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Benchmarking and Fair Pricing Applied to Two Market Models

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This paper considers a market containing both continuous and discrete noise. Modest assumptions ensure the existence of a growth optimal portfolio. Non-negative self-financing trading strategies, when benchmarked by this portfolio, are local martingales under the real-world measure. This justifies the fair pricing approach, which expresses derivative prices in terms of real-world conditional expectations of benchmarked payoffs. Two models for benchmarked primary security accounts are presented, and fair pricing formulas for some common contingent claims are derived.

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1. Introduction

This paper considers a general market where the traded uncertainty is driven by a combination of continuous and discrete noise sources. The basic constituents of the market are a finite number of *primary security accounts*, each of which contains units of some primary asset, with all dividend income reinvested. We examine this model within the *benchmark framework*, see Platen (2004a, 2004b, 2004c). This means that we make only the modest assumptions necessary to ensure the existence of a *growth optimal portfolio (GOP)* (see Kelly (1956), Long (1990) and Bajeux-Besnainou & Portait (1997)) made up of primary security accounts.

The expression of primary security accounts and self-financing portfolios in units of the GOP is referred to as *benchmarking*. An important result is that all non-negative benchmarked primary security accounts and self-financing portfolios are local martingales under the real-world measure. This leads us to introduce the concept of *fair pricing*. Here benchmarked contingent claim prices are martingales and thus expressed in terms of conditional expectations, computed under the real-world measure, of their benchmarked payoffs.

We believe that fair pricing within the benchmark framework offers some important advantages over risk-neutral pricing. If the benchmarked savings account is a martingale under the real-world measure, then it can be used as a Radon-Nikodym derivative process to effect a measure transformation from the real-world measure to an equivalent risk-neutral measure. In this case fair pricing corresponds with risk-neutral pricing; in fact, for continuous markets, the risk-neutral measure constructed in this way is none other than the *minimal martingale measure* introduced in Föllmer & Schweizer (1991). However, if the benchmarked savings account is a strict local martingale under the real-world measure, then such a measure transformation is not possible. Empirical evidence indicates that this might be the case; and so it appears that an alternative to risk-neutral pricing may be justified.

A distinguishing feature of the benchmark approach is that the objects of fundamental interest are the benchmarked primary security accounts. Before modelling their behaviour in detail, we make the simplifying assumption that the GOP is continuous. This is realistic if the domestic currency is a major global currency. Particular models for the benchmarked primary security account dynamics arise by specifying their SDEs. We examine two such models in detail.

The Merton model (MM) This is very much the standard market model, with all parameters constant. It is essentially a modification of the jump-diffusion model introduced in Merton (1976). Each benchmarked primary security account can be expressed as the product of a driftless geometric Brownian motion and an independent jump martingale, and is thus itself a martingale. In particular, the benchmarked savings account is a continuous martingale, and hence a valid Radon-Nikodym derivative process. Consequently, a measure transformation to an equivalent risk-neutral measure is feasible, and the standard risk-neutral pricing theory is applicable.

The minimal market model (MMM) In this case we constrain the parameters associated with the jump parts of the benchmarked primary security accounts to be constant. Their continuous parts are modelled as inverted time-transformed squared Bessel processes of dimension four. Consequently, each benchmarked primary security account is the product of an inverted time-transformed squared Bessel process of dimension four and an independent jump martingale. Since inverted squared Bessel processes of dimension four are strict local martingales, the benchmarked savings account is not a martingale in the MMM, and hence not a viable Radon-Nikodym derivative process. Transformation to an equivalent risk-neutral measure is thus not feasible, and so we advocate fair derivative pricing under the real-world measure.

The difference between the MM and the jump-diffusion model of Merton (1976) is that we allow primary security accounts to be affected by more than one source of jump risk, but keep the jump ratios fixed, while Merton (1976) considered only one source of jump risk, but allowed for random jump ratios. The familiarity of the MM makes it a good vehicle for illustrating the benchmark approach and the fair pricing concept.

The MMM was first introduced in Platen (2001). It is “minimal” in the sense that a time-transformed squared Bessel process of dimension four appears to provide the simplest description of the discounted GOP, involving only a small number of parameters, but which is nevertheless consistent with an optimal market dynamics and certain “stylized empirical facts” about diversified portfolios. We believe the MMM resonates well with observed market behaviour; in particular, we think that the possible non-existence of an equivalent risk-neutral measure should be taken seriously when modelling real markets.

A large part of the paper is devoted to exhibiting the fair pricing approach for common contingent claims. We compute fair prices using both the MM and the MMM. In the MM case, our modest ambition is to demonstrate how fair pricing retrieves the familiar pricing formulas for these instruments. In the case of the MMM, we wish to exhibit derivative pricing formulas for what we believe is a realistic market model. Of course, one could apply standard risk-neutral theory to obtain pricing formulas in the MM, but this would defeat our purpose of illustrating the fair pricing approach.

The structure of the paper is as follows: § 2 introduces the benchmark model with jumps in general terms before presenting the MM and the MMM as specifications; § 3 examines the pricing of a number of common contingent claims in the MM and the MMM; and § 4 concludes. There are also two appendices. Appendix A derives expressions for computing fair derivative prices in the MM; and Appendix B does the same for the MMM.

2. Benchmark Models with Jumps

2.1. Modelling Traded Uncertainty

Let $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, \mathbb{P})$ be a filtered probability space, where $T \in (0, \infty)$ is some fixed finite time horizon. The probability measure $\mathbb{P}$ is thought of as describing the probabilities of observable events, and so we call it the *real-world measure*. We assume that the filtration $\underline{\mathcal{A}} = \{\mathcal{A}_t \,|\, t \in [0, T]\}$ satisfies the usual conditions (see Karatzas & Shreve (1991, p. 10)). Furthermore, we legislate that $\mathcal{A}_0$ is trivial, in that it contains only the sets of measure zero and their complements. From now on, whenever we refer to a filtration, whether implicity or explicitly, it is understood to be $\underline{\mathcal{A}}$.

We start by imagining a market with $d \in \mathbb{N}$ sources of traded uncertainty, represented by $W^k = \{W^k_t \,|\, t \in [0, T]\}$, for $k \in \{1, \dots, d\}$. We take the first $m \leq d$ factors to represent continuously evolving noise, while the last $d - m$ factors are assumed to exhibit discontinuous event-driven randomness. The typical assumption is that $W^1, \dots, W^m$ are independent standard Brownian motions. This accounts for the continuous factors. The discontinuous factors $W^{m+1}, \dots, W^d$ are modelled as independent compensated point processes. In detail, let $N^k = \{N^k_t \,|\, t \in [0,T]\}$, for $k \in \{m+1,\ldots,d\}$, be independent point processes whose respective intensities $\lambda^k = \{\lambda^k_t \,|\, t \in [0,T]\}$, for $k \in \{m+1,\ldots d\}$, are assumed to be independent, strictly positive, predictable processes satisfying

$$\int_0^T \lambda^k_s \, ds < \infty \text{ a.s.}, \tag{2.1}$$

for each $k \in \{m+1,\ldots,d\}$. Then we have

$$W^k_t = N^k_t - \int_0^t \lambda^k_s \, ds, \tag{2.2}$$

for each $k \in \{m+1,\ldots,d\}$ and all $t \in [0,T]$.

2.2. The Market

We consider a market $S = \{S_t = (S^0_t,\ldots,S^d_t)^\top \,|\, t \in [0,T]\}$, whose components we describe as *primary security accounts*. These are best understood as the accumulated values of investments in $d+1$ underlying assets, with all accrued income reinvested. S^0 is chosen to represent a *locally riskless savings account*, whose value is given by

$$S^0_t = \exp\left\{\int_0^t r_s \, ds\right\}, \tag{2.3}$$

for all $t \in [0,T]$, where $r = \{r_t \,|\, t \in [0,T]\}$ is a non-negative adapted *interest rate process*. The remaining d security accounts are assumed to evolve according to the system of SDEs

$$dS^j_t = S^j_{t-}\left\{a^j_t \, dt + \sum_{k=1}^d b^{j,k}_t \, dW^k_t\right\}, \tag{2.4}$$

for each $j \in \{1,\ldots,d\}$ and all $t \in [0,T]$, with $S^j_0 > 0$. We assume that the interest rate process r, the *accumulation rate process* $a = \{a_t = (a^1_t,\ldots,a^d_t)^\top \,|\, t \in [0,T]\}$, the *generalized volatility matrix process* $b = \left\{b_t = [b^{j,k}_t]^d_{j,k=1} \,\middle|\, t \in [0,T]\right\}$, as well as the intensity processes λ^k, for $k \in \{m+1,\ldots,d\}$, satisfy the necessary requirements to ensure that the system (2.4) possesses a unique strong solution.

A properly specified market must preclude appropriately defined arbitrage opportunities (see § 2.7 for a detailed discussion). The following assumption is necessary in order to eliminate the obvious arbitrages that arise when some of the SDEs (2.4) have the same local martingale terms, but different drift rates (see Platen (2004e, p. 290)).

Assumption 2.1. *The generalized volatility matrix b_t is invertible for Lebesgue-almost every $t \in [0,T]$.*

Based on Assumption 2.1, we are free to introduce the *market price of risk process* $\theta = \{\theta_t = (\theta^1_t,\ldots,\theta^d_t)^\top \,|\, t \in [0,T]\}$, by setting

$$\theta_t := b_t^{-1}(a_t - r_t \mathbf{1}), \tag{2.5}$$

for all $t \in [0, T]$, where $\mathbf{1} := (1, \dots, 1)^\top$. In the light of (2.5), we can reformulate (2.4) as

$$dS_t^j = S_{t-}^j \left\{ r_t\, dt + \sum_{k=1}^{d} b_t^{j,k} (\theta_t^k\, dt + dW_t^k) \right\}, \tag{2.6}$$

for each $j \in \{1, \dots, d\}$ and all $t \in [0, T]$.

Let the sequence of stopping times $(\tau_l^k)_{l \in \mathbb{N}}$, taking values in $[0, T]$, denote the jump times of N^k, for $k \in \{m+1, \dots, d\}$. It then follows from (2.6) that

$$S_{\tau_l^k}^j = S_{\tau_l^k -}^j \left(1 + b_{\tau_l^k}^{j,k}\right), \tag{2.7}$$

for each $j \in \{1, \dots, d\}$, each $k \in \{m+1, \dots, d\}$ and all $l \in \mathbb{N}$. To ensure that the primary security accounts are non-negative, we introduce the following assumption.

Assumption 2.2. *The generalized volatility matrix process satisfies the condition*

$$b_t^{j,k} \geq -1, \tag{2.8}$$

for all $j \in \{1, \dots, d\}$, $k \in \{m+1, \dots, d\}$ *and* $t \in [0, T]$.

Note that, for $j \in \{1, \dots, d\}$, $k \in \{m+1, \dots, d\}$ and $l \in \mathbb{N}$, if $b_{\tau_l^k}^{j,k} = -1$, then the jth primary security account will jump to zero at time τ_l^k and remain there indefinitely; while if $b_{\tau_l^k}^{j,k} = 0$, it will not jump at all. Also, negative values of $b_{\tau_l^k}^{j,k}$ correspond with downward jumps, while positive values correspond with upward jumps.

2.3. Trading Strategies and Wealth Processes

We call a predictable process $\delta = \{\delta_t = (\delta_t^0, \dots, \delta_t^d)^\top \mid t \in [0, T]\}$ a *trading strategy* if it is S-integrable (see Protter (2004, p. 163)). Given a trading strategy δ, we interpret δ_t^j as the number of units of the jth primary security account held at time $t \in [0, T]$, for $j \in \{0, \dots, d\}$. A negative value indicates a short position.

Associated with any trading strategy δ is a *wealth process* $S^\delta = \{S_t^\delta \mid t \in [0, T]\}$, determined by

$$S_t^\delta = \sum_{j=0}^{d} \delta_t^j S_t^j, \tag{2.9}$$

for all $t \in [0, T]$. A strategy δ is called *self-financing* if its wealth process satisfies the SDE

$$dS_t^\delta = \sum_{j=0}^{d} \delta_t^j\, dS_t^j, \tag{2.10}$$

for all $t \in [0, T]$. Intuitively, (2.10) specifies that all changes in wealth are attributable to changes in the primary security account values, and are not due to any external flows of capital. As is customary, we restrict our attention to self-financing strategies, and hence omit the phrase “self-financing” altogether.

Let δ be a trading strategy whose corresponding wealth process is almost surely strictly positive. Then the following fractions are well-defined:

$$\pi_t^{\delta,j} := \delta_t^j \frac{S_{t-}^j}{S_{t-}^\delta}, \tag{2.11}$$

for each $j \in \{0, \ldots, d\}$ and all $t \in [0, T]$. Equation (2.11) expresses the fractions of the total wealth invested in each of the primary security accounts as predictable stochastic processes $\pi^{\delta,j} = \{\pi_t^{\delta,j} \,|\, t \in [0, T]\}$, for each $j \in \{0, \ldots, d\}$. By (2.9) it is clear that

$$\sum_{j=0}^{d} \pi_t^{\delta,j} = 1, \tag{2.12}$$

for all $t \in [0, T]$. When representing the trading strategy in terms of these fractions, condition (2.12) indicates that one of them is redundant. By nominating the savings account for omission, the process $\pi^\delta = \{\pi_t^\delta = (\pi_t^{\delta,1}, \ldots, \pi_t^{\delta,d})^\top \,\big|\, t \in [0, T]\}$ provides an alternative description of the trading strategy. By (2.12) the fraction of wealth invested in the savings account at any time $t \in [0, T]$ can be retrieved from the above process as $\pi_t^{\delta,0} = 1 - \sum_{j=1}^d \pi_t^{\delta,j}$. A simple argument using (2.10), (2.3), (2.6), (2.11) and (2.9) now yields the following SDE for the wealth process:

$$dS_t^\delta = S_{t-}^\delta \Big\{ r_t \, dt + \sum_{j=1}^{d} \sum_{k=1}^{d} \pi_t^{\delta,j} b_t^{j,k} (\theta_t^k \, dt + dW_t^k) \Big\}, \tag{2.13}$$

for all $t \in [0, T]$, with $S_0^\delta = \sum_{j=0}^d \delta_0^j S_0^j$.

2.4. The Growth Optimal Portfolio

The *growth optimal portfolio (GOP)* enjoys a distinguished history, starting with Kelly (1956). Among a number of equivalent characterizations, it may be described either as the trading strategy which maximizes the expected logarithm of its associated terminal wealth, for any given time horizon; or as the trading strategy which maximizes the expected growth rate of its wealth process at all times (see Bajeux-Besnainou & Portait (1997, Proposition 1, p. 294)). Much attention has been devoted to establishing the existence of the GOP for various market models (see Becherer (2001), Korn & Schäl (1999), Platen (2002), Goll & Kallsen (2003) and Korn, Oertel & Schäl (2003)).

Assumption 2.1 would be sufficient to avoid arbitrage (see § 2.7) in a continuous market. However, in the current setting which incorporates jumps, arbitrage opportunities may also arise from "exploding" portfolios. The following assumption is necessary to eliminate them (see Platen (2004e, p. 291)).

Assumption 2.3. *The market prices of event risks satisfy*

$$\lambda_t^k > \theta_t^k, \tag{2.14}$$

for each $k \in \{m + 1, \ldots, d\}$ *and all* $t \in [0, T]$.

Subject to Assumption 2.3, Platen (2004c, Proposition 2.1, p. 24) established the existence of a trading strategy δ_* whose wealth process S^{δ_*} is the GOP. Furthermore, this strategy is unique, modulo its initial wealth. For $S_0^{\delta_*} = 1$, δ_* is specified through its fractions of wealth by

$$\pi_t^{\delta_*,j} := \sum_{k=1}^{m} \theta_t^k (b_t^{-1})^{k,j} + \sum_{k=m+1}^{d} \frac{\theta_t^k}{\lambda_t^k - \theta_t^k} (b_t^{-1})^{k,j}, \tag{2.15}$$

for each $j \in \{1, \ldots, d\}$ and all $t \in [0, T]$. A straightforward substitution of the expressions from (2.15) into (2.13) yields the following SDE for the GOP:

$$dS_t^{\delta_*} = S_{t-}^{\delta_*}\Big\{r_t\, dt + \sum_{k=1}^{m} \theta_t^k (\theta_t^k\, dt + dW_t^k) + \sum_{k=m+1}^{d} \frac{\theta_t^k}{\lambda_t^k - \theta_t^k} (\theta_t^k\, dt + dW_t^k)\Big\}, \tag{2.16}$$

for all $t \in [0, T]$, with $S_0^{\delta_*} = 1$.

2.5. Benchmarking

Let δ be an arbitrary trading strategy. An important consequence of Assumption 2.3 is that $S_t^{\delta_*} > 0$ a.s., for all $t \in [0, T]$ (see Platen (2004c, p. 23)). This permits us to make sense of the process $\hat{S}^{\delta} = \{\hat{S}_t^{\delta} \mid t \in [0, T]\}$, defined by

$$\hat{S}_t^{\delta} := \frac{S_t^{\delta}}{S_t^{\delta_*}}, \tag{2.17}$$

for all $t \in [0, T]$. We call $\hat{S}^{\delta}$ a *benchmarked wealth process*. In the case where δ is determined by $\pi_t^{\delta,j} = 1$, for some $j \in \{0, \ldots, d\}$ and all $t \in [0, T]$, with all other fractions identically zero, we employ the notation $\hat{S}^j = \{\hat{S}_t^j \mid t \in [0, T]\}$ and speak of a *benchmarked primary security account process*. An application of Itô's formula to (2.13) and (2.16) yields the following SDE for the benchmarked wealth process:

$$\begin{aligned} d\hat{S}_t^{\delta} = \hat{S}_{t-}^{\delta}\Big\{ &\sum_{k=1}^{m} \Big(\sum_{j=1}^{d} \pi_t^{\delta,j} b_t^{j,k} - \theta_t^k\Big) dW_t^k \\ &+ \sum_{k=m+1}^{d} \Big[\sum_{j=1}^{d} \pi_t^{\delta,j} b_t^{j,k} \Big(1 - \frac{\theta_t^k}{\lambda_t^k}\Big) - \frac{\theta_t^k}{\lambda_t^k}\Big] dW_t^k\Big\}, \end{aligned} \tag{2.18}$$

for all $t \in [0, T]$, with $\hat{S}_0^{\delta} = S_0^{\delta}$. For the case of a benchmarked primary security account process, where $\pi_t^{\delta,j} = 1$, for some $j \in \{0, \ldots, d\}$ and all $t \in [0, T]$, with all other fractions identically zero, (2.18) specializes to

$$d\hat{S}_t^j = \hat{S}_{t-}^j\Big\{\sum_{k=1}^{m} (b_t^{j,k} - \theta_t^k)\, dW_t^k + \sum_{k=m+1}^{d} \Big[b_t^{j,k}\Big(1 - \frac{\theta_t^k}{\lambda_t^k}\Big) - \frac{\theta_t^k}{\lambda_t^k}\Big] dW_t^k\Big\}, \tag{2.19}$$

for all $t \in [0, T]$, with $\hat{S}^j_0 = S^j_0$.

Notice that (2.18) and (2.19) exhibit no drift. Since $W^{m+1}, \ldots, W^d$ contain jumps, this implies that benchmarked wealth processes and benchmarked primary security account processes are sigma-martingales (see Protter (2004, Theorem 89, p. 234)). However, by restricting our attention to non-negative wealth processes and primary security account processes, as we will from now on, it follows that benchmarking yields local martingales (see Protter (2004, Exercise 41, p. 241)). By virtue of Rogers & Williams (2000, Lemma 14.3, p. 22) this means that all non-negative wealth processes and primary security account processes become supermartingales under benchmarking.

2.6. Fair Pricing

The current orthodoxy for contingent claim pricing relies on the interplay between *equivalent local martingale measures (ELMMs)* and *numéraires*, and can be traced back to Harrison & Kreps (1979), and Harrison & Pliska (1981). Given a numéraire (see Geman, El Karoui & Rochet (1995)), the idea is that, under agreeable circumstances, one can identify a *pricing measure* $\tilde{\mathbb{P}}$, equivalent to $\mathbb{P}$, such that the numéraire-denominated wealth processes of all *admissible trading strategies* (see Delbaen & Schachermayer (1994, p. 467)) are local martingales under $\tilde{\mathbb{P}}$. The existence of such a pricing measure depends on the absence of certain kinds of arbitrage. If the market is complete, then it will be unique as well.

Typically, the numéraire is taken to be the savings account; in which case an ELMM is called a *risk-neutral measure*; that is, an equivalent measure under which all discounted admissible wealth processes are local martingales. Another example is when a zero-coupon bond is used as numéraire. In this case a corresponding ELMM is called a *forward-adjusted measure*.

In most applications the numéraire is identified first. Then the measure-theoretic machinery based on Girsanov's theorem is used to infer the existence of a corresponding ELMM. An obvious reason for this is that while numéraires are easily recognized, it is not clear how to obtain a probability measure equivalent to $\mathbb{P}$ directly. This makes going in the opposite direction, by first choosing an equivalent pricing measure and then finding the corresponding numéraire, very difficult, in general. One obvious exception is when we pick the real-world measure $\mathbb{P}$ itself as the pricing measure. Then we are left with having to find a numéraire so that all numéraire-denominated wealth processes are real-world local martingales. As (2.18) attests, the desired numéraire is in fact the GOP (see Long (1990) for the prototype of this result in a discrete-time setting).

In the light of the above, let us now examine risk-neutral pricing. This depends upon the existence of a Radon-Nikodym derivative process for transforming the real-world measure into an equivalent risk-neutral measure. According to Geman, El Karoui & Rochet (1995, Theorem 1, p. 448) or Musiela & Rutkowski (2005, Proposition 8.1.6, p. 293), the natural candidate is the benchmarked savings account $\hat{S}^0$. However, for $\hat{S}^0$ to be a Radon-Nikodym derivative process one must assume that it is a martingale under $\mathbb{P}$, and not merely a local martingale, as indicated by

(2.19). We can identify two arguments that call this assumption into question.

Empirical evidence The hypothesis that $\hat{S}^0$ is a martingale under the real-world measure $\mathbb{P}$ can be examined empirically, since Platen (2004d, Theorem 2.1, p. 517) demonstrates that, subject to reasonable assumptions, any diversified global index approximates the GOP. Alternatively, by Platen (2004a, Theorem 3.8, p. 13), it follows that the global market portfolio is a proxy for the GOP if all investors prefer more for less. Consequently, the historical long-term record of the values of a savings account divided by the corresponding values of a diversified world index, with both sets of data normalized to have initial value one, provides an approximate realized sample path for $\hat{S}^0$. Plotting such a sample path (see Platen (2004d, Figure 8, p. 527) and also Dimson, Marsh & Staunton (2002, Chapter 34)) suggests quite strongly that $\hat{S}^0$ is not a martingale, but rather a strict supermartingale.

Modelling flexibility It is well known that arbitrarily specified markets do not, in general, admit ELMMs (see Karatzas & Shreve (1998, p. 11 and the discussion on p. 33)). So the requirement that the benchmarked savings account should be a martingale under the real-world measure is a constraint on modelling freedom. Recent research on parsimonious market models highlights the importance of this freedom. According to Platen (2004d) it appears that the natural dynamics of the discounted GOP are those of a time-transformed squared Bessel process of dimension four. Benchmarking the savings account with such a GOP does not produce a martingale.

Of course, even if we agree that the benchmarked savings account is not a martingale under the real-world measure, this may, in general, be insufficient for us to infer that no equivalent risk-neutral probability measures exist; but it is certainly enough for us to consider the possibility seriously. We will revisit this question in Remark 3.1. At the very least, we may conclude that there are sufficient grounds to be skeptical of pricing approaches based on ELMM transformations if one is concerned with modelling real markets.

If the arguments above seem plausible, then a pricing theory based upon the existence of ELMMs might not be tenable. This statement applies to risk-neutral pricing, in particular. The need for an alternative approach leads to the introduction of the concept of *fair pricing*.

Definition 2.4. Let $\tau \in [0, T]$ be a stopping time, and suppose that H is an $\mathcal{A}_\tau$-measurable random variable satisfying

$$\mathbb{E}\left[\frac{|H|}{S_\tau^{\delta_*}} \,\middle|\, \mathcal{A}_t\right] < \infty \quad \text{a.s.}, \tag{2.20}$$

for all $t \in [0, \tau]$. Then the *fair price process* $V^H = \{V_t^H \,|\, t \in [0, \tau]\}$ of a contingent claim paying H at its maturity τ is defined by

$$V_t^H = S_t^{\delta_*} \mathbb{E}\left[\frac{H}{S_\tau^{\delta_*}} \,\middle|\, \mathcal{A}_t\right], \tag{2.21}$$

for all $t \in [0, \tau]$.

Thus fair pricing simply involves the computation of real-world expectations of numéraire-denominated payoffs, with the GOP as numéraire. What distinguishes it from the standard numéraire approach is the absence of the assumption that numéraire-denominated wealth processes must be martingales under the pricing measure, which in this case is the real-world measure $\mathbb{P}$.

As demonstrated in Platen (2004c, § 3.1–2), for example, fair pricing generalizes both risk-neutral and actuarial pricing. Furthermore, the fair pricing formula (2.21) depends only on the existence of a GOP, which is a less restrictive requirement than the existence of a risk-neutral measure. This will become important in § 2.9.2, where we describe a market model which admits a GOP but no equivalent risk-neutral measures. Furthermore, it has been pointed out in Heath & Platen (2002) that the fair price corresponds to the minimal replicating portfolio in a complete market. Under such circumstances (2.21) appears to be the most natural way to price derivatives.

Of course, the GOP is not always the most convenient numéraire for obtaining simple pricing formulas. However, fair pricing does offer other advantages. The fact that it involves no measure changes means that the time horizon can be extended to $T = \infty$ with impunity (see Platen (2004b, § 3) for a discussion of this). Thus the pricing of perpetual securities, for example, introduces no extra complications. By contrast, the application of Girsanov's theorem to transform $\mathbb{P}$ into an equivalent measure $\tilde{\mathbb{P}}$, when $T = \infty$, is technically involved (see Karatzas & Shreve (1991, p. 193) and Bichteler (2002, Example 3.9.14, p. 164 and Warning 3.9.20, p. 167)). Furthermore, when expressed in units of the GOP, fair derivative prices are simply martingales.

2.7. Arbitrage

The following notion of arbitrage was used in Platen (2002). It can be justified by appealing to the fundamental and legally enforced principle of *limited liability*, which holds that a market participant whose total wealth becomes negative may no longer transact. Thus, we argue that the only arbitrage opportunities germane to real financial markets occur when an investor can generate positive terminal wealth from zero initial wealth, without falling foul of the limited liability constraint enforced by market regulators.

Definition 2.5. An *arbitrage* is a wealth process S^δ that is a.s. non-negative, with $S_0^\delta = 0$ a.s., and which satisfies

$$\mathbb{P}[S_\tau^\delta > 0] > 0, \tag{2.22}$$

for some stopping time τ taking values in $[0, T]$.

As already noted, non-negative benchmarked wealth processes are supermartingales. A standard argument then shows that the value of a non-negative wealth process must remain zero indefinitely, if its initial value is zero. Thus, arbitrage, in the sense of Definition 2.5, is excluded in the benchmark framework.

Historically, substantial effort has been devoted to teasing out the precise relationship between the non-existence of arbitrage opportunities and the existence of equivalent risk-neutral measures. This enterprise was initiated in Harrison & Kreps (1979), and Harrison & Pliska (1981), and culminated in the definitive formulation of the *fundamental theorem of asset pricing*, established in Delbaen & Schachermayer (1994) for locally bounded semimartingale markets; and extended to the case of general semimartingale markets in Delbaen & Schachermayer (1998). This result states an equivalence between a particular no-arbitrage principle, namely *no free lunches with vanishing risk (NFLVR)*, and the existence of equivalent probability measures under which discounted wealth processes are local martingales, in the locally bounded semimartingale case; or sigma-martingales, in the general semimartingale case.

Note that a no-arbitrage criterion based on Definition 2.5 is weaker than the NFLVR condition. This means that market models, which exclude arbitrage opportunities in the former sense, may in fact not admit any equivalent risk-neutral measures. This is not as serious as it sounds. As argued in Loewenstein & Willard (2000), the real economic content of a no-arbitrage condition lies in the existence of a *competitive equilibrium*, in the sense that an investor who prefers more to less should have an optimal trading strategy. In the case of a market driven by a standard multivariate Brownian motion, these authors demonstrate (see Loewenstein & Willard (2000, Theorem 1, p. 142)) that an agent who prefers more to less and takes prices as given will have an optimal strategy if and only if arbitrages very similar to Definition 2.5 (see Loewenstein & Willard (2000, Definition 1, p. 141)) are prohibited. We may summarize our view by saying that the existence of an equivalent risk-neutral measure is a convenience, but not a necessity; what is important in a market model is that under limited liability investors cannot generate strictly positive wealth from zero initial capital. A no-arbitrage principle weaker than NFLVR may ensure the latter without guaranteeing the former.

Finally, we may point out that the NFLVR condition is not completely robust with respect to changes of numéraire (see Delbaen & Schachermayer (1995)). In fact, even the basic concept of portfolio admissibility, which forms part of the definition of NFLVR, is numéraire dependent. Clearly, the notion of arbitrage encapsulated in Definition 2.5 introduces no numéraire dependencies; if a given portfolio is an arbitrage with respect to one numéraire, the same will be true for all numéraires.

2.8. Specifying a Continuous GOP

If we regard the GOP as representing a very large diversified global portfolio expressed in units of a leading currency, then aggregating all the jumps in the underlying primary security accounts should produce noise which is approximately continuous. In other words, we would expect the jumps to be invisible to an observer of the GOP. This is consistent with empirical data, when a global market index denominated in units of, say, US dollars is used as a proxy for the GOP. According to (2.16), the only way to eliminate jumps from the GOP is by setting the market prices of jump risks equal to zero. Henceforth, the following refinement of

Assumption 2.3 will be in force.

Assumption 2.6. *The market prices of event risks satisfy*

$$\theta_t^k = 0, \tag{2.23}$$

for each $k \in \{m+1, \ldots, d\}$ *and all* $t \in [0, T]$.

For notational convenience we introduce the predictable matrix-valued process $\sigma = \{\sigma_t = [\sigma_t^{j,k}]_{j=0,k=1}^d \,\big|\, t \in [0, T]\}$, defined by

$$\sigma_t^{j,k} := \begin{cases} \theta_t^k & \text{if } j = 0; \\ \theta_t^k - b_t^{j,k} & \text{if } j \in \{1, \ldots, d\}, \end{cases} \tag{2.24}$$

for each $j \in \{0, \ldots, d\}$, each $k \in \{1, \ldots, d\}$ and all $t \in [0, T]$. Substitution of (2.24) into (2.6) yields

$$dS_t^j = S_{t-}^j \Big\{ r_t\, dt + \sum_{k=1}^d (\sigma_t^{0,k} - \sigma_t^{j,k})(\sigma_t^{0,k}\, dt + dW_t^k) \Big\}, \tag{2.25}$$

for each $j \in \{0, \ldots, d\}$ and all $t \in [0, T]$, with $S_0^j > 0$. By an application of Itô's formula for jump diffusions (see Runggaldier (2003, § 2.4)) we can verify the following solution to (2.25):

$$\begin{aligned} S_t^j = &S_0^j \exp\Big\{ \int_0^t \Big(r_s + \frac{1}{2} \sum_{k=1}^m \Big[(\sigma_s^{0,k})^2 - (\sigma_s^{j,k})^2 \Big] \Big) ds \\ &+ \sum_{k=1}^m \int_0^t (\sigma_s^{0,k} - \sigma_s^{j,k})\, dW_s^k \Big\} \\ &\times \exp\Big\{ \int_0^t \sum_{k=m+1}^d \sigma_s^{j,k} \lambda_s^k\, ds \Big\} \prod_{k=m+1}^d \prod_{l=1}^{N_t^k} \Big(1 - \sigma_{\tau_l^k}^{j,k} \Big), \end{aligned} \tag{2.26}$$

for each $j \in \{0, \ldots, d\}$ and all $t \in [0, T]$. Here $(\tau_l^k)_{l \in \mathbb{N}}$ denotes the sequence of jump times of N^k, for each $k \in \{m+1, \ldots, d\}$, as before.

Next, substitution of (2.24) into (2.16) produces the following SDE for the GOP:

$$dS_t^{\delta_*} = S_t^{\delta_*} \Big\{ r_t\, dt + \sum_{k=1}^m \sigma_t^{0,k} (\sigma_t^{0,k}\, dt + dW_t^k) \Big\}, \tag{2.27}$$

for all $t \in [0, T]$, with $S_0^{\delta_*} = 1$. The solution to (2.27) is given by

$$S_t^{\delta_*} = \exp\Big\{ \int_0^t \Big(r_s + \frac{1}{2} \sum_{k=1}^m (\sigma_s^{0,k})^2 \Big) ds + \sum_{k=1}^m \int_0^t \sigma_s^{0,k}\, dW_s^k \Big\}, \tag{2.28}$$

for all $t \in [0,T]$. Finally, the SDEs for the benchmarked primary security accounts are derived by substituting (2.24) into (2.19), yielding

$$d\hat{S}^j_t = -\hat{S}^j_{t-} \sum_{k=1}^{d} \sigma^{j,k}_t \, dW^k_t, \tag{2.29}$$

for all $j \in \{0,\dots,d\}$ and $t \in [0,T]$, with $\hat{S}^j_0 = S^j_0$. To solve (2.29), we simply divide (2.26) by (2.28), to get

$$\begin{aligned}\hat{S}^j_t = S^j_0 \exp\Bigl\{-\frac{1}{2}\int_0^t \sum_{k=1}^{m} (\sigma^{j,k}_s)^2 \, ds - \sum_{k=1}^{m} \int_0^t \sigma^{j,k}_s \, dW^k_s\Bigr\} \\ \times \exp\Bigl\{\int_0^t \sum_{k=m+1}^{d} \sigma^{j,k}_s \lambda^k_s \, ds\Bigr\} \prod_{k=m+1}^{d} \prod_{l=1}^{N^k_t} \bigl(1 - \sigma^{j,k}_{\tau^k_l}\bigr),\end{aligned} \tag{2.30}$$

for each $j \in \{0,\dots,d\}$ and all $t \in [0,T]$. By comparing (2.30) with (2.26) we see that benchmarking only affects the continuous first factor of (2.26), while leaving the discontinuous second factor unchanged. This is a consequence of specifying a continuous GOP. Thus, if we write $S^{j,\mathrm{c}}_t$ for the continuous first factor in (2.26) and $S^{j,\mathrm{d}}_t$ for the discontinuous second factor, for $j \in \{0,\dots,d\}$, then $\hat{S}^j_t = \hat{S}^{j,\mathrm{c}}_t S^{j,\mathrm{d}}_t$, according to (2.30), $t \in [0,T]$.

2.9. Two Models for the Benchmarked Primary Security Accounts

A fundamental insight of the benchmark approach is that the benchmarked primary security accounts are the pivotal objects of study. This is because the savings account and the benchmarked primary security accounts together specify the entire investment universe. For example, $S^{\delta_*}_t = \frac{S^0_t}{\hat{S}^0_t}$, for all $t \in [0,T]$, expresses the GOP in terms of the savings account and the benchmarked savings account. Also, $S^j_t = \hat{S}^j_t S^{\delta_*}_t = \hat{S}^j_t \frac{S^0_t}{\hat{S}^0_t}$, for each $j \in \{1,\dots,d\}$ and all $t \in [0,T]$, expresses each primary security account in terms of the corresponding benchmarked primary security account, the savings account and the benchmarked savings account. In this section we will present two models for the benchmarked primary security accounts. But first we introduce some simplifying notation.

Define the processes $|\sigma^j| = \{|\sigma^j_t| \mid t \in [0,T]\}$, for $j \in \{0,\dots,d\}$, by setting

$$|\sigma^j_t| := \sqrt{\sum_{k=1}^{m} (\sigma^{j,k}_t)^2}, \tag{2.31}$$

for each $j \in \{0,\dots,d\}$ and all $t \in [0,T]$. We will also require the *normalized aggregate continuous noise processes* $\hat{W}^j = \{\hat{W}^j_t \mid t \in [0,T]\}$, for $j \in \{0,\dots,d\}$, defined by

$$\hat{W}^j_t := \sum_{k=1}^{m} \int_0^t \frac{\sigma^{j,k}_s}{|\sigma^j_s|} \, dW^k_s, \tag{2.32}$$

for each $j \in \{0,\dots,d\}$ and all $t \in [0,T]$. By Lévy's characterization of Brownian motion (see Karatzas & Shreve (1991, Theorem 3.16, p. 157)), it follows that $\hat{W}^j$ is a Brownian motion, for each $j \in \{0,\dots,d\}$. Also note Protter (2004, Theorem 17, p. 164 and Theorem 21, p. 165), which allow sums of integrals with respect to $W^1,\dots,W^m$ to be expressed as integrals with respect to the aggregate Brownian motions (2.32).

In both models presented in this section we will assume, for simplicity, that the parameters governing jump behaviour are constant. Thus the point processes N^k, for $k \in \{m+1,\dots,d\}$, are in fact homogenous Poisson processes with constant intensities, so that

$$\lambda_t^k = \lambda^k > 0, \tag{2.33}$$

for each $k \in \{m+1,\dots,d\}$ and all $t \in [0,T]$. Also, the jump ratios are constant, so that

$$\sigma_t^{j,k} = \sigma^{j,k} \leq 1, \tag{2.34}$$

for each $j \in \{0,\dots,d\}$, each $k \in \{m+1,\dots,d\}$ and all $t \in [0,T]$. Notice that Assumption 2.6 ensures that (2.33) does not violate Assumption 2.3. Also, Assumption 2.6 and (2.24) ensure that (2.34) satisfies Assumption 2.2. Using (2.31)–(2.34), we can rewrite (2.30) as

$$\hat{S}_t^j = \hat{S}_t^{j,\mathrm{c}} S_t^{j,\mathrm{d}}, \tag{2.35}$$

where

$$\hat{S}_t^{j,\mathrm{c}} := S_0^j \exp\left\{-\frac{1}{2}\int_0^t \left|\sigma_s^j\right|^2 ds - \int_0^t \left|\sigma_s^j\right| d\hat{W}_s^j\right\} \tag{2.36}$$

and

$$S_t^{j,\mathrm{d}} := \exp\left\{\sum_{k=m+1}^d \sigma^{j,k}\lambda^k t\right\} \prod_{k=m+1}^d (1-\sigma^{j,k})^{N_t^k}, \tag{2.37}$$

for each $j \in \{0,\dots,d\}$ and all $t \in [0,T]$. The two concrete models for the benchmarked primary security accounts which we now present differ in terms of how the continuous processes (2.36) are modelled. The jump processes (2.37) are, for simplicity, the same in both cases.

2.9.1. The Merton Model (MM)

A particular model arises if one assumes that all parameter processes are constant; so that, in addition to (2.33) and (2.34), $r_t = r$ and $\sigma_t^{j,k} = \sigma^{j,k}$, for each $j \in \{0,\dots,d\}$, each $k \in \{1,\dots,m\}$ and all $t \in [0,T]$. Then (2.36) can be written as

$$\hat{S}_t^{j,\mathrm{c}} = S_0^j \exp\left\{-\frac{1}{2}|\sigma^j|^2 t - |\sigma^j|\hat{W}_t^j\right\} \tag{2.38}$$

for each $j \in \{0,\dots,d\}$ and all $t \in [0,T]$. Hence, in this special case, the benchmarked primary security accounts are the products of driftless geometric Brownian motions and compensated Poisson processes. We are thus in a modelling paradigm similar to that first introduced by Samuelson (1965) and extended by Merton (1976) to include jumps. Consequently, we refer to the model, where the benchmarked primary security accounts are described by (2.38) and (2.37) as the *Merton model (MM)*, which is also known as the Merton jump-diffusion model.

2.9.2. The Minimal Market Model (MMM)

Without imposing significant constraints on the parameter processes, and working within the full generality of § 2.8, it has been demonstrated in Platen (2004d, § 3) that the discounted GOP follows a *time-transformed squared Bessel process of dimension four*. Since the discounted GOP is given by $\frac{S_t^{\delta_*}}{S_t^0} = \frac{1}{\hat{S}_t^0}$, for all $t \in [0, T]$, it follows that the benchmarked savings account is an inverted time-transformed squared Bessel process of dimension four. This observation can be extended to all benchmarked primary security accounts, modulo jumps, of course. The *minimal market model (MMM)* for the benchmarked primary security accounts, first introduced in Platen (2001), is obtained by modelling these time transformations as differentiable and deterministic functions. We provide here an outline of this model. Unfortunately, its detailed justification is beyond the scope of this paper and the reader is referred to Platen (2004d, § 3).

For each $j \in \{0, \dots, d\}$, let $\eta^j \in \mathbb{R}$ and define the function $\alpha^j \colon \mathbb{R}_+ \to \mathbb{R}_+$ by setting

$$\alpha^j(t) := \alpha_0^j \exp\{\eta^j t\}, \tag{2.39}$$

for all $t \in [0, T]$, with $\alpha_0^j > 0$. We refer to η^j as the *net growth rate* of the jth primary security account, for $j \in \{0, \dots, d\}$. Next, define the *square root processes* $Y^j = \{Y_t^j \,\big|\, t \in [0, T]\}$, for $j \in \{0, \dots, d\}$, through the system of SDEs

$$dY_t^j = (1 - \eta^j Y_t^j)\, dt + \sqrt{Y_t^j}\, d\hat{W}_t^j, \tag{2.40}$$

for each $j \in \{0, \dots, d\}$ and all $t \in [0, T]$, with $Y_0^j = \frac{1}{\alpha_0^j S_0^j}$. The continuous parts of the benchmarked primary security accounts (2.36) are modelled in terms of these square root processes by setting

$$\hat{S}_t^{j,c} := \frac{1}{\alpha^j(t) Y_t^j}, \tag{2.41}$$

for each $j \in \{0, \dots, d\}$ and all $t \in [0, T]$.

As already mentioned, between jumps the benchmarked primary security accounts are inverted time-transformed squared Bessel processes of dimension four. The time transformations are deterministic in the case of the MMM. In detail, define the continuous strictly increasing functions $\varphi^j \colon \mathbb{R}_+ \to \mathbb{R}_+$, for $j \in \{0, \dots, d\}$, by setting

$$\varphi^j(t) := \varphi_0^j + \frac{1}{4} \int_0^t \alpha^j(s)\, ds, \tag{2.42}$$

for each $j \in \{0, \dots, d\}$ and all $t \in [0, T]$, with $\varphi_0^j \in \mathbb{R}_+$. Continuity and monotonicity imply that φ^j possesses an inverse $(\varphi^j)^{-1} \colon [\varphi_0^j, \infty) \to \mathbb{R}_+$, for each $j \in \{0, \dots, d\}$.

Now define the processes $X^j = \{X^j_\varphi \,\big|\, \varphi \in [\varphi^j(0), \varphi^j(T)]\}$, for each $j \in \{0, \ldots, d\}$, by setting

$$X^j_{\varphi^j(t)} := \alpha^j(t) Y^j_t = \frac{1}{\hat{S}^{j,c}_t}, \tag{2.43}$$

for each $j \in \{0, \ldots, d\}$ and all $t \in [0, T]$. It then follows (see Platen (2004d, § 3)) that X^j is a squared Bessel process of dimension four, for each $j \in \{0, \ldots, d\}$; so that $\frac{1}{\hat{S}^{j,c}}$ is a time-transformed squared Bessel process under the transformation $(\varphi^j)^{-1}$.

Under the MMM benchmarked primary security accounts are strict local martingales, and hence strict supermartingales. This observation holds, in particular, for the benchmarked savings account $\hat{S}^0$. Since this is the putative Radon-Nikodym derivative process employed by Girsanov's theorem to transform from the real-world measure to an equivalent risk-neutral measure, the fact that it is not a martingale rules out this measure transformation. In fact, one can make an even stronger statement: as will be demonstrated in Remark 3.1, the MMM does not admit any equivalent risk-neutral measures. Consequently, risk-neutral derivative pricing is impossible within the MMM, and we will resort to the more general fair pricing approach. Despite this deviation from the standard risk neutral setting, the MMM is attractive for a number of reasons.

1. In the light of accepted "stylized empirical facts", it appears to agree well with the real market. For example, it captures the observed negative correlation between stock value and volatility, known as the *leverage effect* (see Black (1976)).

2. If we accept that α^j is of finite variation, for $j \in \{0, \ldots, d\}$, then squared Bessel processes suggest themselves quite naturally and the derivation of (2.39) from (2.35) follows, as demonstrated in Platen (2004d, § 3).

3. Finally, the MMM represents a parsimonious model resulting from optimal market dynamics. The only free parameters are the interest rate process r and the net growth rates η^j, for $j \in \{0, \ldots, d\}$, together with certain initial values. These are all readily determined from market data (see Platen (2004d, p. 523–524)).

In summary, we think that the MMM manages to capture endogenously many of the features of observed markets. Furthermore, the modest number of parameters make it easy to calibrate and practical to implement.

3. Fair Derivative Pricing Examples

3.1. Zero-Coupon Bonds

We consider first a standard default-free zero-coupon bond paying one unit of the domestic currency at its maturity T. We could allow for any maturity date

$\bar{T} \in [0, T]$, but this added generality has no impact on the calculations that follow. According to the fair pricing formula (2.21), the value of the bond at time t is given by

$$P(t, T) = S_t^{\delta_*} \mathbb{E}\left[\frac{1}{S_T^{\delta_*}} \,\middle|\, \mathcal{A}_t \right] = \frac{1}{\hat{S}_t^0} \mathbb{E}\left[\exp\left\{ -\int_t^T r_s \, ds \right\} \hat{S}_T^0 \,\middle|\, \mathcal{A}_t \right], \tag{3.1}$$

for all $t \in [0, T]$. While (3.1) is completely general and independent of any assumptions, it is also too generic to be directly useful. We will now examine it under the two market models outlined in § 2.9.

3.1.1. The MM Case

Since $\hat{S}^0$ is a martingale in this case, we get

$$P(t, T) = \exp\{-r(T - t)\} \frac{1}{\hat{S}_t^0} \mathbb{E}\left[\hat{S}_T^0 \,\middle|\, \mathcal{A}_t \right] = \exp\{-r(T - t)\}, \tag{3.2}$$

for all $t \in [0, T]$. In other words, we obtain the usual bond pricing formula determined by discounting at the short rate.

3.1.2. The MMM Case

In Miller & Platen (2004, Assumption 2.1, p. 4) it is argued, with some empirical support (see Miller & Platen (2004, Figure 2, p. 5), for example), that the interest rate process and the discounted GOP are independent. If we accept this, and apply it to (3.1), while remembering that $\hat{S}^0 = \hat{S}^{0,c}$, we get

$$\begin{aligned} P(t, T) &= \mathbb{E}\left[\exp\left\{ -\int_t^T r_s \, ds \right\} \,\middle|\, \mathcal{A}_t \right] \frac{1}{\hat{S}_t^0} \mathbb{E}\left[\hat{S}_T^0 \,\middle|\, \mathcal{A}_t \right] \\ &= \mathbb{E}\left[\exp\left\{ -\int_t^T r_s \, ds \right\} \,\middle|\, \mathcal{A}_t \right] \left(1 - \exp\left\{ -\frac{1}{2} \Lambda_t^0 \right\} \right), \end{aligned} \tag{3.3}$$

for all $t \in [0, T]$, from (B.5), with Λ_t^0 as in (B.3) in Appendix B

Remark 3.1. The bond pricing formula (3.3) poses an obvious arbitrage riddle, which we now address. As demonstrated in Heath & Platen (2002, § 5), there exists a trading strategy, whose value is given by (3.3), which hedges the zero-coupon bond under consideration. Imagine now a trading strategy δ consisting of the aforementioned hedge, funded by borrowing $P(0, T)$ from the savings account at initiation. Assuming, for the sake of argument, a zero short rate, the wealth process associated with δ is given by

$$S_t^\delta = P(t, T) - P(0, T) = \exp\left\{ -\frac{1}{2} \Lambda_0^0 \right\} - \exp\left\{ -\frac{1}{2} \Lambda_t^0 \right\}, \tag{3.4}$$

for all $t \in [0, T]$, according to (3.3). The following observations are immediate:

- $S_0^\delta = 0$;

- $S_T^{\delta} = 1 - P(0, T) > 0$ a.s.; and
- $S_t^{\delta} \geq -P(0, T)$ a.s.,

for all $t \in [0, T]$. Thus δ is an admissible strategy, in the sense that its wealth process is uniformly bounded below (see Delbaen & Schachermayer (1994, p. 467)). However, it fails the NFLVR condition. By the fundamental theorem of asset pricing (see Delbaen & Schachermayer (1994, 1998)), we may thus conclude that the MMM admits no equivalent risk-neutral probability measures.

Now fix $t \in [0, T]$. The monotonicity of φ^0 and the relation $X^0_{\varphi^0(0)} = 1$, which follows from (2.3) and (2.43), together imply that

$$\mathbb{P}[S_t^{\delta} < 0] = \mathbb{P}\left[X^0_{\varphi^0(t)} > \frac{\varphi^0(T) - \varphi^0(t)}{\varphi^0(T) - \varphi^0(0)} X^0_{\varphi^0(0)}\right] \geq \mathbb{P}\left[X^0_{\varphi^0(t)} > 1\right] > 0, \tag{3.5}$$

remembering that X^0 is a squared Bessel process of dimension four. So S^{δ} is not an arbitrage in the sense of Definition 2.5; in particular, its associated wealth process can become negative at any time with strictly positive probability. It is an example of what Loewenstein & Willard (2000) call a *free snack*. While the presence of free snacks rules out the existence of equivalent risk-neutral probability measures, Loewenstein & Willard (2000) argue that this alone does not constitute sufficient grounds for dismissing a given market model.

3.2. Forward Contracts

In this section we fix $j \in \{0, \ldots, d\}$ and consider a forward contract with delivery date T on the jth primary security account. According to the fair pricing formula (2.21) the forward price $F^j(t, T)$ at time $t \in [0, T]$ for this contract is implicitly determined by

$$S_t^{\delta_*} \mathbb{E}\left[\frac{F^j(t, T) - S_T^j}{S_T^{\delta_*}} \,\middle|\, \mathcal{A}_t\right] = 0. \tag{3.6}$$

Solving this equation yields the following expression for the forward price:

$$F^j(t, T) = \frac{S_t^{\delta_*} \mathbb{E}\left[\hat{S}_T^j \,\middle|\, \mathcal{A}_t\right]}{S_t^{\delta_*} \mathbb{E}\left[\frac{1}{S_T^{\delta_*}} \,\middle|\, \mathcal{A}_t\right]} = \begin{cases} \frac{S_t^j}{P(t,T)} \frac{1}{\hat{S}_t^j} \mathbb{E}\left[\hat{S}_T^j \,\middle|\, \mathcal{A}_t\right] & \text{if } S_t^j > 0; \\ 0 & \text{if } S_t^j = 0, \end{cases} \tag{3.7}$$

for all $t \in [0, T]$, by (3.1).

3.2.1. The MM Case

With reference to (2.38), the same argument which established that the benchmarked savings account is a continuous martingale also applies to the driftless geometric Brownian motion $\hat{S}^{j,\mathrm{c}}$; while the compensated Poisson process $\hat{S}^{j,\mathrm{d}}$ is a jump martingale. Consequently, $\hat{S}^j$ is the product of independent martingales, and hence itself a martingale. Together with (3.2), this fact enables us to write (3.7) as

$$F^j(t,T) = S^j_t \exp\{r(T-t)\}, \tag{3.8}$$

for all $t \in [0,T]$. Thus, in the MM we recover the standard expression for the forward price.

3.2.2. The MMM Case

In this case, $\hat{S}^{j,c}$ is an inverted time-transformed squared Bessel process of dimension four, according to (2.43), while $S^{j,d}$ is an independent jump martingale, as before. Thus,

$$\frac{1}{\hat{S}^j_t}\mathbb{E}\Big[\hat{S}^j_T \,\Big|\, \mathcal{A}_t\Big] = \frac{1}{\hat{S}^{j,c}_t}\mathbb{E}\Big[\hat{S}^{j,c}_T \,\Big|\, \mathcal{A}_t\Big]\frac{1}{S^{j,d}_t}\mathbb{E}\Big[S^{j,d}_T \,\Big|\, \mathcal{A}_t\Big] = 1 - \exp\Big\{-\frac{1}{2}\Lambda^j_t\Big\} \tag{3.9}$$

for all $t \in [0,T]$, by (B.5). Putting (3.7) together with (3.3) and (3.9) gives the following expression for the forward price:

$$F^j(t,T) = S^j_t \frac{1-\exp\big\{-\frac{1}{2}\Lambda^j_t\big\}}{1-\exp\big\{-\frac{1}{2}\Lambda^0_t\big\}}\mathbb{E}\left[\exp\left\{\int_t^T r_s\,ds\right\}\,\middle|\,\mathcal{A}_t\right], \tag{3.10}$$

for all $t \in [0,T]$.

3.3. Asset-or-Nothing Binaries

Binary options may be regarded as basic building blocks for complex derivatives. This has been exploited in a recent approach to the valuation of exotic options, where a complex payoff is decomposed into series of binaries (see Ingersoll (2000) and Buchen (2004)).

In this section we fix $j \in \{0,\ldots,d\}$ again and consider an *asset-or-nothing binary*, with maturity T and strike $K \in \mathbb{R}_+$, on the jth primary security account. We also fix $k \in \{m+1,\ldots,d\}$ and assume that $\sigma^{j,k} \neq 0$ and $\sigma^{j,l} = 0$, for each $l \in \{m+1,\ldots,d\}$ with $l \neq k$. In other words, we assume that the jth primary security account responds only to the kth jump process. This does not affect the generality of our calculations below, but it does result in more manageable expressions. In addition, we will assume a constant interest rate throughout, so that $r_t = r$, for all $t \in [0,T]$. Although this is already in force in the case of the MM, we require it to obtain a convenient pricing formula under the MMM.

At its maturity the instrument under consideration pays its holder one unit of the jth primary security account if this is greater than the strike amount K, and nothing otherwise. According to the fair pricing formula (2.21), its value is given by

$$
\begin{aligned}
A^{j,k}(t,T,K) &= S_t^{\delta_*}\mathbb{E}\left[\mathbb{I}_{\{S_T^j \geq K\}}\frac{S_T^j}{S_T^{\delta_*}}\,\middle|\,\mathcal{A}_t\right] \\
&= \frac{S_t^j}{\hat{S}_t^j}\mathbb{E}\left[\mathbb{I}_{\{\hat{S}_T^j \geq K(S_T^0)^{-1}\hat{S}_T^0\}}\hat{S}_T^j\,\middle|\,\mathcal{A}_t\right] \\
&= \frac{S_t^j}{\hat{S}_t^{j,\mathrm{c}}}\mathbb{E}\Big[\mathbb{I}_{\{\hat{S}_T^{j,c} \geq \alpha(N_T^k - N_t^k)\hat{S}_T^0\}} \\
&\qquad\qquad \times \exp\{\sigma^{j,k}\lambda^k(T-t)\}(1-\sigma^{j,k})^{N_T^k - N_t^k}\hat{S}_T^{j,\mathrm{c}}\,\Big|\,\mathcal{A}_t\Big] \\
&= \sum_{n=0}^{\infty}\exp\{-\lambda^k(T-t)\}\frac{[\lambda^k(T-t)]^n}{n!}\exp\{\sigma^{j,k}\lambda^k(T-t)\}(1-\sigma^{j,k})^n \\
&\qquad\qquad \times \frac{S_t^j}{\hat{S}_t^{j,\mathrm{c}}}\mathbb{E}\left[\mathbb{I}_{\{\hat{S}_T^{j,c} \geq \alpha(n)\hat{S}_T^0\}}\hat{S}_T^{j,\mathrm{c}}\,\middle|\,\mathcal{A}_t\right],
\end{aligned}
\tag{3.11}
$$

for all $t \in [0,T]$, where

$$
\alpha(n) := \frac{K}{S_t^0 S_t^{j,\mathrm{d}}}\exp\{-(r+\sigma^{j,k}\lambda^k)(T-t)\}(1-\sigma^{j,k})^{-n}, \tag{3.12}
$$

for all $n \in \mathbb{Z}_+$.

3.3.1. The MM Case

Combining (3.11) and (3.12) with (A.18), and remembering that $\hat{S}^{0,\mathrm{c}} = \hat{S}^0$, yields

$$
\begin{aligned}
A^{j,k}(t,T,K) &= \sum_{n=0}^{\infty}\exp\{-\lambda^k(T-t)\}\frac{[\lambda^k(T-t)]^n}{n!} \\
&\qquad \times \exp\{\sigma^{j,k}\lambda^k(T-t)\}(1-\sigma^{j,k})^n S_t^j \mathcal{N}(d_1(n)),
\end{aligned}
\tag{3.13}
$$

for all $t \in [0,T]$, where

$$
d_1(n) := \frac{\ln\frac{S_t^j}{K} + \left(r + \sigma^{j,k}\lambda^k + n\frac{\ln(1-\sigma^{j,k})}{T-t} + \frac{1}{2}(\hat{\sigma}^{0,j})^2\right)(T-t)}{\hat{\sigma}^{0,j}\sqrt{T-t}}, \tag{3.14}
$$

for each $n \in \mathbb{Z}_+$.

3.3.2. The MMM Case

As we have just seen, calculating the fair price of a contingent claim written on a primary security account requires the evaluation of a double integral involving the transition density of a two-dimensional process. This is a consequence of choosing the GOP as numéraire. Closed-form fair derivative prices can be obtained for the MM, but in the case of the MMM things are more difficult, because the transition densities of two-dimensional squared Bessel processes are apparently not known. The natural response to this is to solve the partial integro-differential equation associated with the derivative price numerically. However, to give the reader a feeling for the types of formulas that emerge from applying fair pricing in the MMM, we will now assume the $\hat{S}^0$ and $\hat{S}^{j,c}$ are independent. Combining (3.11) and (3.12) with (B.15), and remembering that $\hat{S}^0 = \hat{S}^{0,c}$, then results in the following:

$$\begin{aligned} &A^{j,k}(t,T,K) \\ &= \sum_{n=0}^{\infty} \exp\{-\lambda^k(T-t)\}\frac{[\lambda^k(T-t)]^n}{n!}\exp\{\sigma^{j,k}\lambda^k(T-t)\}(1-\sigma^{j,k})^n \\ &\qquad \times S_t^j\left[G''_{0,4}\Big((\varphi^0(T)-\varphi^0(t))\alpha(n);\Lambda_t^j,\Lambda_t^0\Big)-\exp\left\{-\frac{1}{2}\Lambda_t^j\right\}\right], \end{aligned} \tag{3.15}$$

for all $t \in [0,T]$. The paragraph preceding (B.15) in Appendix B explains the notation in (3.15).

3.4. Bond-or-Nothing Binaries

In this section we price a *bond-or-nothing binary*, with maturity T and strike $K \in \mathbb{R}_+$, on the jth primary security account, where $j \in \{0,\dots,d\}$ is fixed. As before, let us assume that this primary security account only responds to the kth jump process, where $k \in \{m+1,\dots,d\}$ is fixed. We will also require a constant interest rate for the MMM as well as the MM.

At its maturity the instrument under consideration pays its holder the strike amount K if the value of the jth primary security account is in excess of this, and nothing otherwise. According to the fair pricing formula (2.21), its value is given by

$$\begin{aligned} B^{j,k}(t,T,K) &= S_t^{\delta_*}\mathbb{E}\left[\mathbb{I}_{\{S_T^j \geq K\}}\frac{K}{S_T^{\delta_*}}\,\middle|\,\mathcal{A}_t\right] \\ &= KP(t,T) - KS_t^{\delta_*}\mathbb{E}\left[\mathbb{I}_{\{S_T^j<K\}}\frac{1}{S_T^{\delta_*}}\,\middle|\,\mathcal{A}_t\right] \\ &= KP(t,T) - K\frac{S_t^0}{\hat{S}_t^0}\mathbb{E}\left[\mathbb{I}_{\{\hat{S}_T^0>K^{-1}S_T^0\hat{S}_T^j\}}\frac{\hat{S}_T^0}{S_T^0}\,\middle|\,\mathcal{A}_t\right] \\ &= KP(t,T) - K\exp\{-r(T-t)\}\frac{1}{\hat{S}_t^0}\mathbb{E}\left[\mathbb{I}_{\{\hat{S}_T^0>\alpha(N_T^k-N_t^k)^{-1}\hat{S}_T^{j,c}\}}\hat{S}_T^0\,\middle|\,\mathcal{A}_t\right] \\ &= KP(t,T) - K\exp\{-r(T-t)\}\sum_{n=0}^{\infty}\exp\{-\lambda^k(T-t)\}\frac{[\lambda^k(T-t)]^n}{n!} \\ &\qquad \times\frac{1}{\hat{S}_t^0}\mathbb{E}\left[\mathbb{I}_{\{\hat{S}_T^0>\alpha(n)^{-1}\hat{S}_T^{j,c}\}}\hat{S}_T^0\,\middle|\,\mathcal{A}_t\right], \end{aligned} \tag{3.16}$$

for all $t \in [0, T]$, where $\alpha(n)$ is given by (3.12), for each $n \in \mathbb{Z}_+$.

3.4.1. The MM Case

Applying (A.18) to (3.16), (3.12) and (3.2), and remembering that $\hat{S}_t^0 = \hat{S}_t^{0,c}$, gives

$$\begin{aligned} B^{j,k}(t, T, K) &= K \exp\{-r(T-t)\} \\ &\quad \times \left(1 - \sum_{n=0}^{\infty} \exp\{-\lambda^k (T-t)\} \frac{[\lambda^k (T-t)]^n}{n!} \mathcal{N}(-d_2(n))\right) \\ &= \sum_{n=0}^{\infty} \exp\{-\lambda^k (T-t)\} \frac{[\lambda^k (T-t)]^n}{n!} K \exp\{-r(T-t)\} \mathcal{N}(d_2(n)), \end{aligned} \tag{3.17}$$

for all $t \in [0, T]$, where

$$\begin{aligned} d_2(n) &:= \frac{\ln \frac{S_t^j}{K} + \left(r + \sigma^{j,k}\lambda^k + n\frac{\ln(1-\sigma^{j,k})}{T-t} - \frac{1}{2}(\hat{\sigma}^{0,j})^2\right)(T-t)}{\hat{\sigma}^{0,j}\sqrt{T-t}} \\ &= d_1(n) - \hat{\sigma}^{0,j}\sqrt{T-t}, \end{aligned} \tag{3.18}$$

for each $n \in \mathbb{Z}_+$. To obtain the second equality in (3.17), we have used the relation $\mathcal{N}(-(d_2(n)) = 1 - \mathcal{N}((d_2(n))$, for each $n \in \mathbb{Z}_+$, together with the fact that $\sum_{n=0}^{\infty} \exp\{-\lambda^k (T-t)\} \frac{[\lambda^k (T-t)]^n}{n!} = 1$, as it is the total probability of a Poisson random variable with parameter $\lambda^k (T-t)$.

3.4.2. The MMM Case

Subject to the assumption that $\hat{S}_t^0$ and $\hat{S}_t^{j,c}$ are independent, as before, we can combine (B.15) with (3.16), (3.12) and (3.3), to get

$$\begin{aligned} B^{j,k}(t, T, K) &= K \exp\{-r(T-t)\}\left(1 - \exp\left\{-\frac{1}{2}\Lambda_t^0\right\}\right) \\ &\quad - \sum_{n=0}^{\infty} \exp\{-\lambda^k (T-t)\} \frac{[\lambda^k (T-t)]^n}{n!} K \exp\{-r(T-t)\} \\ &\quad \times \left[G''_{0,4}\left(\frac{\varphi^j(T) - \varphi^j(t)}{\alpha(n)}; \Lambda_t^0, \Lambda_t^j\right) - \exp\left\{-\frac{1}{2}\Lambda_t^0\right\}\right] \\ &= \sum_{n=0}^{\infty} \exp\{-\lambda^k (T-t)\} \frac{[\lambda^k (T-t)]^n}{n!} \\ &\quad \times K \exp\{-r(T-t)\}\left[1 - G''_{0,4}\left(\frac{\varphi^j(T) - \varphi^j(t)}{\alpha(n)}; \Lambda_t^0, \Lambda_t^j\right)\right], \end{aligned} \tag{3.19}$$

for all $t \in [0, T]$. For the second equality in (3.19), we have once again used the fact that $\sum_{n=0}^{\infty} \exp\{-\lambda^k (T-t)\} \frac{[\lambda^k (T-t)]^n}{n!}$ is the total probability of a Poisson random variable with parameter $\lambda^k (T-t)$.

3.5. European Options

In this section we fix $j \in \{0, \dots, d\}$ again and consider a European call option with maturity T and strike $K \in \mathbb{R}_+$ on the jth primary security account. As before, we make the simplifying assumption that the jth primary security account is only sensitive to the kth jump process, for some fixed $k \in \{m+1, \dots, d\}$. We also continue to require a constant interest rate for both market models. According to the fair pricing formula (2.21) the option premium is given by

$$\begin{aligned} C^{j,k}(t,T,K) &= S_t^{\delta_*} \mathbb{E}\left[\frac{(S_T^j - K)^+}{S_T^{\delta_*}} \,\middle|\, \mathcal{A}_t \right] = S_t^{\delta_*} \mathbb{E}\left[\mathbb{I}_{\{S_T^j \geq K\}} \frac{S_T^j - K}{S_T^{\delta_*}} \,\middle|\, \mathcal{A}_t \right] \\ &= A^{j,k}(t,T,K) - B^{j,k}(t,T,K), \end{aligned} \tag{3.20}$$

for all $t \in [0, T]$.

3.5.1. The MM Case

Combining (3.13) and (3.17) gives

$$\begin{aligned} C^{j,k}(t,T,K) = \sum_{n=0}^{\infty} & \exp\{-\lambda^k (T-t)\} \frac{[\lambda^k (T-t)]^n}{n!} \Big[\exp\{\sigma^{j,k} \lambda^k (T-t)\} \\ & \times (1 - \sigma^{j,k})^n S_t^j \mathcal{N}(d_1(n)) - K \exp\{-r(T-t)\} \mathcal{N}(d_2(n)) \Big], \end{aligned} \tag{3.21}$$

for all $t \in [0, T]$, where $d_1(n)$ and $d_2(n)$ are given by (3.14) and (3.18), respectively, for each $n \in \mathbb{Z}_+$.

It is easily seen that (3.21) corresponds with the original pricing formula for a call on a stock whose price follows a jump-diffusion, given by Merton (1976, (19), p. 135). The only difference is that there the jump ratios were taken to be log-normally distributed, while in our case they are constant. Furthermore, since we can have $i, j \neq 0$ in (A.18), this formula can be used to price an option to exchange the jth primary security account for the ith primary security account. In that case, the option pricing formula we obtain in place of (3.21) is essentially that of Margrabe (1978, (7), p. 179), modified for the presence of jumps. Note, in particular, the correspondence between (A.12) and the expression for v^2 in Margrabe (1978, p. 179).

3.5.2. The MMM Case

For the sake of completeness, we present the call option pricing formula under the MMM, obtained by subtracting (3.19) from (3.15), according to (3.20):

$$\begin{aligned} C^{j,k}(t,T,K) = \sum_{n=0}^{\infty} & \exp\{-\lambda^k (T-t)\} \frac{[\lambda^k (T-t)]^n}{n!} \Big[\exp\{\sigma^{j,k} \lambda^k (T-t)\} \\ & \times (1 - \sigma^{j,k})^n S_t^j \Big(G''_{0,4}\Big((\varphi^0(T) - \varphi^0(t)) \alpha(n); \Lambda_t^j, \Lambda_t^0 \Big) - \exp\Big\{ -\frac{1}{2} \Lambda_t^j \Big\} \Big) \\ & - K \exp\{-r(T-t)\} \Big(1 - G''_{0,4}\Big(\frac{\varphi^j(T) - \varphi^j(t)}{\alpha(n)}; \Lambda_t^0, \Lambda_t^j \Big) \Big) \Big], \end{aligned} \tag{3.22}$$

for all $t \in [0, T]$, where $\alpha(n)$ is given by (3.12), for each $n \in \mathbb{Z}_+$.

3.6. Defaultable Zero-Coupon Bonds

Since we have incorporated event risks in our modelling, we have a framework which is appropriate for pricing credit-risky instruments. Here we consider the canonical example of such a contract, namely a defaultable zero-coupon bond with maturity T. To keep the analysis simple, we fix $k \in \{m+1, \ldots, d\}$ and assume that the bond under consideration defaults at the first jump time of N^k, provided that this time is not greater than T. In other words, default occurs if and only if $\tau_1^k \leq T$, in which case τ_1^k is the default time. As a further simplification, we assume zero recovery upon default. According to the fair pricing formula (2.21), the price $\tilde{P}^k(t, T)$ at time t of this instrument is given by

$$\begin{aligned} \tilde{P}^k(t,T) &= S_t^{\delta_*} \mathbb{E}\left[\frac{\mathbb{I}_{\{\tau_1^k > T\}}}{S_T^{\delta_*}} \,\middle|\, \mathcal{A}_t \right] = S_t^{\delta_*} \mathbb{E}\left[\frac{1}{S_T^{\delta_*}} \,\middle|\, \mathcal{A}_t \right] \mathbb{E}\left[\mathbb{I}_{\{\tau_1^k > T\}} \,\middle|\, \mathcal{A}_t \right] \\ &= P(t,T)\, \mathbb{P}[N_T^k = 0 \,|\, \mathcal{A}_t], \end{aligned} \tag{3.23}$$

for all $t \in [0, T]$. Note that the second equality above follows from the independence of the GOP and the underlying point processes, as is clear from (2.27), for example.

Equation (3.23) is naturally interpreted as expressing the price of the defaultable bond as the product of the price of the corresponding default-free bond and the conditional probability of survival. The latter may be further evaluated as follows:

$$\begin{aligned} \mathbb{P}[N_T^k = 0 \,|\, \mathcal{A}_t] &= \mathbb{E}\left[\mathbb{I}_{\{N_t^k = 0\}} \mathbb{I}_{\{N_T^k - N_t^k = 0\}} \,\middle|\, \mathcal{A}_t \right] \\ &= \mathbb{I}_{\{N_t^k = 0\}} \mathbb{P}[N_T^k - N_t^k = 0 \,|\, \mathcal{A}_t] = \mathbb{I}_{\{N_t^k = 0\}} \mathbb{E}\left[\exp\left\{ -\int_t^T \lambda_s^k \, ds \right\} \,\middle|\, \mathcal{A}_t \right], \end{aligned} \tag{3.24}$$

for all $t \in [0, T]$.

One can, of course, combine (3.23) and (3.24) with (3.2) to obtain an explicit pricing formula for the defaultable bond under consideration in the MM. Similarly, one can combine (3.23) and (3.24) with (3.3) to obtain the pricing formula for this instrument in the MMM.

Remark 3.2. Note that the expression obtained by combining (3.23) and (3.24) is similar to the familiar formula for the price of a defaultable zero-coupon bond in a simple reduced-form model for credit risk (see Schönbucher (2003, (5.10), p. 118), for example). The difference is that in the standard formula all expectations are computed with respect to an equivalent risk-neutral measure; in particular, the survival probability is a risk-neutral probability. In (3.23) and (3.24), however, only the real-world measure is in evidence. The crucial advantage of this is that we avoid the undesirable dichotomy between real-world default probabilities, as determined by historical data and credit rating agencies, and risk-neutral default probabilities, as determined by observed credit spreads. Substantial effort has been expended on the problem of reconciling real-world and risk-neutral probabilities of default (see Albanese & Chen (2005), for a recent example). This unresolved problem is, fortunately, avoided altogether with fair pricing under the benchmark approach, since the real-world measure is the pricing measure. In future work we intend to exploit this, to produce credit risk models which can be calibrated both to real-world probabilities of default, derived from historical data and credit ratings information, as well as to traded credit spreads.

Of course, Assumption 2.6 is probably not ideal for a realistic model of credit risk. Nevertheless, formulas analogous to (2.22) and (3.24) would emerge for defaultable bond prices, even if market prices of jump risks were not taken to be identically zero; and the comments above would still apply.

4. Conclusion

In this paper we have set out to achieve two objectives: to introduce benchmarking and the associated concept of fair derivative pricing; and to illustrate fair pricing by valuing some common contingent claims in two different market models. With respect to the first objective, we have demonstrated that the applicability of benchmarking and fair pricing depend only on the modest assumptions necessary to ensure the existence of a *growth optimal portfolio (GOP)*. Furthermore, we hope to have convinced the reader that a case can be made for also considering market models that do not admit equivalent risk-neutral measures. In such models it appears that fair pricing is a natural generalization of risk-neutral pricing, with no need for measure changes.

Contrary to what one might expect, models for which equivalent risk-neutral measures do not exist do not necessarily admit arbitrages; provided that one adopts an appropriate notion of arbitrage. The definition of arbitrage we advocate, namely that no market participant can make something from nothing without the possibility of becoming insolvent in the interim, has a clear economic meaning.

The benchmarking approach is concerned with modelling benchmarked wealth processes. In the two market models we consider, these are jump-diffusions, based, respectively, on geometric Brownian motions and on squared Bessel processes of dimension four. In the former case, we obtain what we call the *Merton model (MM)*; this is very much the standard market model, for which risk-neutral pricing would be appropriate. The model obtained in the latter case is called the *minimal market model (MMM)*; it does not admit equivalent risk-neutral measures, though it does exhibit a number of features which recommend it as a potentially realistic description of observed markets. For the MM we demonstrate that the familiar derivative pricing formulas, usually obtained through risk-neutral pricing, are retrieved by the fair pricing methodology. Fair pricing in the MMM, however, produces *free snacks*, as well as yielding option pricing formulas where the non-central chi-square distribution plays an important role.

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We would like to thank Morten Christensen for valuable discussions about the technicalities and implications of various no-arbitrage conditions.

A. Appendix: Merton Model Calculations

This appendix provides some useful expressions for computing fair prices in the MM. For this section fix $i, j \in \{0, \ldots, d\}$ such that $i \neq j$. Then the function

$$
\begin{aligned}
p^{i,j}_{s,t}(x_i, x_j; y_i, y_j) &:= \frac{1}{2\pi y_i y_j |\sigma^i||\sigma^j|(t-s)\sqrt{1-(\rho^{i,j})^2}} \\
&\quad \times \exp\Bigg\{-\frac{1}{2(1-(\rho^{i,j})^2)}\Bigg[\left(\frac{\ln\frac{y_i}{x_i}+\frac{1}{2}|\sigma^i|^2(t-s)}{|\sigma^i|\sqrt{t-s}}\right)^2 \\
&\qquad - 2\rho^{i,j}\frac{\left(\ln\frac{y_i}{x_i}+\frac{1}{2}|\sigma^i|^2(t-s)\right)\left(\ln\frac{y_j}{x_j}+\frac{1}{2}|\sigma^j|^2(t-s)\right)}{|\sigma^i||\sigma^j|(t-s)} \\
&\qquad + \left(\frac{\ln\frac{y_j}{x_j}+\frac{1}{2}|\sigma^j|^2(t-s)}{|\sigma^j|\sqrt{t-s}}\right)^2\Bigg]\Bigg\},
\end{aligned}
\tag{A.1}
$$

for all $x_i, x_j, y_i, y_j \in (0, \infty)$, where $s, t \in [0, T]$ such that $s \le t$, is the joint transition density of $\hat{S}^{i,c}$ and $\hat{S}^{j,c}$ over the time interval $[s, t]$. The parameter $\rho^{i,j}$ in (A.1) is determined by

$$
\rho^{i,j} := \sum_{k=1}^{m} \frac{\sigma^{i,k}\sigma^{j,k}}{|\sigma^i||\sigma^j|}. \tag{A.2}
$$

It follows from (2.32) that $\rho^{i,j}$ is the correlation between the Brownian motions $\hat{W}^i$ and $\hat{W}^j$.

A.1. An Expression for Computing Fair Binary Prices

Fix $t \in [0, T]$ and let α_t be a non-negative $\mathcal{A}_t$-measurable random variable. We will now evaluate the following expression:

$$
\frac{1}{\hat{S}^{j,c}_t}\mathbb{E}\left[\mathbb{I}_{\{\hat{S}^{j,c}_T \ge \alpha_t \hat{S}^{i,c}_T\}}\hat{S}^{j,c}_T \,\middle|\, \mathcal{A}_t\right] = \int_0^\infty \int_{\alpha_t x}^\infty \frac{y}{\hat{S}^{j,c}_t} p^{i,j}_{t,T}\left(\hat{S}^{i,c}_t, \hat{S}^{j,c}_t; x, y\right) dy\, dx. \tag{A.3}
$$

After the change of variables

$$
\bar{x} := \frac{\ln\frac{x}{\hat{S}^{i,c}_t}+\frac{1}{2}|\sigma^i|^2(T-t)}{|\sigma^i|\sqrt{T-t}}; \tag{A.4}
$$

$$
\bar{y} := \frac{\ln\frac{y}{\hat{S}^{j,c}_t}+\frac{1}{2}|\sigma^j|^2(T-t)}{|\sigma^j|\sqrt{T-t}}, \tag{A.5}
$$

(A.3) becomes

$$\frac{1}{2\pi\sqrt{1-(\rho^{i,j})^2}}\int_{-\infty}^{\infty}\int_{d(\bar{x})}^{\infty}\exp\Big\{-\frac{1}{2(1-(\rho^{i,j})^2)}\Big[\Big(-\bar{x}+\rho^{i,j}|\sigma^j|\sqrt{T-t}\Big)^2 - 2\rho^{i,j}\Big(-\bar{x}+\rho^{i,j}|\sigma^j|\sqrt{T-t}\Big)\Big(-\bar{y}+|\sigma^j|\sqrt{T-t}\Big) + \Big(-\bar{y}+|\sigma^j|\sqrt{T-t}\Big)^2\Big]\Big\}\,d\bar{y}\,d\bar{x}, \tag{A.6}$$

where

$$d(\bar{x}) := \frac{\ln\frac{\alpha_t S_t^{i,c}}{S_t^{j,c}} - \Big(\frac{1}{2}|\sigma^i|^2 - \frac{1}{2}|\sigma^j|^2\Big)(T-t)}{|\sigma^j|\sqrt{T-t}} + \frac{|\sigma^i|}{|\sigma^j|}\bar{x}, \tag{A.7}$$

for all $\bar{x}\in\mathbb{R}$. Another transformation of variables,

$$\tilde{x} := -\bar{x}+\rho^{i,j}|\sigma^j|\sqrt{T-t}; \tag{A.8}$$

$$\tilde{y} := -\bar{y}+|\sigma^j|\sqrt{T-t}, \tag{A.9}$$

allows us to express (A.6) as

$$\frac{1}{2\pi\sqrt{1-(\rho^{i,j})^2}}\int_{-\infty}^{\infty}\int_{-\infty}^{d(\tilde{x})}\exp\Big\{-\frac{1}{2(1-(\rho^{i,j})^2)}\Big[\tilde{x}^2-2\rho^{i,j}\tilde{x}\tilde{y}+\tilde{y}^2\Big]\Big\}\,d\tilde{y}\,d\tilde{x}, \tag{A.10}$$

where

$$d(\tilde{x}) := \frac{\ln\frac{S_t^{j,c}}{\alpha_t S_t^{i,c}} + \frac{1}{2}(\hat{\sigma}^{i,j})^2(T-t)}{|\sigma^j|\sqrt{T-t}} + \frac{|\sigma^i|}{|\sigma^j|}\tilde{x} =: a+b\tilde{x}, \tag{A.11}$$

for all $\tilde{x}\in\mathbb{R}$, with

$$\hat{\sigma}^{i,j} := \sqrt{|\sigma^i|^2 - 2\rho^{i,j}|\sigma^i||\sigma^j| + |\sigma^j|^2}. \tag{A.12}$$

After the transformation of variables

$$\hat{y} := \tilde{y} - b\tilde{x}, \tag{A.13}$$

for all $\tilde{x}\in\mathbb{R}$ and $\tilde{y}\in(-\infty, d(\tilde{x}))$, (A.10) becomes

$$\frac{1}{2\pi\sqrt{1-(\rho^{i,j})^2}}\int_{-\infty}^{a}\int_{-\infty}^{\infty}\exp\Big\{-\frac{1}{2(1-(\rho^{i,j})^2)} \times\Big[\tilde{x}^2-2\rho^j\tilde{x}(\hat{y}+b\tilde{x})+(\hat{y}+b\tilde{x})^2\Big]\Big\}\,d\tilde{x}\,d\hat{y}. \tag{A.14}$$

Now, performing the change of variables

$$\hat{x} := \sqrt{\frac{1-2b\rho^j+b^2}{1-(\rho^{i,j})^2}}\Big(\tilde{x}+\frac{b-\rho^{i,j}}{1-2b\rho^{i,j}+b^2}\hat{y}\Big), \tag{A.15}$$

for all $\tilde{x} \in \mathbb{R}$, transforms (A.14) into

$$\frac{1}{\sqrt{1-2b\rho^{i,j}+b^2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} \exp\left\{-\frac{1}{2} \frac{\hat{y}^2}{1-2b\rho^{i,j}+b^2}\right\} d\hat{y}. \tag{A.16}$$

Finally, we set

$$z := \frac{\hat{y}}{\sqrt{1-2b\rho^{i,j}+b^2}}, \tag{A.17}$$

for all $\hat{y} \in (-\infty, a)$, so that (A.16) becomes

$$\begin{aligned}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{a}{\sqrt{1-2b\rho^{i,j}+b^2}}} \exp\left\{-\frac{1}{2}z^2\right\} dz &= \mathcal{N}\left(\frac{a}{\sqrt{1-2b\rho^{i,j}+b^2}}\right) \\ &= \mathcal{N}\left(\frac{\ln \frac{S_t^{j,c}}{\alpha_t S_t^{i,c}} + \frac{1}{2}(\hat{\sigma}^{i,j})^2(T-t)}{\hat{\sigma}^{i,j}\sqrt{T-t}}\right),\end{aligned} \tag{A.18}$$

where $\mathcal{N}$ is the standard cumulative normal distribution function.

B. Appendix: MMM Calculations

In this appendix we perform some calculations that are useful for deriving expressions for fair prices in the MMM. The derivations are based upon similar work done in Platen (2003, Appendix) and Miller & Platen (2005). Since X^j, defined by (2.43), is a squared Bessel processes of dimension four, for $j \in \{0, \dots, d\}$, it is a time-homogenous diffusion with transition density

$$\begin{aligned}q_{\varphi_1,\varphi_2}(x,y) := \frac{1}{2(\varphi_2-\varphi_1)} \sqrt{\frac{y}{x}} \exp\left\{-\frac{x+y}{2(\varphi_2-\varphi_1)}\right\} \\ \times \sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(n+2)} \left(\frac{\sqrt{xy}}{2(\varphi_2-\varphi_1)}\right)^{2n+1},\end{aligned} \tag{B.1}$$

for all $\varphi_1, \varphi_2 \in [\varphi^j(0), \varphi^j(T)]$ with $\varphi_1 < \varphi_2$ and all $x, y \in (0, \infty)$ (see Revuz & Yor (1999, Corollary 1.4, p. 441)). In (B.1) Γ denotes the *gamma function* (see Davis (1970)). We recall the important identity

$$\Gamma(n+1) = n!\,, \tag{B.2}$$

for each $n \in \mathbb{Z}_+$.

For each $j \in \{0, \dots, d\}$, define the process $\Lambda^j = \{\Lambda_t^j \,\big|\, t \in [0, T)\}$ by setting

$$\Lambda_t^j := \frac{X_{\varphi^j(t)}^j}{\varphi^j(T) - \varphi^j(t)}, \tag{B.3}$$

for all $t \in [0, T)$. We will typically be interested in the process $\exp\{-\frac{1}{2}\Lambda^j\}$, for $j \in \{0, \ldots, d\}$. It is easily extended to $[0, T]$, by setting

$$\exp\Big\{-\frac{1}{2}\Lambda^j_T\Big\} := \exp\Big\{-\frac{1}{2}\Lambda^j_{T-}\Big\} = 0 \text{ a.s.,} \tag{B.4}$$

for each $j \in \{0, \ldots, d\}$.

B.1. An Expression for Computing Fair Bond Prices

Let $j \in \{0, \ldots, d\}$ and $t \in [0, T)$. We now use the Markov property of X^j to compute the following:

$$\begin{aligned}
\frac{1}{\hat{S}^{j,c}_t}\mathbb{E}\Big[\hat{S}^{j,c}_T \,\Big|\, \mathcal{A}_t\Big] &= X^j_{\varphi^j(t)}\mathbb{E}\Bigg[\frac{1}{X^j_{\varphi^j(T)}}\,\Bigg|\,\mathcal{A}_t\Bigg] \\
&= \int_0^\infty \frac{X^j_{\varphi^j(t)}}{y} q_{\varphi^j(t),\varphi^j(T)}\Big(X^j_{\varphi^j(t)}, y\Big)\,dy \\
&= \frac{1}{2(\varphi^j(T) - \varphi^j(t))}\exp\Big\{-\frac{1}{2}\Lambda^j_t\Big\}\sum_{n=0}^{\infty}\frac{1}{n!\,\Gamma(n+2)} \\
&\quad \times \frac{\Big(X^j_{\varphi^j(t)}\Big)^{n+1}}{[2(\varphi^j(T) - \varphi^j(t))]^{2n+1}}\int_0^\infty \exp\Big\{-\frac{y}{2(\varphi^j(T) - \varphi^j(t))}\Big\}y^n\,dy \qquad \text{(B.5)} \\
&= \exp\Big\{-\frac{1}{2}\Lambda^j_t\Big\}\sum_{n=0}^{\infty}\frac{1}{n!\,\Gamma(n+2)}\frac{\Big(X^j_{\varphi^j(t)}\Big)^{n+1}}{[2(\varphi^j(T) - \varphi^j(t))]^{2n+1}} \\
&\qquad\qquad\qquad \times [2(\varphi^j(T) - \varphi^j(t))]^n\Gamma(n+1) \\
&= \exp\Big\{-\frac{1}{2}\Lambda^j_t\Big\}\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\Big(\frac{1}{2}\Lambda^j_t\Big)^{n+1} \\
&= \exp\Big\{-\frac{1}{2}\Lambda^j_t\Big\}\Big(\exp\Big\{\frac{1}{2}\Lambda^j_t\Big\} - 1\Big) = 1 - \exp\Big\{-\frac{1}{2}\Lambda^j_t\Big\}.
\end{aligned}$$

In the derivation above, the fourth equality follows from the definition of the gamma function; the fifth equality is an application of (B.2); and the sixth equality is an application of the MacLaurin series expansion of the exponential function. According to (B.4) we can extend (B.5) to include the case $t = T$.

B.2. An Expression for Fair Binary Prices

Let $i, j \in \{0, \ldots, d\}$ such that $i \neq j$. Also fix $t \in [0, T)$ and let α_t be a non-negative $\mathcal{A}_t$-measurable random variable. If we make the assumption that $\hat{S}^{i,c}$ and

$\hat{S}^{j,c}$ are independent processes, then it follows that

$$
\begin{aligned}
\frac{1}{\hat{S}^{j,c}_t}\mathbb{E}\left[\mathbb{I}_{\{\hat{S}^{j,c}_T \geq \alpha_t \hat{S}^{i,c}_T\}}\hat{S}^{j,c}_T \,\middle|\, \mathcal{A}_t\right] &= X^j_{\varphi^j(t)}\mathbb{E}\left[\mathbb{I}_{\{X^j_{\varphi^j(T)} \leq \alpha_t X^i_{\varphi^i(T)}\}}\frac{1}{X^j_{\varphi^j(T)}} \,\middle|\, \mathcal{A}_t\right] \\
&= \int_0^\infty \int_0^{\alpha_t x} \frac{X^j_{\varphi^j(t)}}{y} q_{\varphi^i(t),\varphi^i(T)}\left(X^i_{\varphi^i(t)}, x\right) q_{\varphi^j(t),\varphi^j(T)}\left(X^j_{\varphi^j(t)}, y\right) dy\, dx \\
&= \int_0^\infty \int_0^{\alpha_t x} \frac{1}{y} \exp\left\{-\frac{1}{2}\Lambda^j_t\right\} \frac{1}{\varphi^i(T)-\varphi^i(t)} \exp\left\{-\frac{1}{2}\Lambda^i_t\right\} \\
&\quad \times \left[\sum_{l=0}^\infty \frac{1}{l!\,\Gamma(l+2)2^{l+1}}\left(\frac{1}{2}\Lambda^j_t\right)^{l+1}\left(\frac{y}{\varphi^j(T)-\varphi^j(t)}\right)^{l+1}\right. \\
&\qquad \left. \times \exp\left\{-\frac{y}{2(\varphi^j(T)-\varphi^j(t))}\right\}\right] \\
&\quad \times \left[\sum_{q=0}^\infty \frac{1}{q!\,\Gamma(q+2)2^{q+2}}\left(\frac{1}{2}\Lambda^i_t\right)^{q}\left(\frac{x}{\varphi^i(T)-\varphi^i(t)}\right)^{q+1}\right. \\
&\qquad \left. \times \exp\left\{-\frac{x}{2(\varphi^i(T)-\varphi^i(t))}\right\}\right] dy\, dx \\
&= \exp\left\{-\frac{1}{2}(\Lambda^j_t + \Lambda^i_t)\right\} \sum_{l=0}^\infty \frac{(\frac{1}{2}\Lambda^j_t)^{l+1}}{l!\,\Gamma(l+2)2^{l+1}} \sum_{q=0}^\infty \frac{(\frac{1}{2}\Lambda^i_t)^{q}}{q!\,\Gamma(q+2)2^{q+2}} \\
&\qquad \times \int_0^\infty \int_0^{\bar{\alpha}_t \bar{x}} \exp\left\{-\frac{1}{2}(\bar{x}+\bar{y})\right\} \bar{y}^l \bar{x}^{q+1}\, d\bar{y}\, d\bar{x}.
\end{aligned}
\tag{B.6}
$$

Here we have made the substitutions

$$\bar{x} := \frac{x}{\varphi^i(T)-\varphi^i(t)}; \tag{B.7}$$

$$\bar{y} := \frac{y}{\varphi^j(T)-\varphi^j(t)}. \tag{B.8}$$

The constant in the upper limit of the inner integral in the last line of (B.6) is thus given by

$$\bar{\alpha}_t := (\varphi^i(T)-\varphi^i(t))\alpha_t. \tag{B.9}$$

With the aid of another change of variables, namely

$$\tilde{y} := \frac{\bar{y}}{\bar{x}}, \tag{B.10}$$

(B.6) now becomes

$$
\begin{aligned}
&\exp\left\{-\frac{1}{2}(\Lambda_t^j+\Lambda_t^i)\right\}\sum_{l=0}^{\infty}\frac{(\frac{1}{2}\Lambda_t^j)^{l+1}}{l!\,\Gamma(l+2)2^{l+1}}\sum_{q=0}^{\infty}\frac{(\frac{1}{2}\Lambda_t^i)^q}{q!\,\Gamma(q+2)2^{q+2}}\\
&\qquad\times\int_0^{\bar{\alpha}_t}\int_0^{\infty}\exp\left\{-\frac{1}{2}\bar{x}(1+\tilde{y})\right\}\tilde{y}^l\bar{x}^{q+l+2}\,d\bar{x}\,d\tilde{y}\\
&\quad=\exp\left\{-\frac{1}{2}(\Lambda_t^j+\Lambda_t^i)\right\}\sum_{l=0}^{\infty}\frac{(\frac{1}{2}\Lambda_t^j)^{l+1}}{l!\,\Gamma(l+2)2^{l+1}}\sum_{q=0}^{\infty}\frac{(\frac{1}{2}\Lambda_t^i)^q}{q!\,\Gamma(q+2)2^{q+2}}\\
&\qquad\times\Gamma(q+l+3)2^{q+l+3}\int_0^{\bar{\alpha}_t}\frac{\tilde{y}^l}{(1+\tilde{y})^{q+l+3}}\,d\tilde{y}\\
&\quad=\exp\left\{-\frac{1}{2}(\Lambda_t^j+\Lambda_t^i)\right\}\sum_{m=1}^{\infty}\frac{(\frac{1}{2}\Lambda_t^j)^m}{m!}\sum_{q=0}^{\infty}\frac{(\frac{1}{2}\Lambda_t^i)^q}{q!}\frac{\Gamma(q+m+2)}{\Gamma(m)\Gamma(q+2)}\\
&\qquad\times\int_0^{\bar{\alpha}_t}\frac{\tilde{y}^{m-1}}{(1+\tilde{y})^{q+m+2}}\,d\tilde{y} \qquad\qquad (B.11)\\
&\quad=\exp\left\{-\frac{1}{2}(\Lambda_t^j+\Lambda_t^i)\right\}\sum_{m=1}^{\infty}\frac{(\frac{1}{2}\Lambda_t^j)^m}{m!}\sum_{q=0}^{\infty}\frac{(\frac{1}{2}\Lambda_t^i)^q}{q!}\frac{\Gamma(q+m+2)}{\Gamma(m)\Gamma(q+2)}\\
&\qquad\times\frac{\bar{\alpha}_t^m}{m}\,{}_2F_1(m,q+m+2;m+1;-\bar{\alpha}_t)\\
&\quad=\sum_{m=1}^{\infty}\frac{\exp\{-\frac{1}{2}\Lambda_t^j\}(\frac{1}{2}\Lambda_t^j)^m}{m!}\sum_{q=0}^{\infty}\frac{\exp\{-\frac{1}{2}\Lambda_t^i\}(\frac{1}{2}\Lambda_t^i)^q}{q!}\frac{\Gamma(q+m+2)}{\Gamma(m+1)\Gamma(q+2)}\bar{\alpha}_t^m\\
&\qquad\times{}_2F_1(m,q+m+2;m+1;-\bar{\alpha}_t), \qquad\qquad (B.12)
\end{aligned}
$$

where ${}_2F_1$ is *Gauss' hypergeometric function* (see Oberhettinger (1970)). The second and fourth equalities above were obtained using Mathematica's symbolic integration facility, while the last equality is an application of (B.2). Note that

$$
\begin{aligned}
&\frac{\exp\{-\frac{1}{2}\Lambda_t^j\}(\frac{1}{2}\Lambda_t^j)^0}{0!}\sum_{q=0}^{\infty}\frac{\exp\{-\frac{1}{2}\Lambda_t^i\}(\frac{1}{2}\Lambda_t^i)^q}{q!}\frac{\Gamma(q+0+2)}{\Gamma(0+1)\Gamma(q+2)}\bar{\alpha}_t^0\\
&\qquad\qquad\times{}_2F_1(0,q+0+2;0+1;-\bar{\alpha}_t) \qquad\qquad (B.13)\\
&=\exp\left\{-\frac{1}{2}\Lambda_t^j\right\}\sum_{q=0}^{\infty}\frac{\exp\{-\frac{1}{2}\Lambda_t^i\}(\frac{1}{2}\Lambda_t^i)^q}{q!}=\exp\left\{-\frac{1}{2}\Lambda_t^j\right\}.
\end{aligned}
$$

The first equality follows from (B.2) and the properties of the hypergeometric function; for the second equality, note that $\sum_{q=0}^{\infty}\frac{\exp\{-\frac{1}{2}\Lambda_t^i\}(\frac{1}{2}\Lambda_t^i)^q}{q!}$ is the total probability of a Poisson random variable with parameter $\frac{1}{2}\Lambda_t^i$. Thus, putting (B.12) and

(B.13) together, and then rewinding from (B.12) to (B.11), we see that (B.6) can be written as

$$\begin{aligned}
&\Bigg[\sum_{m=0}^{\infty}\frac{\exp\{-\frac{1}{2}\Lambda_t^j\}(\frac{1}{2}\Lambda_t^j)^m}{m!}\sum_{q=0}^{\infty}\frac{\exp\{-\frac{1}{2}\Lambda_t^i\}(\frac{1}{2}\Lambda_t^i)^q}{q!}\frac{\Gamma(q+m+2)}{\Gamma(m+1)\Gamma(q+2)}\bar{\alpha}_t^m \\
&\qquad\times {}_2F_1(m,q+m+2;m+1;-\bar{\alpha}_t)\Bigg]-\exp\Big\{-\frac{1}{2}\Lambda_t^j\Big\} \\
&=\Bigg[\sum_{m=0}^{\infty}\frac{\exp\{-\frac{1}{2}\Lambda_t^j\}(\frac{1}{2}\Lambda_t^j)^m}{m!}\sum_{q=0}^{\infty}\frac{\exp\{-\frac{1}{2}\Lambda_t^i\}(\frac{1}{2}\Lambda_t^i)^q}{q!}\frac{\Gamma(q+m+2)}{\Gamma(m)\Gamma(q+2)} \\
&\qquad\times\int_0^{\bar{\alpha}_t}\frac{\tilde{y}^{m-1}}{(1+\tilde{y})^{q+m+2}}\,d\tilde{y}\Bigg]-\exp\Big\{-\frac{1}{2}\Lambda_t^j\Big\}.
\end{aligned} \tag{B.14}$$

Finally, we are able to conclude with the remarkable observation that if $Z_1 \sim \chi'^2_0(\Lambda_t^j)$ and $Z_2 \sim \chi'^2_4(\Lambda_t^i)$, where $\chi'^2_\nu(\lambda)$ denotes the *non-central chi-square distribution* with dimension ν and non-centrality parameter λ, then the expression in the square brackets in (B.14) is $\mathbb{P}[\frac{Z_1}{Z_2} \le \bar{\alpha}_t]$, according to Johnson, Kotz & Balakrishnan (1995, (30.49), p. 499). It is not clear whether the distribution of $\frac{Z_1}{Z_2}$ has an established name; however, from its relationship to a *doubly non-central F-distribution* (see Johnson, Kotz & Balakrishnan (1995, p. 499)), it seems appropriate to call it an *unscaled doubly non-central F-distribution*. In keeping with the notation of Johnson, Kotz & Balakrishnan (1995, p. 499), we will write $G''_{0,4}(\,\cdot\,;\Lambda_t^j,\Lambda_t^i)$ for its cumulative distribution function, so that

$$G''_{0,4}\Big((\varphi^i(T)-\varphi^i(t))\alpha_t;\Lambda_t^j,\Lambda_t^i\Big)-\exp\Big\{-\frac{1}{2}\Lambda_t^j\Big\} \tag{B.15}$$

provides an interesting expression for (B.14).

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