<table>
<thead>
<tr>
<th>Title</th>
<th>Effective mass and mass renormalization of nonrelativistic QED (Applications of Renormalization Group Methods in Mathematical Sciences)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Hiroshima, Fumio</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1386: 20-34</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2004-07</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/24874">http://hdl.handle.net/2433/24874</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Effective mass and mass renormalization of nonrelativistic QED

Fumio Hiroshima*†

November 29, 2003

Abstract

The effective mass \( m_{\text{eff}} \) of the nonrelativistic QED is considered. \( m_{\text{eff}} \) is defined as the inverse of curvature of the ground state energy with total momentum zero. The effective mass \( m_{\text{eff}} = m_{\text{eff}}(e^2, \Lambda, \kappa, m) \) is a function of bare mass \( m > 0 \), ultraviolet cutoff \( \Lambda > 0 \), infrared cutoff \( \kappa > 0 \), and the square of charge \( e \) of an electron. Introduce a scaling \( m \to m(\Lambda) = (b\Lambda)^\beta, \beta < 0 \). Then asymptotics behavior of \( m_{\text{eff}} \) as \( \Lambda \to \infty \) is studied.

1 Introduction

1.1 The Pauli-Fierz Hamiltonian

This is a joint work with Herbert Spohn.¹ We consider a single, spinless free electron coupled to a quantized radiation field (photons). The Hilbert space of states of photons is the symmetric Fock space:

\[
\mathcal{F} = \bigoplus_{n=0}^\infty \bigotimes^n L^2(\mathbb{R}^3 \times \{1, 2\})
\]

where \( \bigotimes^n L^2(\mathbb{R}^3 \times \{1, 2\}) \) denotes the \( n \)-fold symmetric tensor product of \( L^2(\mathbb{R}^3 \times \{1, 2\}) \) with \( \bigotimes^0 L^2(\mathbb{R}^3 \times \{1, 2\}) = \mathbb{C} \). The inner product in \( \mathcal{F} \) is denoted by \((\cdot, \cdot)\) and the Fock vacuum by \( \Omega \). On \( \mathcal{F} \) we introduce the Bose field

\[
a(f) = \sum_{j=1,2} \int f(k,j)^* a(k,j) dk, \quad f \in L^2(\mathbb{R}^3 \times \{1, 2\}), \tag{1.1}
\]

*Department of Mathematics and Physics, Setsunan University, 572-8508, Osaka, Japan. email: hiroshima@mpg.setsunan.ac.jp
†This work is partially supported by Grant-in-Aid for Science Reserch C 1554019 from MEXT.
¹Zentrum Mathematik and Physik Department, TU München, D-80290, München, Germany. email: spohn@ma.tum.de
where \( a(f) \) and \( a^*(f) = a(\bar{f})^* \) are densely defined and satisfy the CCR

\[
[a(f), a^*(g)] = (f, g)_{L^2(\mathbb{R}^3 \times \{1, 2\})},
[a(f), a(g)] = 0,
[a^*(f), a^*(g)] = 0.
\]

The free Hamiltonian of \( \mathcal{F} \) is read as

\[
H_{f} = \sum_{j=1,2} \int \omega(k)a^*(k,j)a(k,j)dk,
\]
where the dispersion relation is given by

\[\omega(k) = |k|.\]

The free Hamiltonian \( H_{f} \) acts as

\[
H_{f}\Omega = 0,
H_{f}a_{f_{1}}\cdots a_{f_{n}}\Omega = \sum_{j=1}^{n}a^{*}(f_{1})\cdots a^{*}(\omega f_{j})\cdots a^{*}(f_{n})\Omega.
\]

The Pauli-Fierz Hamiltonian \( H \) is defined as a self-adjoint operator acting on

\[
\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathcal{F} \cong \int_{\mathbb{R}^3}^\oplus \mathcal{F} \, dx
\]
by

\[
H = \frac{1}{2m}(p_{x} \otimes 1 - eA_{\hat{\phi}})^2 + V \otimes 1 + 1 \otimes H_{f},
\]
where \( m \) and \( e \) denote the mass and charge of electron, respectively,

\[
p_{x} = \left( -i \frac{\partial}{\partial x_{1}}, -i \frac{\partial}{\partial x_{2}}, -i \frac{\partial}{\partial x_{3}} \right)
\]
and \( V \) an external potential. The quantized radiation field \( A_{\hat{\phi}} \) is defined by

\[
A_{\hat{\phi}} = \frac{1}{\sqrt{2}} \int_{\mathbb{R}^3}^\oplus (a(f_{x}) + a^{*}(\bar{f}_{x}))dx,
\]
where

\[
f_{x}(k, j) = \frac{1}{\sqrt{\omega}}\hat{\phi}(k)e(k, j)e^{ikx};
\]
\( e(k, 1), e(k, 2), k/|k| \) form a right-handed dreibain, and \( \hat{\phi} \) is a form factor. \( A_{\hat{\phi}} \) acts for \( \Psi \in \mathcal{H} \) as

\[
(A_{\hat{\phi}}\Psi)(x) = (a(f_{x}) + a^{*}(\bar{f}_{x}))\Psi(x), \quad x \in \mathbb{R}^3.
\]

**Theorem 1.1** Assume that \( \hat{\phi}/\omega, \hat{\phi}/\sqrt{\omega}, \sqrt{\omega}\hat{\phi} \in L^2(\mathbb{R}^3) \) and \( V \) is relatively bounded with respect to \(-\Delta\) with a relative bound \(< 1\). Then, for arbitrary values of \( e \), \( H \) is self-adjoint on \( D(\Delta \otimes 1) \cap D(1 \otimes H_{f}) \) and bounded from below.

**Proof:** See Hiroshima [3, 4].
1.2 Effective mass

The momentum of the photon field is given by

$$P_t = \sum_{j=1,2} \int k a^*(k,j) a(k,j) dk$$

(1.5)

and the total moment by

$$P_{\text{total}} = p_x \otimes 1 + 1 \otimes P_t.$$

Let us assume that

$$V \equiv 0.$$

Then we see that

$$[H, P_{\text{total} \mu}] = 0, \quad \mu = 1, 2, 3.$$

Hence $H$ and $\mathcal{H}$ can be decomposable with respect to $\text{Spec}(P_{\text{total}}) = \mathbb{R}^3$, i.e.,

$$\mathcal{H} = \int_{\mathbb{R}^3} \mathcal{H}(p)dp,$$

$$H = \int_{\mathbb{R}^3} H(p)dp.$$

Note that

$$e^{-iz \otimes P_t} P_{\text{total}} e^{iz \otimes P_t} = p_x,$$

$$e^{-iz \otimes P_t} H e^{iz \otimes P_t} = \frac{1}{2m} (p_x \otimes 1 - 1 \otimes P_t - e \otimes A_{\hat{\varphi}}(0)) + 1 \otimes H_t,$$

where

$$A_{\hat{\varphi}}(0) = \frac{1}{\sqrt{2}} (a(\bar{f}_0) + a(f_0)).$$

From this we obtain that for each $p \in \mathbb{R}^3$,

$$\mathcal{H}(p) \cong \mathcal{F},$$

$$H(p) \cong \frac{1}{2m} (p - P_t - eA_{\hat{\varphi}}(0)) + H_t,$$

Let

$$E_{m, \Lambda}(p) = \inf \text{Spec}(H(p)).$$

(1.6)

Let us assume sharp ultraviolet cutoff $\Lambda$ and infrared cutoff $\kappa$, which means

$$\hat{\varphi}(k) = \begin{cases} 0 & \text{for } |k| < \kappa, \\ (2\pi)^{-3/2} & \text{for } \kappa \leq |k| \leq \Lambda, \\ 0 & \text{for } |k| > \Lambda. \end{cases}$$

(1.7)

Lemma 1.2 There exists constants $p_*$ and $e_*$ such that for

$$(p, e) \in \mathcal{O} = \{(p, e) \in \mathbb{R}^3 \times \mathbb{R} ||p| < p_*, |e| < e^*\},$$

$H(p)$ has a ground state $\psi_g(p)$ and it is unique. Moreover $\psi_g(p) = \psi_g(p, e)$ is strongly analytic and $E_{m, \Lambda}(p) = E_{m, \Lambda}(p, e)$ analytic with respect to $(p, e) \in \mathcal{O}$. 


Proof: See Hiroshima and Spohn [6, 7].

In what follows we assume that \((p,e) \in \mathcal{O}\).

**Definition 1.3** The effective mass \(m_{\text{eff}} = m_{\text{eff}}(e^2, \Lambda, \kappa, m)\) is defined by

\[
\frac{1}{m_{\text{eff}}} = \frac{1}{3} \Delta p E(p, e)|_{p=0}.
\]

(1.8)

### 1.3 Mass renormalization

Removal of the ultraviolet cutoff \(\Lambda\) through mass renormalization means to find sequences\

\[
\Lambda \to \infty, \quad m \to 0
\]

(1.9)

such that \(E_{m,\Lambda}(p) - E_{m,\Lambda}(0)\) has a nondegenerate limit. To achieve this, as a first step we want to find constants

\[
\beta < 0, \quad 0 < b
\]

such that

\[
\lim_{\Lambda \to \infty} m_{\text{eff}}(e^2, \Lambda, \kappa \Lambda^\beta, (b\Lambda)^\beta) = m_{\text{ph}},
\]

(1.10)

where \(m_{\text{ph}}\) is a given constant. Actually \(m_{\text{ph}}\) is a physical mass. Namely in the mass renormalization the scaled bare mass goes to zero and the effective mass goes to a physical mass as the ultraviolet cutoff \(\Lambda\) goes to infinity.

We will see later that \(m_{\text{eff}}/m\) is a function of \(e^2, \Lambda/m\) and \(\kappa/m\). Let

\[
\frac{m_{\text{eff}}}{m} = f(e^2, \Lambda/m, \kappa/m),
\]

(1.11)

where \(f(0, \Lambda/m, \kappa/m) = 1\) holds. An analysis of (1.10) can be reduced to investigate the asymptotic behavior of \(f\) as \(\Lambda \to \infty\). Namely we want to find constants

\[
0 \leq \gamma < 1, \quad 0 < b
\]

such that

\[
\lim_{\Lambda \to \infty} \frac{f(e^2, \Lambda/m, \kappa/m)}{(\Lambda/m)^\gamma} = b_0.
\]

(1.12)

If we succeed to find constants \(\gamma\) and \(b_0\) such as in (1.12) then by

\[
m_{\text{eff}}(e^2, \Lambda, \kappa, m) = mf(e^2, \Lambda/m, \kappa/m),
\]

we have

\[
m_{\text{eff}}(e^2, \Lambda, \kappa \Lambda^\beta, (b\Lambda)^\beta) = (b\Lambda)^\beta f(e^2, \Lambda/(b\Lambda)^\beta, \kappa/b^\beta) \approx b_0 (b\Lambda)^\beta (\Lambda/(b\Lambda)^\beta)^\gamma.
\]

(1.13)

Taking

\[
\beta = \frac{-\gamma}{1 - \gamma} < 0, \quad b = 1/b_1^{1/\gamma},
\]
we see that by (1.13)

\[
\lim_{\Lambda \to \infty} m_{\text{eff}}(e^2, \Lambda, \kappa \Lambda^\beta, (b\Lambda)^\beta) = \lim_{\Lambda \to \infty} b_0 \left( \frac{\Lambda}{b_1^{1/\gamma}} \right)^\beta \left( \Lambda / (b_1)^{1/\gamma} \right)^\gamma = b_0 b_1,
\]

where \( b_1 \) is a parameter, which is adjusted such as

\[ b_0 b_1 = m_{\text{ph}}. \]

Hence we will be able to establish (1.10). It is easily seen that

\[ f(e^2, \Lambda/m, \kappa/m) = 1 + \alpha \frac{8}{3\pi} \log(\frac{\Lambda}{m} + 2) + O(\alpha^2), \]

where \( \alpha = e^2/4\pi \), which suggests

\[ f(e^2, \Lambda/m, \kappa/m) \approx (\Lambda/m)^{8\alpha/3\pi}, \]

for sufficiently small \( \alpha \) and large \( \Lambda \), and therefore

\[ \gamma = 8\alpha/3\pi. \]

One may assume that

\[ f(e^2, \Lambda/m, \kappa/m) \approx (\Lambda/m)^{\alpha(8/3\pi) + \alpha^2 b} \]

for sufficiently small \( \alpha \) with some constant \( b \). Then by expanding \( m_{\text{eff}}/m \) to order \( \alpha^2 \) one may expect that

\[
f(e^2, \Lambda/m, \kappa/m) \approx 1 + \alpha \frac{8}{3\pi} \log(\frac{\Lambda}{m}) + \frac{1}{2} \alpha^2 \left( \frac{8}{3\pi} \log(\frac{\Lambda}{m}) \right)^2 + b\alpha^2 \log(\frac{\Lambda}{m}) + O(\alpha^3)
\]

for sufficiently small \( \alpha \) and large \( \Lambda \). It is, however, that (1.14) is not confirmed. Instead of (1.14) we prove that there exists a constant \( C > 0 \) such that

\[
f(e^2, \Lambda/m, \kappa/m) = 1 + \alpha \frac{8}{3\pi} \log(\frac{\Lambda/m + 2}{\kappa/m + 2}) + \alpha^2 C \sqrt{\Lambda/m} + O(\alpha^3).
\]

The effective mass and its renormalization have been studied from a mathematical point of view by many authors. Spohn [10] investigates the effective mass of the Nelson model [9] from a functional integral point of view. Lieb and Loss [8] studied mass renormalization and binding energies of models of matter coupled to radiation fields including the Pauli-Fierz model. Hainzl and Seiringer [2] computed exactly the leading order in \( \alpha \) of the effective mass of the Pauli-Fierz Hamiltonian with spin.
2 Perturbative expansions

The effective masses for $H(p)$ and

$$\frac{1}{2m}:(p - P_f - eA\phi(0))^2 + H_f$$

are identical. Then in what follows we redefine $H(p)$ as

$$H(p) = \frac{1}{2m}:(p - P_f - eA\phi(0))^2 + H_f.$$  
Furthermore for notational convenience we write $A$ and $E(p)$ for $A\phi(0)$ and $E_{m, \Lambda}(p)$, respectively.

2.1 Formulae

Lemma 2.1 We have

$$\frac{m}{m_{\text{eff}}} = 1 - \frac{2}{3} \sum_{\mu=1,2,3} \frac{(\psi_{g}(0), (P_{f} + eA)_{\mu}(H(0) - E(0))^{-1}(P_{f} + eA)_{\mu}\psi(0))}{(\psi_{g}(0), \psi_{g}(0))}.$$  

Proof: It is seen that $E(p, e) = E(p, -e) = E(-p, e)$. Then

$$\frac{\partial}{\partial p_{\mu}}E(p, e)\Big|_{p_{\mu}=0} = 0, \quad \mu = 1, 2, 3, \quad (2.1)$$

follows. Moreover it is seen that $E(p, e)$ is a function of $e^2$ and

$$\frac{d^{2m-1}}{de^{2m-1}}E(p, e)\Big|_{e=0} = 0. \quad (2.2)$$

In this proof, $f'(p)_{\mu}$ means the strong derivative of $f(p)$ with respect to $p_{\mu}$. Since

$$H(p)\psi_{g}(p) = E(p)\psi_{g}(p),$$

we have

$$H'(p)_{\mu}\psi_{g}(p) + H(p)\psi'_{g}(p)_{\mu} = E'(p)_{\mu}\psi_{g}(p) + E(p)\psi'_{g}(p)_{\mu} \quad (2.3)$$

and

$$H''(p)_{\mu}\psi_{g}(p) + 2H'(p)_{\mu}\psi'_{g}(p)_{\mu} + H(p)\psi''_{g}(p)_{\mu} = E''(p)_{\mu}\psi_{g}(p) + 2E'(p)_{\mu}\psi'_{g}(p)_{\mu} + E(p)\psi''_{g}(p)_{\mu}. \quad (2.4)$$

By (2.1) it follows that $E'(0)_{\mu} = 0$, and by (2.3) with $p = 0$,

$$(P_{f} + eA)_{\mu}\psi_{g}(0) \in D((H(0) - E(0))^{-1}),$$
$$\psi'_{g}(0)_{\mu} = (H(0) - E(0))^{-1}(P_{f} + eA)_{\mu}\psi_{g}(0).$$
Then we have by (2.3) and (2.4),
\[
\frac{m}{m_{\text{eff}}} = \frac{1}{3} \sum_{\mu=1,2,3} \frac{(\psi_{g}(0), E''(0)_{\mu}\psi_{g}(0))}{(\psi_{g}(0), \psi_{g}(0))} = 1 - \frac{2}{3} \sum_{\mu=1,2,3} \frac{((P_{\mu} + eA)_{\mu}\psi_{g}(0), (H(0) - E(0))^{-1}(P_{\mu} + eA)_{\mu}\psi_{g}(0))}{(\psi_{g}(0), \psi_{g}(0))}
\]
Thus the lemma follows. \(\square\)

Let
\[
\psi_{g}(0) = \sum_{n=0}^{\infty} \frac{e^{n}}{n!} \varphi_{n}, \quad E(0) = \sum_{n=0}^{\infty} \frac{e^{2n}}{(2n)!} E_{2n}.
\]
Note that \(\varphi_{2m} \in \bigoplus_{m=0}^{\infty} \mathcal{F}^{(2m)}, \quad \varphi_{2m+1} \in \bigoplus_{m=0}^{\infty} \mathcal{F}^{(2m+1)}\).

We want to get the explicit form of \(\varphi_{n}\). Let
\[
\mathcal{F}_{\text{fin}} = \left\{ \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F} | \Psi^{(m)} = 0 \text{ for } m \geq \ell \right\},
\]
\[
\mathcal{F}_{0} = \left\{ \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_{\text{fin}} \left| \begin{array}{l}
(i) \Psi^{(0)} = 0,
(ii) \supp_{(k_{1},\ldots,k_{n})\in \mathbb{R}^{3n}} \Psi^{(n)}(k_{1},\ldots,k_{n},j_{1},\ldots,j_{n}) \neq \{(0,\ldots,0)\}
\end{array} \right. \right\}.
\]

**Lemma 2.2** We see that \(\mathcal{F}_{0} \subset D(H_{0}^{-1})\).

**Proof:** Let \(\Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_{0}\). Since
\[
(H_{0}\Psi)^{(n)}(k_{1},\ldots,k_{n},j_{1},\ldots,j_{n}) = \left[ \frac{1}{2} (k_{1} + \cdots + k_{n})^{2} + \sum_{j=1}^{n} \omega(k_{j}) \right] \Psi^{(n)}(k_{1},\ldots,k_{n},j_{1},\ldots,j_{n})
\]
we see that
\[
(H_{0}^{-1}\Psi)^{(n)}(k_{1},\ldots,k_{n},j_{1},\ldots,j_{n}) = \left[ \frac{1}{2} (k_{1} + \cdots + k_{n})^{2} + \sum_{j=1}^{n} \omega(k_{j}) \right]^{-1} \Psi^{(n)}(k_{1},\ldots,k_{n},j_{1},\ldots,j_{n}).
\]
Since \(\supp_{(k_{1},\ldots,k_{n})\in \mathbb{R}^{3n}} \Psi^{(n)}(k_{1},\ldots,k_{n},j_{1},\ldots,j_{n}) \neq \{(0,\ldots,0)\}\), we obtain that
\[
\|H_{0}^{-1}\Psi\|_{\mathcal{F}}^{2} = \sum_{n=1}^{\text{finite}} \|(H_{0}^{-1}\Psi)^{(n)}\|_{\mathcal{F}^{(n)}}^{2} < \infty.
\]

Then the lemma follows. \(\square\)

We split \(H(0)\) as
\[
H(0) = H_{0} + eH_{1} + \frac{e^{2}}{2} H_{2},
\]
where
\[
\begin{align*}
H_0 &= \frac{1}{2} P_t^2 + H_t, \\
H_1 &= \frac{1}{2} (P_t \cdot A + A \cdot P_t) = P_t \cdot A = A \cdot P_t, \\
H_2 &= A^2.
\end{align*}
\]

**Lemma 2.3** We have \( E_0 = E_1 = E_2 = E_3 = 0 \) and
\[
\varphi_0 = \Omega, \quad \varphi_1 = 0, \quad \varphi_2 = -H_0^{-1}H_2\Omega, \quad \varphi_3 = 3H_0^{-1}H_1H_0^{-1}H_2\Omega.
\]
In particular \( \varphi_2 \in \mathcal{F}^{(2)} \) and \( \varphi_3 \in \mathcal{F}^{(1)} \cap \mathcal{F}^{(3)} \).

**Proof:** Let us set \( H(0), E(0) \) and \( \psi_0(0) \) as \( H, E \) and \( \psi_0 \), respectively. It is obvious that \( E_0 = 0 \) and \( \varphi_0 = a\Omega \) with arbitrary \( a \in \mathbb{C} \), and by (2.2), \( E_1 = E_3 = 0 \). Set \( a = 1 \). We denote the strong derivative of \( f = f(e) \) with respect to \( e \) by \( f' \). We have
\[
H'\psi_g + H\psi_g' = E'\psi_g + E\psi_g'
\]
(2.5)
and
\[
H''\psi_g + 2H'\psi_g' + H\psi_g'' = E''\psi_g + 2E'\psi_g' + E\psi_g''.
\]
(2.6)
From (2.6) it follows that
\[
(\psi_g, H''\psi_g) + (\psi_g, 2H'\psi_g') + (\psi_g, H\psi_g'') = E''(\psi_g, \psi_g) + (\psi_g, 2E'\psi_g') + (\psi_g, E\psi_g'').
\]
(2.7)
Put \( e = 0 \) in (2.7). Then
\[
(\Omega, H_2\Omega) + (\Omega, 2H_1\Omega) + (\Omega, H_0\varphi_2) = E_2(\Omega, \Omega).
\]
(2.8)
Since the left-hand side of (2.8) vanishes, we have \( E_2 = 0 \). From (2.5) with \( e = 0 \) and the fact \( E_0 = E_1 = 0 \), it follows that
\[
H_1\Omega + H_0\varphi_1 = 0,
\]
from which it holds that \( H_0\varphi_1 = 0 \). Since \( H_0 \) has the unique eigenvector \( \Omega \) (the ground state) with eigenvalue zero, it follows that \( \varphi_1 = b\Omega \) with some constant \( b \). \( \varphi_1 \in \bigoplus_{m=0}^{\infty} \mathcal{F}^{(2m+1)} \) which implies \( b = 0 \). Hence \( \varphi_1 = 0 \) follows. By (2.6) with \( e = 0 \), we have
\[
H_2\Omega + 2H_1\varphi_1 + H_0\varphi_2 = 0.
\]
Since \( H_2\Omega \in \mathcal{F}_0 \), we see that by Lemma 2.2, \( H_2\Omega \in D(H_0^{-1}) \). Thus we have \( \varphi_2 = -H_0^{-1}H_2\Omega \). From the identity
\[
H''\psi_g + 3H'\psi_g' + 3H\psi_g'' + H\psi_g''' = E''\psi_g + 3E'\psi_g' + 3E\psi_g'' + E\psi_g'''
\]
(2.9)
it follows that at \( e = 0 \),
\[
3H_1\varphi_2 + H_0\varphi_3 = 0.
\]
Since \( H_1\varphi_2 = -H_1H_0^{-1}H_2\Omega \in \mathcal{F}_0 \), Lemma 2.2 ensures that \( H_1\varphi_2 \in D(H_0^{-1}) \). Hence \( \varphi_3 = -3H_0^{-1}H_1\varphi_2 = 3H_0^{-1}H_1H_0^{-1}H_2\Omega \). Then the lemma is proven.
\[\square\]
2.2 Order $e^4$

In this subsection we expand $m/m_{\text{eff}}$ up to order $e^4$. We define $A^{-}$ and $A^{+}$ by

$$A^{-} = \frac{1}{\sqrt{2}}a(f), \quad A^{+} = \frac{1}{\sqrt{2}}a^{*}(f).$$

Then $A = A^{+} + A^{-}$.

**Lemma 2.4** We have

$$\frac{m}{m_{\text{eff}}} = 1 - e^2 \frac{2}{3} \sum_{\mu=1}^{3} (\Omega, A_{\mu} H_{0}^{-1} A_{\mu} \Omega)$$

$$- e^4 \frac{2}{3} \sum_{\mu=1}^{3} \left\{ 2 \left( \Psi_{3}^\mu, H_{0}^{-1} \Psi_{1}^\mu \right) + \left( \Psi_{2}^\mu, H_{0}^{-1} \Psi_{2}^\mu \right) - 2 \left( \Psi_{2}^\mu, H_{0}^{-1} H_{1} H_{0}^{-1} \Psi_{1}^\mu \right) \right\} + O(e^6), \tag{2.10}$$

where

$$\Psi_{1}^\mu = A_{\mu} \Omega,$$

$$\Psi_{2}^\mu = -\frac{1}{2} P_{\mu} H_{0}^{-1} (A^{+} \cdot A^{+}) \Omega,$$

$$\Psi_{3}^\mu = \frac{1}{2} \left\{ -A_{\mu} H_{0}^{-1} (A^{+} \cdot A^{+}) \Omega + \frac{1}{2} P_{\mu} H_{0}^{-1} (P_{\mu} \cdot A + A \cdot P_{\mu}) H_{0}^{-1} (A^{+} \cdot A^{+}) \Omega \right\}.$$

**Proof:** In Lemma 2.1 we have seen that

$$\frac{m}{m_{\text{eff}}} = 1 - \frac{2}{3} \sum_{\mu=1,2,3} \frac{(P_{\mu} + e A)_{\mu} \psi_{g}(0), (H(0) - E(0))^{-1}(P_{\mu} + e A)_{\mu} \psi_{g}(0))}{(\psi_{g}(0), \psi_{g}(0))}.$$ \tag{2.11}

We can strongly expand $(H(0) - E(0))^{-1}$ as

$$(H(0) - E(0))^{-1} = H_{0}^{-1} - e H_{0}^{-1} H_{1} H_{0}^{-1}$$

$$+ e^2 \left( -\frac{1}{2} H_{0}^{-1} H_{2} H_{0}^{-1} + H_{0}^{-1} H_{1} H_{0}^{-1} H_{1} H_{0}^{-1} \right) + O(e^3). \tag{2.12}$$

Here we set

$$H_{j} = \begin{cases} H_{j}, & j = 1, 2, \\ -E_{j}, & j \geq 3. \end{cases}$$

Note that

$$\varphi_{0} \in \mathcal{F}^{(0)}, \varphi_{2} \in \mathcal{F}^{(2)}, \varphi_{3} \in \mathcal{F}^{(3)} \cap \mathcal{F}^{(1)}, \varphi_{4} \in \mathcal{F}^{(4)} \cap \mathcal{F}^{(2)}.$$
In particular
\[
\frac{1}{(\psi_{g}, \psi_{g})} = 1 - e^4 \left( \frac{1}{2} \varphi_{2}, \frac{1}{2} \varphi_{2} \right) - e^4 (\Omega, \frac{1}{24} \varphi_{4}) + O(e^6) = 1 - e^4 \frac{1}{4} (\varphi_{2}, \varphi_{2}) + O(e^6).
\] (2.13)

Moreover we have
\[
(P_f + eA)_{\mu} \psi_{g}(0) = eA_{\mu} \Omega + e^2 (\frac{1}{2} P_{f\mu} \varphi_{2}) + e^3 (\frac{1}{2} A_{\mu} \varphi_{2} + \frac{1}{6} P_{f\mu} \varphi_{3}) + O(e^4)
\]
\[
= e \Psi_{1}^\mu + e^2 \Psi_{2}^\mu + e^3 \Psi_{3}^\mu + O(e^4). \] (2.14)

Substitute (2.12), (2.13) and (2.14) into (2.11). Then the lemma follows. \(\square\)

For each \(k \in \mathbb{R}^3\) let us define the projection \(Q(k)\) on \(\mathbb{R}^3\) by
\[
Q(k) = \sum_{j=1,2} |e_j(k)\rangle\langle e_j(k)|.
\]

We set
\[
\hat{\varphi}_j = \hat{\varphi}(k), \quad \omega_j = \omega(k), \quad Q(k) = Q_j, \quad j = 1, 2.
\]

Let
\[
\frac{1}{F_j} = \frac{1}{r_j^2/2 + r_j^2}, \quad j = 1, 2,
\]
\[
\frac{1}{F_{12}} = \frac{1}{(r_1^2 + 2r_1r_2X + r_2^2)/2 + r_1 + r_2}, \quad r_1, r_2 \geq 0, \quad -1 \leq X \leq 1.
\]

Lemma 2.5. We have
\[
\frac{m}{m_{\text{eff}}} = 1 - a_{1}(\Lambda/m, \kappa/m) - a_{2}(\Lambda/m, \kappa/m) + O(a^3),
\]
where
\[
a_{1}(\Lambda/m, \kappa/m) = \frac{8}{3\pi} \log \left( \frac{\Lambda/m + 2}{\kappa/m + 2} \right)
\] (2.15)

and
\[
a_{2}(\Lambda/m, \kappa/m)
\]
\[
= \frac{(4\pi)^2}{(2\pi)^3} \int_{-1}^{1} dX \int_{\kappa/m}^{\Lambda/m} dr_1 \int_{\kappa/m}^{\Lambda/m} dr_2 r_1 r_2 \times
\]
\[
\times \left\{ - \left( \frac{1}{F_1} + \frac{1}{F_2} \right) \frac{1}{F_{12}} (1 + X^2) + \left( \frac{1}{F_{12}} \right)^3 \frac{r_1^2 + 2r_1r_2X + r_2^2}{2} (1 + X^2) + \left( \frac{1}{F_1} + \frac{1}{F_2} \right)^2 (1 + X^2) + \frac{1}{F_1 F_2} (1 + X^2) \right\}.
\] (2.16)
Proof: Note that

\[ a_1(\Lambda, \kappa) = \frac{2}{3} (\sqrt{4\pi})^2 (A_\mu^+ \Omega, H_0^{-1} A_\mu^+ \Omega) \]

\[ = \frac{8}{3\pi} \log \left( \frac{\Lambda/m + 2}{\kappa/m + 2} \right). \]

Thus (2.15) follows. To see \( a_2(\Lambda, \kappa) \) we exactly compute the five terms on the right-hand side of (2.10) separately. Let

\[ \frac{1}{E_j} = \frac{1}{|k_j|^2/2 + \omega_j}, \quad j = 1, 2, \]

\[ \frac{1}{E_{12}} = \frac{1}{|k_1 + k_2|^2/2 + \omega_1 + \omega_2}. \]

(1) We have

\[ 2 \left( \Psi_3^\mu, H_0^{-1} \Psi_4^\mu \right) = \left( \Omega, -(A^- \cdot A^-) H_0^{-1} A_\mu H_0^{-1} A_\mu^+ \Omega \right) \]

\[ + \frac{1}{2} \left( \Omega, (A^- \cdot A^-) H_0^{-1} (P_1 \cdot A + A \cdot P_1) H_0^{-1} (A^+ \cdot A^+ \Omega) \right). \]

\[ = - \iint dk_1^3 dk_2^3 \frac{1}{2\omega_1} \frac{1}{2\omega_2} \frac{1}{E_{12}} \left( \frac{1}{E_1} + \frac{1}{E_2} \right) \text{tr}(Q_1 Q_2). \]  

(2.17)

(2) We have

\[ \left( \Psi_2^\mu, H_0^{-1} \Psi_2^\mu \right) \]

\[ = \left( \frac{1}{2} \right)^2 \left( P_\mu H_0^{-1} (A^+ \cdot A^+) \Omega, H_0^{-1} P_\mu H_0^{-1} (A^+ \cdot A^+) \Omega \right) \]

\[ = \left( \frac{1}{2} \right)^2 \iint dk_1^3 dk_2^3 \frac{1}{2\omega_1} \frac{1}{2\omega_2} \left( \frac{1}{E_{12}} \right)^2 \frac{1}{E_1} \text{tr}(Q_1 Q_2). \]  

(2.18)

(3) We have

\[ -2 \left( \Psi_2^\mu, H_0^{-1} H_1 H_0^{-1} \Psi_1^\mu \right) \]

\[ = \frac{1}{2} \left( P_\mu H_0^{-1} (A^+ \cdot A^+) \Omega, H_0^{-1} (P_1 \cdot A + A \cdot P_1) H_0^{-1} A_\mu^+ \Omega \right) \]

\[ = \iint dk_1^3 dk_2^3 \frac{1}{2\omega_1} \frac{1}{2\omega_2} \left( \frac{1}{E_{12}} \right)^2 \frac{1}{E_1} \frac{1}{E_2} \text{tr}(Q_1 Q_2). \]  

(2.19)

(4) We have

\[ -\frac{1}{2} \left( \Psi_1^\mu, H_0^{-1} H_2 H_0^{-1} \Psi_1^\mu \right) \]

\[ = -\frac{1}{2} \left( A_\mu^+ \Omega, H_0^{-1} ((A^+ \cdot A^+) + 2(A^+ \cdot A^-) + (A^- \cdot A^-)) H_0^{-1} A_\mu^+ \Omega \right) \]

\[ = -\iint dk_1^3 dk_2^3 \frac{1}{2\omega_1} \frac{1}{2\omega_2} \frac{1}{E_1} \frac{1}{E_2} \text{tr}(Q_1 Q_2). \]  

(2.20)
(5) We have
\[
\left( \Psi_{1}^{\mu}, H_{0}^{-1}H_{1}H_{0}^{-1}H_{1}H_{0}^{-1}\Psi_{1}^{\mu} \right)
= \left( \frac{1}{2} \right)^{2} \left( A_{\mu}^{+} \Omega, H_{0}^{-1}(P_{t} \cdot A + A \cdot P_{t})H_{0}^{-1}(P_{t} \cdot A + A \cdot P_{t})H_{0}^{-1}A_{\mu}^{+} \Omega \right)
= \int \int d\mathbf{k}_{1}^{3} d\mathbf{k}_{2}^{3} \left| \hat{\varphi}_{1} \right|^{2} \left| \hat{\varphi}_{2} \right|^{2} \frac{1}{2 \omega_{1}} \frac{1}{2 \omega_{2}} \frac{1}{E_{12}} \left\{ \left( \frac{1}{E_{1}} \right)^{2} (k_{1}, Q_{2}k_{1}) + \left( \frac{1}{E_{2}} \right)^{2} (k_{2}, Q_{1}k_{2}) \right\}
+ \int \int d\mathbf{k}_{1}^{3} d\mathbf{k}_{2}^{3} \left| \hat{\varphi}_{1} \right|^{2} \left| \hat{\varphi}_{2} \right|^{2} \frac{1}{2 \omega_{1}} \frac{1}{2 \omega_{2}} \frac{1}{E_{12}} \frac{1}{E_{1}} \frac{1}{E_{2}} \left( k_{2}, Q_{1}Q_{2}k_{1} \right).
\]
(2.21)

Changing variables to the polar coordinate, we obtain (2.16) from Lemma 2.4, (2.17), (2.18), (2.19), (2.20), (2.21) and the facts
\[
\text{tr}[Q_{1}Q_{2}] = 1 + (\hat{k}_{1}, \hat{k}_{2})^{2},
(k_{1}, Q_{2}Q_{1}k_{2}) = (k_{1}, k_{2})((\hat{k}_{1}, \hat{k}_{2})^{2} - 1),
(k_{1}, Q_{2}k_{1}) = | \mathcal{C}_{1} |^{2} (1 - (\hat{k}_{1}, \hat{k}_{2})^{2}).
\]
Thus the proof is complete. \(\square\)

3 Main theorem

The main theorem is as follows.

**Theorem 3.1** There exist strictly positive constants \( C_{\text{min}} \) and \( C_{\text{max}} \) such that
\[
C_{\text{min}} \leq \lim_{\Lambda \to \infty} \frac{a_{2}(\Lambda/m, \kappa/m)}{\sqrt{\Lambda/m}} \leq C_{\text{max}}.
\]

**Proof:** We show an outline of a proof. See Hiroshima and Spohn [7] for details. By (2.16) we can see that
\[
a_{2}(\Lambda, \kappa) = \frac{(4\pi)^{2}}{(2\pi)^{6}} \sum_{j=1}^{6} b_{j}(\Lambda/m), \tag{3.1}
\]
where
\[
b_{1}(\Lambda/m) = - \int (1 + X^{2}) \left( \frac{1}{F_{1}} + \frac{1}{F_{2}} \right) \frac{1}{F_{12}},
b_{2}(\Lambda/m) = \int (1 + X^{2}) \left( \frac{1}{F_{12}} \right)^{3} \frac{r_{1}^{2} + 2r_{1}r_{2}X + r_{2}^{2}}{2},
b_{3}(\Lambda/m) = \int X(-1 + X^{2})r_{1}r_{2} \left( \frac{1}{F_{1}} + \frac{1}{F_{2}} \right) \left( \frac{1}{F_{12}} \right)^{2},
b_{4}(\Lambda/m) = - \int (1 + X^{2}) \frac{1}{F_{1}} \frac{1}{F_{2}},
b_{5}(\Lambda/m) = \int (1 - X^{2}) \left( \frac{r_{1}^{2}}{F_{1}^{2}} + \frac{r_{2}^{2}}{F_{2}^{2}} \right) \frac{1}{F_{12}},
b_{6}(\Lambda/m) = \int X(-1 + X^{2})r_{1}r_{2} \frac{1}{F_{1}} \frac{1}{F_{2}} \frac{1}{F_{12}},
\]
\[
\int = \int_{-1}^{1} \d X \int_{\kappa/m}^{\Lambda/m} \d r_1 \int_{\kappa/m}^{\Lambda/m} \d r_2 \pi r_1 r_2.
\]

Let \( \rho_{\Lambda}(\cdot, \cdot) : [0, \infty) \times [-1, 1] \to \mathbb{R} \) be defined by

\[
\rho_{\Lambda} = \rho_{\Lambda}(r, X) = r^2 + 2\Lambda r X + \Lambda^2 + 2r + 2\Lambda = (r + \Lambda X + 1)^2 + \Delta,
\]

where

\[
\Delta = \Lambda^2(1 - X^2) + 2\Lambda(1 - X) - 1.
\]

Then we can show that there exist constants \( C_1, C_2, C_3 \) and \( C_4 \) such that for sufficiently large \( \Lambda > 0 \),

\[
\begin{align*}
(1) & \quad \int_{-1}^{1} \d X \int_{0}^{\Lambda} \d r \frac{1}{\rho_{\Lambda}(r, X)} \leq C_1 \frac{1}{\Lambda}, \\
(2) & \quad \int_{-1}^{1} \d X \int_{0}^{\Lambda} \d r \left( \frac{1}{\rho_{\Lambda}(r, X)} \right)^2 \leq C_2 \frac{1}{\Lambda^{5/2}}, \\
(3) & \quad \int_{-1}^{1} \d X \int_{0}^{\Lambda} \d r \frac{1}{\rho_{\Lambda}(r, X)} \frac{1}{r+2} \leq C_3 \frac{\log \Lambda}{\Lambda^2}, \\
(4) & \quad \int_{-1}^{1} \d X \int_{0}^{\Lambda} \d r \left( \frac{1}{\rho_{\Lambda}(r, X)} \right)^2 (1 - X^2) \leq C_4 \frac{1}{\Lambda^{3/2}}.
\end{align*}
\]

Using (1)–(4) we can prove that there exists a constant \( C > 0 \) such that

\[
|b_j(\Lambda/m)| \leq C [\log(\Lambda/m)]^2, \quad j = 1, 4, \\
|b_2(\Lambda/m)| \leq C(\Lambda/m)^{1/2}, \\
|b_j(\Lambda/m)| \leq C \log(\Lambda/m), \quad j = 3, 5, 6.
\]

Hence there exists a constant \( C_{\text{max}} \) such that

\[
\lim_{\Lambda \to \infty} \frac{a_2(\Lambda/m, \kappa/m)}{\sqrt{\Lambda/m}} \leq C_{\text{max}}.
\]

Next we can show that there exists a positive constant \( \xi > 0 \) such that

\[
\lim_{\Lambda \to \infty} \sqrt{\Lambda/m} \frac{d}{d(\Lambda/m)} b_2(\Lambda/m) > \xi,
\]

which implies that there exists a constant \( \xi' \) such that

\[
\xi' \leq \lim_{\Lambda \to \infty} \frac{b_2(\Lambda/m)}{\sqrt{\Lambda/m}}.
\]

Thus we have

\[
C_{\text{min}} \leq \lim_{\Lambda \to \infty} \frac{a_2(\Lambda/m, \kappa/m)}{\sqrt{\Lambda/m}} \leq C_{\text{max}}.
\]
Remark 3.2 Theorem 3.1 may suggest \( \gamma \geq 1/2 \) uniformly in \( e \) but \( e \neq 0 \).

Remark 3.3 (1) \( a_2(\Lambda/m, \kappa/m)/\sqrt{\Lambda/m} \) converges to a nonnegative constant as \( \Lambda \to \infty \). (2) By (3.1), we can define \( a_2(\Lambda/m, 0) \) since \( b_j(\Lambda/m) \) with \( \kappa = 0 \) are finite. Moreover \( a_2(\Lambda/m, 0) \) also satisfies Theorem 3.1. (3) In the case of \( \kappa = 0 \), Chen [1] established that \( H(0) \) has a ground state \( \psi_0(0) \) but does not for \( H(p) \) with \( p \neq 0 \).

4 Concluding remarks

The Pauli-Fierz Hamiltonian with the dipole approximation, \( H_{\text{dip}} \), is defined by \( H \) with \( A_\phi \) replaced by \( 1 \otimes A_\phi(0) \), i.e.,

\[
H_{\text{dip}} = \frac{1}{2m}(p \otimes 1 - e1 \otimes A_\phi(0))^2 + V \otimes 1 + 1 \otimes H_t.
\]

Set \( V \equiv 0 \). Note that

\[
[H_{\text{dip}}, P_{\text{total}}] \neq 0.
\]

It is established in [5] that there exists a unitary operator \( U : \mathcal{H} \to \mathcal{H} \) such that

\[
UH_{\text{dip}}U^{-1} = -\frac{1}{2(m + \delta m)} \Delta \otimes 1 + 1 \otimes H_t + e^2 G,
\]

where

\[
\delta m = m + e^2 \frac{2}{3} ||\hat{\phi}/\omega||^2,
\]

\[
G = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{t^2 ||\hat{\phi}/(t^2 + \omega^2)||^2}{m + (2e^2/3)||\hat{\phi}/\sqrt{t^2 + \omega^2}||^2} dt.
\]

Hence

\[
[UH_{\text{dip}}U^{-1}, P_{\text{total}}] = 0.
\]

Then we can define the effective mass \( m_{\text{eff}} \) for \( UH_{\text{dip}}U^{-1} \), and which is

\[
m_{\text{eff}}/m = 1 + \alpha \frac{4}{3\pi} (\Lambda/m - \kappa/m).
\]

Hence \( \gamma = 1 \), then the mass renormalization for \( H_{\text{dip}} \) is not available.

References


