intersection complex of a toric degeneration [2]. In the case of hypersurfaces, it coincides with the central part of a tropical Calabi–Yau hypersurface $V(\mathrm{trop}(f))$ [1].



2. Goal

To describe the asymptotics of Hodge structure of a degenerating family of Calabi–Yau hypersurfaces by use of the tropical Calabi–Yau hypersurface $V(\mathrm{trop}(f)) \simeq B$.

3. Radiance obstructions

- B : an integral affine manifold with singularities
- $\iota \colon B_0 \hookrightarrow B$: the smooth part
- $\mathcal{T}_{\mathbb{Z}}$: the sheaf of integral tangent vectors on B_0
- $\mathcal{T}_Q := \mathcal{T}_{\mathbb{Z}} \otimes_{\mathbb{Z}} Q$ for $Q = \mathbb{R}, \mathbb{C}$
- $\{U_i\}_i$: a sufficiently fine open covering of B
- $\left\{ s_i \in \Gamma(U_i \cap B_0, T^{\mathrm{aff}} B_0) \right\}_i$

Definition (Goldman–Hirsch '84)

The radiance obstruction $c_B \in H^1(B, \iota_* \mathcal{T}_{\mathbb{R}})$ is defined by
$$c_B((U_i, U_j)) := s_j - s_i$$
for each 1-simplex (U_i, U_j) of $\{U_i\}_i$.

4. Main results

- $\Delta \subset M_{\mathbb{R}}, \check{\Delta} \subset N_{\mathbb{R}}$: reflexive polytopes dual to each other
- $f = \sum_{m \in \Delta \cap M} k_m x^m \in K[x_1^{\pm}, \cdots, x_{d+1}^{\pm}]$
- B : an integral affine d-sphere obtained by contracting the tropical hypersurface $V(\mathrm{trop}(f))$
- $H^\bullet(B, \iota_* \bigwedge^\bullet \mathcal{T}_{\mathbb{Z}}) := \bigoplus_{i=0}^d H^i\left(B, \iota_* \bigwedge^i \mathcal{T}_{\mathbb{Z}}\right)$
- $\check{\Sigma} \subset M_{\mathbb{R}}$: a subdivision of the normal fan of $\check{\Delta}$ that gives rise to a crepant resolution
- $X_{\check{\Sigma}}$: the complex toric variety associated with $\check{\Sigma}$
- D_ρ : the toric divisor on $X_{\check{\Sigma}}$ corresponding to $\rho \in \check{\Sigma}(1)$
- $Y \subset X_{\check{\Sigma}}$: an anti-canonical hypersurface
- $H^{2i}_{\mathrm{amb}}(Y, \mathbb{Z}) := \mathrm{Im}\left\{\iota^* \colon H^{2i}(X_{\check{\Sigma}}, \mathbb{Z}) \to H^{2i}(Y, \mathbb{Z})\right\}$
- $H^{\bullet}_{\mathrm{amb}}(Y, \mathbb{Z}) := \bigoplus_{i=0}^d H^{2i}_{\mathrm{amb}}(Y, \mathbb{Z})$

Theorem 1 (Y.)

1. There is an injective graded ring homomorphism
$$\psi \colon H^\bullet_{\mathrm{amb}}(Y, \mathbb{Z}) \hookrightarrow H^\bullet(B, \iota_* \textstyle\bigwedge^\bullet \mathcal{T}_{\mathbb{Z}}).$$
2. The radiance obstruction c_B is given by
$$c_B = \sum_{\rho \in \Sigma(1)} \left\{ h(m_\rho) - h(0) \right\} \psi(D_\rho).$$

- $H^{\mathrm{amb}}_{A,\mathbb{Z}} := \left\{ \left(2\pi\sqrt{-1}\right)^{-d} \widehat{\Gamma}_Y \cup \left(2\pi\sqrt{-1}\right)^{\frac{\deg}{2}} \mathrm{ch}(\iota^* \mathcal{E}) \middle| \mathcal{E} \in K(X_{\check{\Sigma}}) \right\}$
- $H^\bullet_{\psi,\mathbb{Z}}(B, \iota_* \bigwedge^\bullet \mathcal{T}_{\mathbb{C}})$: the image of $H^{\mathrm{amb}}_{A,\mathbb{Z}}$ by
$$\psi \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{C}} \colon H^\bullet_{\mathrm{amb}}(Y, \mathbb{C}) \hookrightarrow H^\bullet(B, \iota_* \textstyle\bigwedge^\bullet \mathcal{T}_{\mathbb{C}})$$

Definition

The tropical period of B is the following polarized logarithmic Hodge structure $(H_{\mathbb{Z}}, Q, F)$ on the standard log point $\{0\}$:

- the locally constant sheaf $H_{\mathbb{Z}}$ on $\{0\}^{\log} \cong S^1$ whose stalk is isomorphic to $H^\bullet_{\psi,\mathbb{Z}}(B, \iota_* \bigwedge^\bullet \mathcal{T}_{\mathbb{C}})$ and the monodromy is given by the cup product with $\exp\left(-2\pi\sqrt{-1} c_B\right)$,
- the $(-1)^d$-symmetric pairing
$$Q \colon H^\bullet_{\psi,\mathbb{Z}}(B, \iota_* \textstyle\bigwedge^\bullet \mathcal{T}_{\mathbb{C}}) \times H^\bullet_{\psi,\mathbb{Z}}(B, \iota_* \bigwedge^\bullet \mathcal{T}_{\mathbb{C}}) \to H^d\left(B, \iota_* \bigwedge^d \mathcal{T}_{\mathbb{C}}\right) \cong \mathbb{C}$$
induced by the wedge product,
- the decreasing filtration $\{F^p\}_{p=1}^d$ of $\mathcal{O}^{\log}_{\{0\}} \otimes_{\mathbb{Z}} H_{\mathbb{Z}} \cong \mathcal{O}^{\log}_{\{0\}} \otimes_{\mathbb{Z}} H^\bullet_{\psi,\mathbb{Z}}(B, \iota_* \bigwedge^\bullet \mathcal{T}_{\mathbb{C}})$ given by
$$F^p := \mathcal{O}^{\log}_{\{0\}} \otimes_{\mathbb{Z}} \left(\bigoplus_{i=0}^{d-p} H^i_{\psi,\mathbb{Z}}\left(B, \iota_* \textstyle\bigwedge^i \mathcal{T}_{\mathbb{C}}\right)\right).$$

- $D_\varepsilon := \{z \in \mathbb{C} \mid |z| < \varepsilon\}$
- $q \in D_\varepsilon \setminus \{0\} \quad \rightsquigarrow \quad f_q := f|_{t=q} \in \mathbb{C}[x_1^{\pm}, \cdots, x_{d+1}^{\pm}]$
- $\left\{V_q\right\}_q := \left\{f_q = 0\right\}_q$: a family of Calabi–Yau hypersurfaces
  $\rightsquigarrow$ the residual B-model VHS of Iritani [3] on $D_\varepsilon \setminus \{0\}$
  $\rightsquigarrow$ the logarithmic VHS of Kato–Usui [4] on D_ε

Theorem 2 (Y.)

The restriction of this logarithmic VPH to $\{0\}$ is isomorphic to the tropical period of B.

References

[1] Mark Gross, *Toric degenerations and Batyrev-Borisov duality*, Math. Ann. 333 (2005), no. 3, 645-688.

[2] Mark Gross and Bernd Siebert, *Mirror symmetry via logarithmic degeneration data. I*, J. Differential Geom. 72 (2006), no. 2, 169-338.

[3] Hiroshi Iritani, *Quantum cohomology and periods*, Ann. Inst. Fourier (Grenoble) 61 (2011), no. 7, 2909-2958.

[4] Kazuya Kato and Sampei Usui, *Classifying spaces of degenerating polarized Hodge structures*, Annals of Mathematics Studies, vol. 169, Princeton University Press, Princeton, NJ, 2009.