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Effective mass and mass renormalization of nonrelativistic QED

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Abstract
The effective mass $m_{\text{eff}}$ of the nonrelativistic QED is considered. $m_{\text{eff}}$ is defined as the inverse of curvature of the ground state energy with total momentum zero. The effective mass $m_{\text{eff}} = m_{\text{eff}}(e^2, \Lambda, \kappa, m)$ is a function of bare mass $m > 0$, ultraviolet cutoff $\Lambda > 0$, infrared cutoff $\kappa > 0$, and the square of charge $e$ of an electron. Introduce a scaling $m \to m(\Lambda) = (b\Lambda)^\beta$, $\beta < 0$. Then asymptotics behavior of $m_{\text{eff}}$ as $\Lambda \to \infty$ is studied.

1 Introduction
1.1 The Pauli-Fierz Hamiltonian
This is a joint work with Herbert Spohn. We consider a single, spinless free electron coupled to a quantized radiation field (photons). The Hilbert space of states of photons is the symmetric Fock space:

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \bigotimes_{s}^{n} L^2(\mathbb{R}^3 \times \{1, 2\})$$

where $\bigotimes_{s}^{n} L^2(\mathbb{R}^3 \times \{1, 2\})$ denotes the $n$-fold symmetric tensor product of $L^2(\mathbb{R}^3 \times \{1, 2\})$ with $\bigotimes_{\mathit{8}}^{0} L^2(\mathbb{R}^3 \times \{1, 2\}) = \mathbb{C}$. The inner product in $\mathcal{F}$ is denoted by $(\cdot, \cdot)$ and the Fock vacuum by $\Omega$. On $\mathcal{F}$ we introduce the Bose field

$$a(f) = \sum_{j=1,2} \int f(k, j)^* a(k, j) dk, \quad f \in L^2(\mathbb{R}^3 \times \{1, 2\}),$$

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where $a(f)$ and $a^*(f) = a(\bar{f})^*$ are densely defined and satisfy the CCR

\[
[a(f), a^*(g)] = (f, g)_{L^2(\mathbb{R}^3 \times \{1, 2\})},
\]
\[
[a(f), a(g)] = 0,
\]
\[
[a^*(f), a^*(g)] = 0.
\]

The free Hamiltonian of $\mathcal{F}$ is read as

\[
H_t = \sum_{j=1,2} \int \omega(k) a^*(k,j)a(k,j)dk,
\]

where the dispersion relation is given by

\[
\omega(k) = |k|.
\]

The free Hamiltonian $H_t$ acts as

\[
H_t \Omega = 0,
\]
\[
H_t a^*(f_1) \cdots a^*(f_n) \Omega = \sum_{j=1}^{n} a^*(f_1) \cdots a^*(\omega f_j) \cdots a^*(f_n) \Omega.
\]

The Pauli-Fierz Hamiltonian $H$ is defined as a self-adjoint operator acting on

\[
\mathcal{H} = L^2(\mathbb{R}^3) \otimes F \cong \int_{\mathbb{R}^3} \mathcal{F} dx
\]

by

\[
H = \frac{1}{2m}(p_x \otimes 1 - eA_{\hat{\phi}})^2 + V \otimes 1 + 1 \otimes H_t,
\]

where $m$ and $e$ denote the mass and charge of electron, respectively,

\[
p_x = (-i \frac{\partial}{\partial x_1}, -i \frac{\partial}{\partial x_2}, -i \frac{\partial}{\partial x_3} )
\]

and $V$ an external potential. The quantized radiation field $A_{\hat{\phi}}$ is defined by

\[
A_{\hat{\phi}} = \frac{1}{\sqrt{2}} \int_{\mathbb{R}^3} (a(f_x) + a^{*}(\bar{f}_x)) dx,
\]

where

\[
f_x(k,j) = \frac{1}{\sqrt{\omega}} \hat{\phi}(k)e(k,j)e^{ikx};
\]

$e(k,1), e(k,2), k/|k|$ form a right-handed dreibain, and $\hat{\phi}$ is a form factor. $A_{\hat{\phi}}$ acts for $\Psi \in \mathcal{H}$ as

\[
(A_{\hat{\phi}} \Psi)(x) = (a(f_x) + a^{*}(\bar{f}_x))\Psi(x), \quad x \in \mathbb{R}^3.
\]

**Theorem 1.1** Assume that $\hat{\phi}/\omega, \hat{\phi}/\sqrt{\omega}, \sqrt{\omega}\hat{\phi} \in L^2(\mathbb{R}^3)$ and $V$ is relatively bounded with respect to $-\Delta$ with a relative bound $< 1$. Then, for arbitrary values of $e$, $H$ is self-adjoint on $D(\Delta \otimes 1) \cap D(1 \otimes H_t)$ and bounded from below.

**Proof:** See Hiroshima [3, 4].

\[\square\]
1.2 Effective mass

The momentum of the photon field is given by

\[ P_t = \sum_{j=1,2} \int ka^*(k,j) a(k,j) dk \]  

(1.5)

and the total moment by

\[ P_{\text{total}} = p_x \otimes 1 + 1 \otimes P_t. \]

Let us assume that

\[ V \equiv 0. \]

Then we see that

\[ [H, P_{\text{total}_\mu}] = 0, \quad \mu = 1, 2, 3. \]

Hence \( H \) and \( \mathcal{H} \) can be decomposable with respect to \( \text{Spec}(P_{\text{total}}) = \mathbb{R}^3 \), i.e.,

\[
\mathcal{H} = \int_{\mathbb{R}^3} \mathcal{H}(p) dp, \\
H = \int_{\mathbb{R}^3} H(p) dp.
\]

Note that

\[
e^{-iz \otimes P_t} P_{\text{total}} e^{iz \otimes P_t} = p_x, \\
e^{-iz \otimes P_t} H e^{iz \otimes P_t} = \frac{1}{2m} (p_x \otimes 1 - 1 \otimes P_t - e1 \otimes A_\phi(0)) + 1 \otimes H_t,
\]

where

\[ A_\phi(0) = \frac{1}{\sqrt{2}} (a(f_0) + a(\bar{f}_0)). \]

From this we obtain that for each \( p \in \mathbb{R}^3 \),

\[
\mathcal{H}(p) \cong \mathcal{F}, \\
H(p) \cong \frac{1}{2m} (p - P_t - e A_\phi(0)) + H_t,
\]

Let

\[ E_{m,\Lambda}(p) = \inf \text{Spec}(H(p)). \]  

(1.6)

Let us assume sharp ultraviolet cutoff \( \Lambda \) and infrared cutoff \( \kappa \), which means

\[ \phi(k) = \begin{cases} 
0 & \text{for } |k| < \kappa, \\
(2\pi)^{-3/2} & \text{for } \kappa \leq |k| \leq \Lambda, \\
0 & \text{for } |k| > \Lambda.
\end{cases} \]  

(1.7)

**Lemma 1.2** There exists constants \( p_* \) and \( e_* \) such that for

\[ (p, e) \in \mathcal{O} = \{(p, e) \in \mathbb{R}^3 \times \mathbb{R} | |p| < p_*, |e| < e_*\}, \]

\( H(p) \) has a ground state \( \psi_\phi(p) \) and it is unique. Moreover \( \psi_\phi(p) = \psi_\phi(p, e) \) is strongly analytic and \( E_{m,\Lambda}(p) = E_{m,\Lambda}(p, e) \) analytic with respect to \( (p, e) \in \mathcal{O} \).
Proof: See Hiroshima and Spohn [6, 7].

In what follows we assume that \((p, e) \in \mathcal{O}\).

Definition 1.3 The effective mass \(m_{\text{eff}} = m_{\text{eff}}(e^2, \Lambda, \kappa, m)\) is defined by

\[
\frac{1}{m_{\text{eff}}} = \frac{1}{3} \Delta \phi E(p, e)|_{p = 0}.
\]

(1.8)

1.3 Mass renormalization

Removal of the ultraviolet cutoff \(\Lambda\) through mass renormalization means to find sequences

\[
\Lambda \to \infty, \quad m \to 0
\]

(1.9)
such that \(E_{m, \Lambda}(p) - E_{m, \Lambda}(0)\) has a nondegenerate limit. To achieve this, as a first step we want to find constants

\[\beta < 0, \quad 0 < b\]

such that

\[
\lim_{\Lambda \to \infty} m_{\text{eff}}(e^2, \Lambda, \kappa \Lambda^\beta, (b \Lambda)^\beta) = m_{\text{ph}},
\]

(1.10)

where \(m_{\text{ph}}\) is a given constant. Actually \(m_{\text{ph}}\) is a physical mass. Namely in the mass renormalization the scaled bare mass goes to zero and the effective mass goes to a physical mass as the ultraviolet cutoff \(\Lambda\) goes to infinity.

We will see later that \(m_{\text{eff}}/m\) is a function of \(e^2, \Lambda/m\) and \(\kappa/m\). Let

\[
\frac{m_{\text{eff}}}{m} = f(e^2, \Lambda/m, \kappa/m),
\]

(1.11)

where \(f(0, \Lambda/m, \kappa/m) = 1\) holds. An analysis of (1.10) can be reduce to investigate the asymptotic behavior of \(f\) as \(\Lambda \to \infty\). Namely we want to find constants

\[0 \leq \gamma < 1, \quad 0 < b_0\]

such that

\[
\lim_{\Lambda \to \infty} \frac{f(e^2, \Lambda/m, \kappa/m)}{(\Lambda/m)\gamma} = b_0.
\]

(1.12)

If we succeed to find constants \(\gamma\) and \(b_0\) such as in (1.12) then by

\[
m_{\text{eff}}(e^2, \Lambda, \kappa, m) = mf(e^2, \Lambda/m, \kappa/m),
\]

we have

\[
m_{\text{eff}}(e^2, \Lambda, \kappa \Lambda^\beta, (b \Lambda)^\beta) = (b \Lambda)^\beta f(e^2, \Lambda/(b \Lambda)^\beta, \kappa/(b^\beta)) \approx b_0 (b \Lambda)^\beta (\Lambda/(b \Lambda)^\beta)^\gamma.
\]

(1.13)

Taking

\[
\beta = \frac{-\gamma}{1 - \gamma} < 0, \quad b = 1/b_1^{1/\gamma},
\]
we see that by (1.13)

\[
\lim_{\Lambda \to \infty} m_{\text{eff}}(\epsilon^2, \Lambda, \kappa \Lambda^\beta, (b\Lambda)^\beta) = \lim_{\Lambda \to \infty} b_0 \left( \frac{\Lambda}{b_1^{1/\gamma}} \right)^\beta \left( \frac{\Lambda}{(\Lambda/(b_1)^{1/\gamma})^{\beta}} \right)^\gamma = b_0 b_1,
\]

where \(b_1\) is a parameter, which is adjusted such as

\[b_0 b_1 = m_{\text{ph}}.\]

Hence we will be able to establish (1.10). It is easily seen that

\[f(\epsilon^2, \Lambda/m, \kappa/m) = 1 + \alpha \frac{8}{3\pi} \log(\frac{\Lambda/m + 2}{\kappa/m + 2}) + O(\alpha^2),\]

where \(\alpha = \epsilon^2/4\pi\), which suggests

\[f(\epsilon^2, \Lambda/m, \kappa/m) \approx (\Lambda/m)^{8\alpha/3\pi},\]

for sufficiently small \(\alpha\) and large \(\Lambda\), and therefore

\[\gamma = 8\alpha/3\pi.\]

One may assume that

\[f(\epsilon^2, \Lambda/m, \kappa/m) \approx (\Lambda/m)^{a(8/3\pi) + \alpha^2 b}\]

for sufficiently small \(\alpha\) with some constant \(b\). Then by expanding \(m_{\text{eff}}/m\) to order \(\alpha^2\) one may expect that

\[f(\epsilon^2, \Lambda/m, \kappa/m) \approx 1 + \alpha \frac{8}{3\pi} \log(\frac{\Lambda}{m}) + \frac{1}{2} \alpha^2 \left( \frac{8}{3\pi} \log(\frac{\Lambda}{m}) \right)^2 + \alpha^2 \log(\frac{\Lambda}{m}) + O(\alpha^3)\]

(1.14)

for sufficiently small \(\alpha\) and large \(\Lambda\). It is, however, that (1.14) is not confirmed. Instead of (1.14) we prove that there exists a constant \(C > 0\) such that

\[f(\epsilon^2, \Lambda/m, \kappa/m) = 1 + \alpha \frac{8}{3\pi} \log(\frac{\Lambda/m + 2}{\kappa/m + 2}) + \alpha^2 C \sqrt{\Lambda/m} + O(\alpha^3).\]

The effective mass and its renormalization have been studied from a mathematical point of view by many authors. Spohn [10] investigates the effective mass of the Nelson model [9] from a functional integral point of view. Lieb and Loss [8] studied mass renormalization and binding energies of models of matter coupled to radiation fields including the Pauli-Fierz model. Hainzl and Seiringer [2] computed exactly the leading order in \(\alpha\) of the effective mass of the Pauli-Fierz Hamiltonian with spin.
2 Perturbative expansions

The effective masses for $H(p)$ and

$$\frac{1}{2m} : (p - P + eA_{\phi}(0))^{2} + H_{f}$$

are identical. Then in what follows we redefine $H(p)$ as

$$H(p) = \frac{1}{2m} : (p - P + eA_{\phi}(0))^{2} + H_{f}.$$ 

Furthermore for notational convenience we write $A$ and $E(p)$ for $A_{\phi}(0)$ and $E_{m,\Lambda}(p)$, respectively.

2.1 Formulae

Lemma 2.1 We have

$$\frac{m}{m_{\text{eff}}} = 1 - \frac{2}{3} \sum_{\mu=1,2,3} (\psi_{g}(0), (P_{f} + eA)_{\mu}(H(0) - E(0))^{-1}(P_{f} + eA)_{\mu}\psi(0))$$

Proof: It is seen that $E(p, e) = E(p, -e) = E(-p, e)$. Then

$$\frac{\partial}{\partial p_{\mu}} E(p, e) \{ p_{\mu} = 0, \mu = 1, 2, 3 \} \text{ (2.1)}$$

follows. Moreover it is seen that $E(p, e)$ is a function of $e^2$ and

$$\frac{d^{2m-1}}{de^{2m-1}} E(p, e) \{ e = 0 \} \text{ (2.2)}$$

In this proof, $f'(p)_{\mu}$ means the strong derivative of $f(p)$ with respect to $p_{\mu}$. Since

$$H(p)\psi_{g}(p) = E(p)\psi_{g}(p),$$

we have

$$H'_{\mu} \psi_{g}(p) + H(p)\psi'_{g}(p)_{\mu} = E'(p)_{\mu}\psi_{g}(p) + E(p)\psi'_{g}(p)_{\mu} \text{ (2.3)}$$

and

$$H''_{\mu} \psi_{g}(p) + 2H'(p)_{\mu}\psi'_{g}(p)_{\mu} + H(p)\psi''_{g}(p)_{\mu}$$

$$= E''(p)_{\mu}\psi_{g}(p) + 2E'(p)_{\mu}\psi'_{g}(p)_{\mu} + E(p)\psi''_{g}(p)_{\mu}. \text{ (2.4)}$$

By (2.1) it follows that $E'(0)_{\mu} = 0$, and by (2.3) with $p = 0$,

$$(P_{f} + eA)_{\mu}\psi_{g}(0) \in D((H(0) - E(0))^{-1}),$$

$$\psi'_{g}(0)_{\mu} = (H(0) - E(0))^{-1}(P_{f} + eA)_{\mu}\psi_{g}(0).$$
Then we have by (2.3) and (2.4),
\[
\frac{m}{m_{\text{eff}}} = \frac{1}{3} \sum_{\mu=1,2,3} \frac{(\psi_\mu(0), E''(0)_{\mu} \psi_\mu(0))}{(\psi_\mu(0), \psi_\mu(0))}
\]
\[
= 1 - \frac{2}{3} \sum_{\mu=1,2,3} \frac{((P_\mu + eA)_{\mu} \psi_\mu(0), (H(0) - E(0))^{-1}(P_\mu + eA)_{\mu} \psi_\mu(0))}{(\psi_\mu(0), \psi_\mu(0))}
\]
Thus the lemma follows. \(\square\)

Let
\[
\psi_\mu(0) = \sum_{n=0}^{\infty} \frac{e^n}{n!} \varphi_n, \quad E(0) = \sum_{n=0}^{\infty} \frac{e^{2n}}{(2n)!} E_{2n}.
\]
Note that
\[
\varphi_{2m} \in \bigoplus_{m=0}^{\infty} \mathcal{F}^{(2m)}, \quad \varphi_{2m+1} \in \bigoplus_{m=0}^{\infty} \mathcal{F}^{(2m+1)}.
\]
We want to get the explicit form of \(\varphi_n\). Let
\[
\mathcal{F}_{\text{fin}} = \{\{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F} | \Psi^{(m)} = 0 \text{ for } m \geq \ell \text{ with some } \ell\},
\]
\[
\mathcal{F}_0 = \{\{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_{\text{fin}} | (i) \Psi^{(0)} = 0,
(ii) \text{ supp}(k_1,\ldots,k_n) \in \mathbb{R}^m \Psi^{(n)}(k_1,\ldots,k_n,j_1,\ldots,j_n) \neq \{(0,\ldots,0)\}\}.
\]

**Lemma 2.2** We see that \(\mathcal{F}_0 \subset D(H_0^{-1})\).

**Proof:** Let \(\Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_0\). Since
\[
(H_0 \Psi)^{(n)}(k_1,\ldots,k_n,j_1,\ldots,j_n)
= \left[ \frac{1}{2} (k_1 + \cdots + k_n)^2 + \sum_{j=1}^{n} \omega(k_j) \right] \Psi^{(n)}(k_1,\ldots,k_n,j_1,\ldots,j_n),
\]
we see that
\[
(H_0^{-1} \Psi)^{(n)}(k_1,\ldots,k_n,j_1,\ldots,j_n)
= \left[ \frac{1}{2} (k_1 + \cdots + k_n)^2 + \sum_{j=1}^{n} \omega(k_j) \right]^{-1} \Psi^{(n)}(k_1,\ldots,k_n,j_1,\ldots,j_n).
\]
Since \(\text{supp}(k_1,\ldots,k_n) \in \mathbb{R}^m \Psi^{(n)}(k_1,\ldots,k_n,j_1,\ldots,j_n) \neq \{(0,\ldots,0)\}\), we obtain that
\[
\|H_0^{-1} \Psi\|_F^2 = \sum_{n=1}^{\text{finite}} \|H_0^{-1} \Psi^{(n)}\|_{\mathcal{F}^{(n)}}^2 < \infty.
\]
Then the lemma follows. \(\square\)

We split \(H(0)\) as
\[
H(0) = H_0 + eH_1 + \frac{e^2}{2} H_2.
\]
where
\[
H_0 = \frac{1}{2}P_t^2 + H_f,
\]
\[
H_1 = \frac{1}{2}(P_t \cdot A + A \cdot P_t) = P_t \cdot A = A \cdot P_t,
\]
\[
H_2 = A^2.
\]

**Lemma 2.3** We have \( E_0 = E_1 = E_2 = E_3 = 0 \) and
\[
\varphi_0 = \Omega, \quad \varphi_1 = 0, \quad \varphi_2 = -H_0^{-1}H_2\Omega, \quad \varphi_3 = 3H_0^{-1}H_1H_0^{-1}H_2\Omega.
\]
In particular \( \varphi_2 \in \mathcal{F}^{(2)} \) and \( \varphi_3 \in \mathcal{F}^{(1)} \cap \mathcal{F}^{(3)} \).

**Proof:** Let us set \( H(0), E(0) \) and \( \psi_g(0) \) as \( H, E \) and \( \psi_g \), respectively. It is obvious that \( E_0 = 0 \) and \( \varphi_0 = a\Omega \) with arbitrary \( a \in \mathbb{C} \), and by (2.2), \( E_1 = E_3 = 0 \). Set \( a = 1 \). We denote the strong derivative of \( f = f(e) \) with respect to \( e \) by \( f' \). We have
\[
H'\psi_g + H\psi_g' = E'\psi_g + E\psi_g'
\] (2.5)
and
\[
H''\psi_g + 2H'\psi_g' + H\psi_g'' = E''\psi_g + 2E'\psi_g' + E\psi_g''.
\] (2.6)
From (2.6) it follows that
\[
(\psi_g, H''\psi_g) + (\psi_g, 2H'\psi_g') + (\psi_g, H\psi_g'') = E''(\psi_g, \psi_g) + (\psi_g, 2E'\psi_g') + (\psi_g, E\psi_g'').
\] (2.7)
Put \( e = 0 \) in (2.7). Then
\[
(\Omega, H_2\Omega) + (\Omega, 2H_1\Omega) + (\Omega, H_0\varphi_2) = E_2(\Omega, \Omega).
\] (2.8)
Since the left-hand side of (2.8) vanishes, we have \( E_2 = 0 \). From (2.5) with \( e = 0 \) and the fact \( E_0 = E_1 = 0 \), it follows that
\[
H_1\Omega + H_0\varphi_1 = 0,
\]
from which it holds that \( H_0\varphi_1 = 0 \). Since \( H_0 \) has the unique eigenvector \( \Omega \) (the ground state) with eigenvalue zero, it follows that \( \varphi_1 = b\Omega \) with some constant \( b \). \( \varphi_1 \in \bigoplus_{m=0}^{\infty} \mathcal{F}^{(2m+1)} \) which implies \( b = 0 \). Hence \( \varphi_1 = 0 \) follows.

By (2.6) with \( e = 0 \), we have
\[
H_2\Omega + 2H_1\varphi_1 + H_0\varphi_2 = 0.
\]
Since \( H_2\Omega \in \mathcal{F}_0 \), we see that by Lemma 2.2, \( H_2\Omega \in D(H_0^{-1}) \). Thus we have \( \varphi_2 = -H_0^{-1}H_2\Omega \). From the identity
\[
H''\psi_g + 3H''\psi_g' + 3H'\psi_g'' + H\psi_g''' = E''\psi_g + 3E'\psi_g' + 3E\psi_g'' + E\psi_g'''
\] (2.9)
it follows that at \( e = 0 \),
\[
3H_1\varphi_2 + H_0\varphi_3 = 0.
\]
Since \( H_1\varphi_2 = -H_1H_0^{-1}H_2\Omega \in \mathcal{F}_0 \), Lemma 2.2 ensures that \( H_1\varphi_2 \in D(H_0^{-1}) \).

Hence \( \varphi_3 = -3H_0^{-1}H_1\varphi_2 = 3H_0^{-1}H_1H_0^{-1}H_2\Omega \). Then the lemma is proven.

\( \square \)
2.2 Order $e^4$

In this subsection we expand $m/m_{\text{eff}}$ up to order $e^4$. We define $A^-$ and $A^+$ by

$$A^- = \frac{1}{\sqrt{2}} a(f), \quad A^+ = \frac{1}{\sqrt{2}} a^*(f).$$

Then $A = A^+ + A^-$. 

**Lemma 2.4** We have

$$\frac{m}{m_{\text{eff}}} = 1 - e^2 \sum_{\mu=1}^{3} \left( \Omega, A_{\mu} H_0^{-1} A_{\mu} \Omega \right)$$

$$- e^4 \sum_{\mu=1}^{3} \left\{ 2 \left( \Psi_1^{\mu}, H_0^{-1} \Psi_1^{\mu} \right) + \left( \Psi_2^{\mu}, H_0^{-1} \Psi_2^{\mu} \right) - 2 \left( \Psi_2^{\mu}, H_0^{-1} H_1 H_0^{-1} \Psi_2^{\mu} \right) \right\}.$$

(2.10)

where

$$\Psi_1^{\mu} = A_{\mu} \Omega,$$

$$\Psi_2^{\mu} = -\frac{1}{2} P_{\mu} H_0^{-1} (A^+ \cdot A^+) \Omega,$$

$$\Psi_3^{\mu} = \frac{1}{2} \left\{ -A_{\mu} H_0^{-1} (A^+ \cdot A^+) \Omega + \frac{1}{2} P_{\mu} H_0^{-1} (P_{\mu} \cdot A + A_{\mu} P_{\mu}) H_0^{-1} (A^+ \cdot A^+) \Omega \right\}.$$

**Proof:** In Lemma 2.1 we have seen that

$$\frac{m}{m_{\text{eff}}} = 1 - \frac{2}{3} \sum_{\mu=1,2,3} \frac{((P_{\mu} + eA)_{\mu} \varphi_{g}(0), (H(0) - E(0))^{-1} (P_{\mu} + eA)_{\mu} \varphi_{g}(0))}{(\varphi_{g}(0), \varphi_{g}(0))}.$$

(2.11)

We can strongly expand $(H(0) - E(0))^{-1}$ as

$$(H(0) - E(0))^{-1} = H_0^{-1} - eH_0^{-1} H_1 H_0^{-1}$$

$$+ e^2 \left( -\frac{1}{2} H_0^{-1} H_2 H_0^{-1} + H_0^{-1} H_1 H_0^{-1} H_1 H_0^{-1} \right) + O(e^3).$$

(2.12)

Here we set

$$H_j = \begin{cases} H_j, & j = 1, 2, \\ -E_j, & j \geq 3. \end{cases}$$

Note that

$$\varphi_0 \in \mathcal{F}^{(0)}, \varphi_2 \in \mathcal{F}^{(2)}, \varphi_3 \in \mathcal{F}^{(3)} \cap \mathcal{F}^{(1)}, \varphi_4 \in \mathcal{F}^{(4)} \cap \mathcal{F}^{(2)}.$$
In particular
\[ \frac{1}{(\psi_{g},\psi_{g})} = 1 - e^{4}(\frac{1}{2}\varphi_{2}, \frac{1}{2}\varphi_{2}) - e^{4}(\Omega, \frac{1}{24}\varphi_{4}) + O(e^{6}) = 1 - e^{4}\frac{1}{4}(\varphi_{2}, \varphi_{2}) + O(e^{6}). \]

Moreover we have
\[ (P_{f} + eA)_{\mu} \psi_{g}(0) = eA_{\mu}\Omega + e^{2}(\frac{1}{2}P_{f\mu}\varphi_{2}) + e^{3}(\frac{1}{2}A_{\mu}\varphi_{2} + \frac{1}{6}P_{f\mu}\varphi_{3}) + O(e^{4}) = e\Psi_{1}^{\mu} + e^{2}\Psi_{2}^{\mu} + e^{3}\Psi_{3}^{\mu} + O(e^{4}). \]

Substitute (2.12), (2.13) and (2.14) into (2.11). Then the lemma follows.

For each $k \in \mathbb{R}^{3}$ let us define the projection $Q(k)$ on $\mathbb{R}^{3}$ by
\[ Q(k) = \sum_{j=1,2} |e_{j}(k)\rangle\langle e_{j}(k)|. \]

We set
\[ \tilde{\varphi}_{j} = \varphi(k_{j}), \quad \omega_{j} = \omega(k_{j}), \quad Q(k_{j}) = Q_{j}, \quad j = 1, 2. \]

Let
\[ \frac{1}{F_{j}} = \frac{1}{r_{j}^{2}/2 + r_{j}}, \quad j = 1, 2, \]
\[ \frac{1}{F_{12}} = \frac{1}{(r_{1}^{2} + 2r_{1}r_{2}X + r_{2}^{2})/2 + r_{1} + r_{2}}, \]
\[ r_{1}, r_{2} \geq 0, \quad -1 \leq X \leq 1. \]

Lemma 2.5 We have
\[ \frac{m}{m_{\text{eff}}} = 1 - a_{1}(\Lambda/m, \kappa/m) - a^{2}a_{2}(\Lambda/m, \kappa/m) + O(a^{3}), \]
where
\[ a_{1}(\Lambda/m, \kappa/m) = \frac{8}{3\pi} \log \left( \frac{\Lambda/m + 2}{\kappa/m + 2} \right) \]
and
\[ a_{2}(\Lambda/m, \kappa/m) = \frac{(4\pi)^{2} 2}{(2\pi)^{6} 3} \int_{-1}^{1} dX \int_{\kappa/m}^{\Lambda/m} dr_{1} \int_{\kappa/m}^{\Lambda/m} dr_{2} \pi r_{1} r_{2} \times \]
\[ \left\{ - \left( \frac{1}{F_{1}} + \frac{1}{F_{2}} \right) \frac{1}{F_{12}} (1 + X^{2}) + \left( \frac{1}{F_{12}} \right)^{3} \frac{r_{1}^{2} + 2r_{1}r_{2}X + r_{2}^{2}}{2} (1 + X^{2}) \right. \]
\[ + \left. \left( \frac{1}{F_{1}} + \frac{1}{F_{2}} \right) \left( \frac{1}{F_{12}} \right)^{2} r_{1}r_{2}X(-1 + X^{2}) - \frac{1}{F_{1}} F_{2} (1 + X^{2}) \right. \]
\[ + \left. \left( \frac{1}{F_{2}} + \frac{1}{F_{2}} \right) \frac{1}{F_{12}} (1 - X^{2}) + \frac{1}{F_{1}} \frac{1}{F_{2}} \frac{1}{F_{12}} r_{2}X(-1 + X^{2}) \right\}. \]
Proof: Note that
\[
a_1(\Lambda, \kappa) = \frac{2}{3} (\sqrt{4\pi})^2 (A^+_\mu \Omega, H_0^{-1} A^+_\mu \Omega) \\
= \frac{8}{3\pi} \log \left( \frac{\Lambda/m + 2}{\kappa/m + 2} \right).
\]
Thus (2.15) follows. To see \(a_2(\Lambda, \kappa)\) we exactly compute the five terms on the right-hand side of (2.10) separately. Let
\[
\frac{1}{E_j} = \frac{1}{|k_j|^2/2 + \omega_j}, \quad j = 1, 2, \\
\frac{1}{E_{12}} = |k_1 + k_2|^2/2 + \omega_1 + \omega_2.
\]

(1) We have
\[
2 \left( \Psi_3^\mu, H_0^{-1} \Psi_3^\mu \right) = \left( \Omega, -(A^- \cdot A^-) H_0^{-1} A_\mu H_0^{-1} A^+_\mu \Omega \right) \\
+ \frac{1}{2} \left( \Omega, (A^- \cdot A^-) H_0^{-1} (P_1 \cdot A + A \cdot P_1) H_0^{-1} P_\mu H_0^{-1} A^+_\mu \Omega \right).
\]
\[
= - \int \int dk_1^3 dk_2^3 \frac{1}{E_1} \frac{1}{E_2} \frac{1}{E_{12}} \left( \frac{1}{E_1} + \frac{1}{E_2} \right) \text{tr}(Q_1 Q_2).
\] (2.17)

(2) We have
\[
\left( \Psi_2^\mu, H_0^{-1} \Psi_2^\mu \right)
= \left( \frac{1}{2} \right)^2 \left( P_{\mu} H_0^{-1} (A^+ \cdot A^+) \Omega, H_0^{-1} P_\mu H_0^{-1} (A^+ \cdot A^+) \Omega \right)
= \left( \frac{1}{2} \right)^2 \int \int dk_1^3 dk_2^3 \frac{1}{E_1} \frac{1}{E_2} \frac{1}{E_{12}} \left( \frac{1}{E_1} + \frac{1}{E_2} \right) |k_1 + k_2|^2 \text{tr}(Q_1 Q_2).
\] (2.18)

(3) We have
\[
-2 \left( \Psi_2^\mu, H_0^{-1} H_1 H_0^{-1} \Psi_2^\mu \right)
= \frac{1}{2} \left( P_{\mu} H_0^{-1} (A^+ \cdot A^+) \Omega, H_0^{-1} (P_1 \cdot A + A \cdot P_1) H_0^{-1} A^+_\mu \Omega \right)
= \int \int dk_1^3 dk_2^3 \frac{1}{E_1} \frac{1}{E_2} \frac{1}{E_{12}} \left( \frac{1}{E_1} + \frac{1}{E_2} \right) (Q_1 Q_2 k_1). \] (2.19)

(4) We have
\[
-\frac{1}{2} \left( \Psi_2^\mu, H_0^{-1} H_2 H_0^{-1} \Psi_2^\mu \right)
= -\frac{1}{2} \left( A^+_\mu \Omega, H_0^{-1} ((A^+ \cdot A^+) + 2(A^+ \cdot A^-) + (A^- \cdot A^-)) H_0^{-1} A^+_\mu \Omega \right)
= - \int \int dk_1^3 dk_2^3 \left( \frac{1}{E_1} \frac{1}{E_2} \frac{1}{E_{12}} \right) \text{tr}(Q_1 Q_2).
\] (2.20)
(5) We have
\[
\left( \Psi_{1}^{\mu}, H_{0}^{-1}H_{1}H_{0}^{-1}H_{1}H_{0}^{-1}\Psi_{1}^{\mu} \right)
= \left( \frac{1}{2} \right)^{2} A_{\mu}^{\pm}\Omega, H_{0}^{-1}(P_{t} \cdot A + A \cdot P_{t})H_{0}^{-1}(P_{t} \cdot A + A \cdot P_{t})H_{0}^{-1}A_{\mu}^{\pm}\Omega \right)
\]
\[
= \int \int dk_{1}^{2}dk_{2}^{2} \frac{1}{2 \omega_{1}} \frac{1}{2 \omega_{2}} \frac{1}{E_{12}} \left\{ \frac{1}{E_{1}}^{2} (k_{1}, Q_{2}k_{1}) + \frac{1}{E_{2}}^{2} (k_{2}, Q_{1}k_{2}) \right\}
+ \int \int \int \mathrm{d}k_{1}^{3}\mathrm{d}k_{2}^{3} \frac{1}{2 \omega_{1}} \frac{1}{2 \omega_{2}} \frac{1}{E_{12}} \frac{1}{E_{1}} \frac{1}{E_{2}} (k_{2}, Q_{1}Q_{2}k_{1}).
\]

Changing variables to the polar coordinate, we obtain (2.16) from Lemma 2.4, (2.17), (2.18), (2.19), (2.20), (2.21) and the facts
\[
\begin{align*}
\text{tr}[Q_{1}Q_{2}] &= 1 + (k_{1}, \hat{k}_{2})^{2}, \\
(k_{1}, Q_{2}Q_{1}k_{2}) &= (k_{1}, k_{2})((\hat{k}_{1}, \hat{k}_{2})^{2} - 1), \\
(k_{1}, Q_{2}k_{2}) &= |7\mathrm{c}_{1}|^{2}(1 - (\hat{k}_{1}, \hat{k}_{2})^{2}).
\end{align*}
\]
Thus the proof is complete. \(\square\)

3 Main theorem

The main theorem is as follows.

**Theorem 3.1** There exist strictly positive constants \(C_{\min}\) and \(C_{\max}\) such that
\[
C_{\min} \leq \lim_{\Lambda \to \infty} \frac{a_{2}(\Lambda/m, \kappa/m)}{\sqrt{\Lambda/m}} \leq C_{\max}.
\]

**Proof:** We show an outline of a proof. See Hiroshima and Spohn [7] for details. By (2.16) we can see that
\[
a_{2}(\Lambda, \kappa) = \frac{(4\pi)^{2}}{(2\pi)^{6}} 2 \sum_{j=1}^{6} b_{j}(\Lambda/m), \quad (3.1)
\]
where
\[
\begin{align*}
b_{1}(\Lambda/m) &= - \int (1 + X^{2}) \left( \frac{1}{F_{1}} + \frac{1}{F_{2}} \right) \frac{1}{F_{12}}, \\
b_{2}(\Lambda/m) &= \int (1 + X^{2}) \left( \frac{1}{F_{12}} \right)^{2} r_{1}^{2} + 2r_{1}r_{2}X + r_{2}^{2} \frac{2}{2}, \\
b_{3}(\Lambda/m) &= \int X(-1 + X^{2})r_{1}r_{2} \left( \frac{1}{F_{1}} + \frac{1}{F_{2}} \right) \left( \frac{1}{F_{12}} \right)^{2}, \\
b_{4}(\Lambda/m) &= - \int (1 + X^{2}) \frac{1}{F_{1}} \frac{1}{F_{2}}, \\
b_{5}(\Lambda/m) &= \int (1 - X^{2}) \left( \frac{r_{1}^{2}}{F_{1}^{2}} + \frac{r_{2}^{2}}{F_{2}^{2}} \right) \frac{1}{F_{12}}, \\
b_{6}(\Lambda/m) &= \int X(-1 + X^{2})r_{1}r_{2} \frac{1}{F_{1}} \frac{1}{F_{2}} \frac{1}{F_{12}}.
\end{align*}
\]
where
\[
\int = \int_{-1}^{1} \mathrm{d}X \int_{\kappa/m}^{\Lambda/m} \mathrm{d}r_1 \int_{\kappa/m}^{\Lambda/m} \mathrm{d}r_2 \pi r_1 r_2.
\]
Let \( \rho_{\Lambda}(\cdot, \cdot) : [0, \infty) \times [-1, 1] \rightarrow \mathbb{R} \) be defined by
\[
\rho_{\Lambda} = \rho_{\Lambda}(r, X) = r^2 + 2\Lambda r X + \Lambda^2 + 2r + 2\Lambda = (r + \Lambda X + 1)^2 + \Delta,
\]
where
\[
\Delta = \Lambda^2(1 - X^2) + 2\Lambda(1 - X) - 1.
\]
Then we can show that there exist constants \( C_1, C_2, C_3 \) and \( C_4 \) such that for sufficiently large \( \Lambda > 0 \),
\[
\begin{align*}
(1) \quad & \int_{-1}^{1} \mathrm{d}X \int_0^{\Lambda} \mathrm{d}r \frac{1}{\rho_{\Lambda}(r, X)} \leq C_1 \frac{1}{\Lambda}, \\
(2) \quad & \int_{-1}^{1} \mathrm{d}X \int_0^{\Lambda} \mathrm{d}r \left( \frac{1}{\rho_{\Lambda}(r, X)} \right)^2 \leq C_2 \frac{1}{\Lambda^{5/2}}, \\
(3) \quad & \int_{-1}^{1} \mathrm{d}X \int_0^{\Lambda} \mathrm{d}r \frac{1}{\rho_{\Lambda}(r, X)} \frac{1}{r+2} \leq C_3 \frac{\log \Lambda}{\Lambda^2}, \\
(4) \quad & \int_{-1}^{1} \mathrm{d}X \int_0^{\Lambda} \mathrm{d}r \left( \frac{1}{\rho_{\Lambda}(r, X)} \right)^2 (1 - X^2) \leq C_4 \frac{1}{\Lambda^{3}}.
\end{align*}
\]
Using (1)–(4) we can prove that there exists a constant \( C > 0 \) such that
\[
|b_j(\Lambda/m)| \leq C[\log(\Lambda/m)]^2, \quad j = 1, 4,
\]
\[
|b_j(\Lambda/m)| \leq C(\Lambda/m)^{1/2}, \quad j = 3, 5, 6.
\]
Hence there exists a constant \( C_{\text{max}} \) such that
\[
\lim_{\Lambda \rightarrow \infty} \frac{a_2(\Lambda/m, \kappa/m)}{\sqrt{\Lambda/m}} \leq C_{\text{max}}.
\]
Next we can show that there exists a positive constant \( \xi > 0 \) such that
\[
\lim_{\Lambda \rightarrow \infty} \sqrt{\Lambda/m} \frac{d}{d(\Lambda/m)} b_2(\Lambda/m) > \xi,
\]
which implies that there exists a constant \( \xi' \) such that
\[
\xi' \leq \lim_{\Lambda \rightarrow \infty} \frac{b_2(\Lambda/m)}{\sqrt{\Lambda/m}}.
\]
Thus we have
\[
C_{\text{min}} \leq \lim_{\Lambda \rightarrow \infty} \frac{a_2(\Lambda/m, \kappa/m)}{\sqrt{\Lambda/m}} \leq C_{\text{max}}.
\]
\(\square\)
Remark 3.2 Theorem 3.1 may suggests $\gamma \geq 1/2$ uniformly in $e$ but $e \neq 0$.

Remark 3.3 (1) $a_2(\Lambda/m, \kappa/m)/\sqrt{\Lambda/m}$ converges to a nonnegative constant as $\Lambda \to \infty$. (2) By (3.1), we can define $a_2(\Lambda/m, 0)$ since $b_j(\Lambda/m)$ with $\kappa = 0$ are finite. Moreover $a_2(\Lambda/m, 0)$ also satisfies Theorem 3.1. (3) In the case of $\kappa = 0$, Chen [1] established that $H(0)$ has a ground state $\psi_0(0)$ but does not for $H(p)$ with $p \neq 0$.

4 Concluding remarks

The Pauli-Fierz Hamitonian with the dipole approximation, $H_{\text{dip}}$, is defined by $H$ with $A_{\hat{\varphi}}$ replaced by $1 \otimes A_{\hat{\varphi}}(0)$, i.e.,

$$H_{\text{dip}} = \frac{1}{2m} (p \otimes 1 - e1 \otimes A_{\hat{\varphi}}(0))^2 + V \otimes 1 + 1 \otimes H_t.$$ 

Set $V \equiv 0$. Note that

$$[H_{\text{dip}}, P_{\text{total}}] \neq 0.$$ 

It is established in [5] that there exists a unitary operator $U : \mathcal{H} \to \mathcal{H}$ such that

$$U H_{\text{dip}} U^{-1} = -\frac{1}{2(m + \delta m)} \Delta \otimes 1 + 1 \otimes H_t + e^2 G,$$

where

$$\delta m = m + e^2 \frac{2}{3} ||\hat{\varphi}/\omega||^2,$$

$$G = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{t^2 ||\hat{\varphi}/(t^2 + \omega^2)||^2}{m + (2e^2/3)||\hat{\varphi}/\sqrt{t^2 + \omega^2}||^2} dt.$$

Hence

$$[U H_{\text{dip}} U^{-1}, P_{\text{total}}] = 0.$$ 

Then we can define the effective mass $m_{\text{eff}}$ for $U H_{\text{dip}} U^{-1}$, and which is

$$m_{\text{eff}}/m = 1 + \alpha \frac{4}{3\pi} (\Lambda/m - \kappa/m).$$

Hence $\gamma = 1$, then the mass renormalization for $H_{\text{dip}}$ is not available.

References


