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Some new results on Brownian Directed Polymers in Random Environment

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Abstract

We prove some new results on Brownian directed polymers in random environment recently introduced by the authors. The directed polymer in this model is a d-dimensional Brownian motion (up to finite time t) viewed under a Gibbs measure which is built up with a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d$ (=time $\times$ space). Here, the Poisson random measure plays the role of the random environment which is independent both in time and in space. We prove that

(i) For $d \geq 3$ and the inverse temperature β smaller than a certain positive value β_0, the central limit theorem for the directed polymer holds almost surely with respect to the environment.

(ii) If $d = 1$ and $\beta \neq 0$, the variance of the free energy diverges with a magnitude not smaller than $t^{1/8}$ as t goes to infinity. The argument leading to this result strongly supports the inequalities $\chi(1) \geq 1/5$ for the fluctuation exponent for the free energy, and $\xi(1) \geq 3/5$ for the wandering exponent.

We provide necessary background by reviewing some results in the previous paper [CY03].

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1 Introduction

1.1 The Brownian directed polymers in random environment

The model we consider in this article is defined in terms of Brownian motion and of a Poisson random measure. Before introducing the polymer measure, we first fix some notations. In what follows, $\mathbb{R}_+ = [0,\infty)$, d denotes a positive integer and $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$ the class of Borel sets in $\mathbb{R}_+ \times \mathbb{R}^d$.

• *The Brownian motion:* Let $(\{\omega_t\}_{t\geq 0}, \{P^x\}_{x\in\mathbb{R}^d})$ denote a d-dimensional standard Brownian motion. Specifically, we let the measurable space $(\Omega,\mathcal{F})$ be the path space $C(\mathbb{R}_+ \to \mathbb{R}^d)$ with the cylindrical σ-field, and P^x be the Wiener measure on $(\Omega,\mathcal{F})$ such that $P^x\{\omega_0 = x\} = 1$.

• *The space-time Poisson random measure:* Let η denote the Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d$ with unit intensity, defined on a probability space $(\mathcal{M},\mathcal{G},Q)$. Then, η is an integer valued random measure characterized by the following property: If $A_1,...,A_n \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$ are disjoint and bounded, then

$$Q\left(\bigcap_{j=1}^{n}\{\eta(A_j)=k_j\}\right) = \prod_{j=1}^{n}\exp(-|A_j|)\frac{|A_j|^{k_j}}{k_j!} \quad \text{for } k_1,...,k_n \in \mathbb{N}. \tag{1.1}$$

Here, $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^{1+d}$. For $t > 0$, it is natural and convenient to introduce

$$\eta_t(A) = \eta(A \cap ((0,t]\times\mathbb{R}^d)), \quad A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d) \tag{1.2}$$

and the sub σ-field

$$\mathcal{G}_t = \sigma[\eta_t(A)\ ;\ A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)]\ . \tag{1.3}$$

• *The polymer measure:* We let V_t denote a "tube" around the graph $\{(s,\omega_s)\}_{0<s\leq t}$ of the Brownian path,

$$V_t = V_t(\omega) = \{(s,x)\ ;\ s\in(0,t],\ x\in U(\omega_s)\}, \tag{1.4}$$

where $U(x) \subset \mathbb{R}^d$ is the closed ball with the unit volume, centered at $x \in \mathbb{R}^d$. For any $t > 0$ and $x \in \mathbb{R}^d$, define a probability measure μ_t^x on the path space $(\Omega,\mathcal{F})$

$$\mu_t^x(d\omega) = \frac{\exp\left(\beta\eta(V_t)\right)}{Z_t^x} P^x(d\omega), \tag{1.5}$$

where $\beta \in \mathbb{R}$ is a parameter and

$$Z_t^x = P^x[\exp\left(\beta\eta(V_t)\right)]\ . \tag{1.6}$$

is the normalizing constant (the partition function). Note that μ_t^x and Z_t^x contain $\eta \in \mathcal{M}$ as a parameter and hence that they are random objects on the probability space $(\mathcal{M},\mathcal{G},Q)$. We will denote by $P, \mu_t, Z_t, \cdots$, the quantities $P^x, \mu_t^x, Z_t^x, \cdots$ with $x = 0$.

Under the measure μ_t^x, the graph $\{(s,\omega_s)\}_{0\leq s\leq t}$ may be interpreted as a polymer chain living in the $(1+d)$-dimensional space, constrained to stretch in the direction of the first coordinate (t-axis). At the heuristic level, the polymer measure is governed by the formal Hamiltonian

$$\beta \mathbf{H}_t^\eta(\omega) = \frac{1}{2}\int_0^t |\dot{\omega}_s|^2 ds - \beta\ \#\Big\{\text{points } (s,x) \text{ in } \eta : s \leq t, x \in U(\omega_s)\Big\} \tag{1.7}$$

on the path space. The path ω is attracted to Poisson points when $\beta > 0$, and repelled by them when $\beta < 0$. The sets $\{s\} \times U(x)$ with (s, x) a point of the Poisson field η, appear as "rewards" in the first case, and "soft obstacles" in the second one. Note that the obstacles stretches in the transverse direction (x-hyperplane): This is a key technical point, allowing a simple use of stochastic calculus with respect to the Poisson field.

Let us finish the definition of the model with some remarks on the notation we use. An important parameter is

$$\lambda = \lambda(\beta) = e^{\beta} - 1 \in (-1, \infty)\,, \tag{1.8}$$

which is in fact the logarithmic moment generating function of a mean-one Poisson distribution. When we want to stress the dependence of λ on $\beta \in \mathbb{R}$, we will use the notation $\lambda(\beta)$. But otherwise, we will simply write λ.

Remark 1.1.1 The Brownian directed polymer we discuss in this article has a discrete model as its ancestor. We call the discrete model the *simple random walk model of directed polymers.* The discrete model was originally introduced in physics literature [HuHe85] to mimic the phase boundary of Ising model subject to random impurities. Later on, the model reached the mathematics community [ImSp88, Bol89], where it was reformulated as follows. Let $\{\omega_n\}_{n\geq 0}$ be the simple random walk in the d-dimensional integer lattice $\mathbb{Z}^d$, defined on a probability space $(\Omega, \mathcal{F}, P)$. The random environment is introduced as a sequence $\eta = \{\eta(n, x) : n \in \mathbb{N},\ x \in \mathbb{Z}^d\}$ which are real valued, non-constant, and i.i.d.(independent identically distributed) r.v.'s defined on a probability space $(H, \mathcal{G}, Q)$ such that

$$Q[\exp(\beta\eta(n, x))] < \infty \quad \text{for all } \beta \in \mathbb{R}. \tag{1.9}$$

For any $n > 0$, we define the polymer measure μ_n on the path space $(\Omega, \mathcal{F})$ by

$$\mu_n(d\omega) = \frac{1}{Z_n} \exp\left(\beta \sum_{1\leq j\leq n} \eta(j, \omega_j)\right) P(d\omega), \tag{1.10}$$

where $\beta \in \mathbb{R}$ is a parameter (the inverse temperature) and

$$Z_n = P\left[\exp\left(\beta \sum_{1\leq j\leq n} \eta(j, \omega_j)\right)\right] \tag{1.11}$$

is the normalizing constant (the partition function).

Therefore, the Brownian directed polymer discussed in this article can be thought of as a natural transposition of simple random walk model into continuum setting. The simple random walk model has already been studied for more than a decade and by many authors. See for example [ImSp88, Bol89, SoZh96, Piz97, CaHu02, CSY03]. See also a review paper [CSY04].

1.2 The weak and strong disorder phases

The feature of the results we can expect to obtain for the directed polymer in random environment is different, depending on which of the following situation we consider:

$$d = 1, 2 \text{ and } \beta \neq 0, \tag{1.12}$$

$$d \geq 1 \text{ and } \beta \text{ is large enough}, \tag{1.13}$$

$$d \geq 3 \text{ and } \beta \in (-\infty, \beta_0(d)) \text{ with some } \beta_0(d) > 0. \tag{1.14}$$

In the former two cases (1.12) and (1.13), the system is in "strong disorder phase", in which the presence of the random environment is supposed to make qualitative difference in the large time behavior the Brownian polymer. On the other hand, in the last case (1.14), the system is in "the weak disorder phase" in which the presence of the random environment is irrelevant and the large time behavior of the Brownian polymer is essentially the same as the original Brownian motion.

As we explain below, the weak and strong disorder phases are defined in terms of a zero-one law for the limiting normalized partition function and are also characterized by the decay rate of the replica overlap.

• *The normalized partition function*: We now introduce an important martingale on $(\mathcal{M}, \mathcal{G}, Q)$ ((1.15) below). In fact, the large time behavior of this martingale somehow characterizes the phase diagram of this model.

For any fixed path ω, the process $\{\eta(V_t)\}_{t\geq 0}$ has independent, Poissonian increments, hence it is itself a standard Poisson process on the half-line, and $\{\exp(\beta\eta(V_t) - \lambda t)\}_{t\geq 0}$ is its exponential martingale. Therefore, the normalized partition function

$$W_t = e^{-\lambda t} Z_t, \quad t \geq 0 \tag{1.15}$$

is itself a mean-one, right-continuous and left-limited, positive martingale on $(\mathcal{M}, \mathcal{G}, Q)$, with respect to the filtration $(\mathcal{G}_t)_{t\geq 0}$ defined by (1.3). In particular, the following limit exists Q-a.s.:

$$W_\infty \stackrel{\text{def.}}{=} \lim_{t\nearrow\infty} W_t \,. \tag{1.16}$$

Since $\exp(\beta\eta(V_t)) > 0$ Q-a.s. for all $0 \leq t < \infty$ and all $\omega \in \Omega$, the event $\{W_\infty = 0\}$ is measurable with respect to the tail σ-field

$$\bigcap_{t\geq 1} \sigma[\eta|_{[t,\infty)\times\mathbb{R}^d}] \,,$$

and therefore by Kolmogorov's 0-1 law, we only have the two contrasting situations:

$$Q\{W_\infty = 0\} = 1 \,, \tag{1.17}$$

or

$$Q\{W_\infty > 0\} = 1 \,, \tag{1.18}$$

We define the former case (1.17) as the *strong disorder phase*, and the latter case (1.18) as the *weak disorder phase*. As we will see in Theorem 1.2.1 below, this definition is consistent with the introduction at the beginning of this subsection.

• *The replica overlap*: On the product space $(\Omega^2, \mathcal{F}^{\otimes 2})$, we consider the probability measure $\mu_t = \mu_t^{\otimes 2}(d\omega, d\widetilde{\omega})$, that we will view as the distribution of the couple $(\omega, \widetilde{\omega})$ with $\widetilde{\omega}$ an independent copy of ω with law μ_t. We introduce a random variable I_t, $t \geq 0$, given by

$$I_t = \mu_t^{\otimes 2}\left[|U(\omega_t) \cap U(\widetilde{\omega}_t)|\right]. \tag{1.19}$$

Here we have used the notation $|\cdot|$ for the Lebesgue measure on $\mathbb{R}^d$. Note that for some constant $c_1 = c_1(d) \in (0, 1)$,

$$c_1 \sup_{y\in\mathbb{R}^d} \mu_t\left[\omega_t \in U(y)\right]^2 \leq I_t \leq \sup_{y\in\mathbb{R}^d} \mu_t\left[\omega_t \in U(y)\right] \,. \tag{1.20}$$

The maximum appearing in the above bounds should be viewed as the probability of the favorite "location" for ω_t, under the polymer measure μ_t.

We collect some of the basic facts from [CY03] in the following Theorem 1.2.1. Roughly speaking, it says that

$$\begin{aligned}(1.12),(1.13) &\implies \text{strong disorder} \iff \text{slow decay of } I_t \text{ in } t,\\ (1.14) &\implies \text{weak disorder} \iff \text{fast decay of } I_t \text{ in } t.\end{aligned}$$

Theorem 1.2.1 (a) *Let $\beta \neq 0$. Then,*

$$\{W_\infty > 0\} = \left\{ \int_0^\infty I_s ds < \infty \right\}, \quad Q\text{-a.s.} \tag{1.21}$$

(b) *The system is in the strong disorder phase i.e., (1.17) holds in cases (1.12) and (1.13). Moreover, in case (1.13), the localization occurs : there exists a constant $c = c(d, \beta) > 0$ such that*

$$\overline{\lim_{t\nearrow\infty}} I_t \geq c, \quad Q\text{-a.s.} \tag{1.22}$$

(c) *For $d \geq 3$, there exist $\beta_0(d) > 0$ with $\lim_{d\nearrow\infty} \beta_0(d) = \infty$ such that the system is in the weak disorder phase, i.e., (1.18) holds for $\beta \in (-\infty, \beta_0(d))$.*

Remark 1.2.1 For the simple random walk model, results corresponding to Theorem 1.2.1(a), (b) are obtained in [CaHu02] (in the case $\eta(n,x)$ is the Gaussian r.v.) and in [CSY03] (for any $\eta(n,x)$ that satisfies (1.9)). It should be mentioned that a corresponding results to Theorem 1.2.1(b) for the simple random walk is shown also in the case (1.12):

$$\overline{\lim_{n\nearrow\infty}} I_n \geq c, \quad Q\text{-a.s.}$$

where

$$I_n = \mu_{n-1}^{\otimes 2}\left(\omega_n = \widetilde{\omega}_n\right). \tag{1.23}$$

The result corresponding to Theorem 1.2.1(c) for the simple random walk model is also known, e.g., [Bol89, SoZh96].

2 Results

2.1 The central limit theorem and the delocalization in the weak disorder phase

The following theorem sheds more light on the weak disorder phase of the Brownian directed polymer.

Theorem 2.1.1 *For $d \geq 3$, there exist $\beta_0(d) > 0$ with $\lim_{d\nearrow\infty} \beta_0(d) = \infty$ such that the following conclusions hold for $\beta \in (-\infty, \beta_0(d))$:*

(a) *The central limit theorem holds: for all $f \in C(\mathbb{R}^d)$ with at most polynomial growth at infinity,*

$$\lim_{t\nearrow\infty} \mu_t\left[f\left(\omega_t/\sqrt{t}\right)\right] = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) \exp(-|x|^2/2) dx, \quad Q\text{-a.s.} \tag{2.1}$$

In particular,

$$\lim_{t\nearrow\infty} \mu_t\left(\omega_t/\sqrt{t} \in \cdot\right) = (2\pi)^{-d/2} \exp(-|x|^2/2) dx, \ \textit{weakly}, \ Q\text{-a.s.} \tag{2.2}$$

(b) *Delocalization occurs:* $I_t = \mathcal{O}(t^{-d/2})$ *in* Q*-probability in the sense that*

$$Q\{t^{d/2} I_t \in \cdot\},\ t > 0 \text{ are tight.} \tag{2.3}$$

The proof is presented in section 3.1.

Remark 2.1.1 For the simple random walk model, results corresponding to Theorem 2.1.1 (a) are obtained by J. Imbrie, T. Spencer, E. Bolthausen, R. Song and X. Y. Zhou [ImSp88, Bol89, SoZh96]. The following weaker form of Theorem 2.1.1(b) for the simple random walk model can be found in [CSY03]: for $d \geq 3$, there exists $c = c(d, \beta) \geq 0$ such that $\lim_{\beta \to 0} c(d, \beta) = d/2$ and that $I_n = \mathcal{O}(n^{-c})$ in Q-probability, cf. (1.23). The present result (2.3) for the Brownian motion model is sharper, since we are able to prove the delocalization with the correct power $d/2$ for all $\beta \in (-\infty, \beta_0(d))$.

2.2 Power divergence of the energy fluctuation in $d = 1$

We now state the following estimate for the longitudinal fluctuation of the free energy.

Theorem 2.2.1 **(a)** *For all* $d \geq 1$ *and* $\beta \in \mathbb{R}$,

$$\mathrm{Var}_Q(\ln Z_t) \leq Ct, \quad t \geq 0, \tag{2.4}$$

where $C = \lambda(|\beta|)^2$.

(b) *If* $d = 1$ *and* $\beta \neq 0$*, then for any* $\varepsilon > 0$,

$$\mathrm{Var}_Q(\ln Z_t) \geq ct^{\frac{1}{4}-\varepsilon}, \quad t \geq 0. \tag{2.5}$$

where the positive constant c *depends only on* β *and* ε.

The first estimate (2.4) is proved in [CY03]. The second one (2.5) is new and the proof is given in section 3.2.

We now interpret some of our results from the view point of fluctuation exponents. We write $\xi(d)$ for the "wandering exponent",i.e., the exponent for the transversal fluctuation of the path, and $\chi(d)$ for the exponent for the longitudinal fluctuation of the free energy. Their definitions are roughly

$$|\omega_t| \approx t^{\xi(d)} \quad \text{and} \quad \ln Z_t - Q[\ln Z_t] \approx t^{\chi(d)} \quad \text{as } t \nearrow \infty. \tag{2.6}$$

There are various ways to define rigorously these exponents, e.g. (0.6) and (0.10-11) in [Wut98a], (2.4) and (2.6-7-8) in [Piz97], and the equivalence between these specific definitions are often non trivial. Here, we do not go into such subtleties and take (2.6) as "definitions". The polymer is said to be *diffusive* if $\xi(d) = 1/2$ and *super-diffusive* if $\xi(d) > 1/2$.

These exponents are investigated in the context of various other models and in a large number of papers. In particular, it is conjectured in physics literature that the *scaling identity* holds in any dimension,

$$\chi(d) = 2\xi(d) - 1, \quad d \geq 1, \tag{2.7}$$

and that the polymer is super-diffusive in dimension one;

$$\chi(1) = 1/3, \quad \xi(1) = 2/3. \tag{2.8}$$

See, e.g., [HuHe85],[FiHu91, (3.4),(5.11),(5.12)], [KrSp91, (5.19),(5.28)].

On the other hand, other rigorous results prove (or suggest) for example that

$$\begin{aligned} \chi(d) &\leq 1/2 \text{ for all } d \geq 1, & (2.9)\\ \chi(d) &\geq 2\xi(d) - 1 \text{ for all } d \geq 1, & (2.10)\\ \xi(d) &\leq 3/4 \text{ for all } d \geq 1, & (2.11)\\ \xi(1) &> 1/2 \text{ if } \beta \neq 0, & (2.12)\\ \chi(1) &> 0 \text{ if } \beta \neq 0, & (2.13) \end{aligned}$$

cf. Remark 2.2.1 below. For the Brownian directed polymer model, the central limit theorem (2.1) implies that $\xi(d) = 1/2$ in the weak disorder phase, or more precisely, in a region of the weak disorder phase for which the assumption of Theorem 2.1.1 is valid. On the other hand, Theorem 2.2.1 implies (2.9) and (2.13) with a lower bound $\chi(1) \geq 1/8$ for $\beta \neq 0$. If we insert $\chi(1) \geq 1/8$ in (2.7), we get the super-diffusivity (2.12) with a lower bound $\xi(1) \geq 9/16$ for $\beta \neq 0$. In Remark 3.2.2 below, we give explanations for (2.10), (2.11), $\chi(1) \geq 1/5$ ($\beta \neq 0$) and $\xi(1) \geq 3/5$ ($\beta \neq 0$) in the context of the Brownian directed polymer model.

Remark 2.2.1 M. Piza [Piz97] discusses (2.9) –(2.13) for the simple random walk model. In particular, the following estimate is obtained there: for $d = 1$ and $\beta \neq 0$,

$$\mathrm{Var}_Q(\ln Z_n) \geq c \ln n, \quad n = 1, 2, \ldots. \tag{2.14}$$

Thus, our estimate (2.5) for the Brownian case improves (2.14). For the Gaussian random walk model, M. Petermann [Pet00] proves that $\xi(1) \geq 3/5$, a stronger statement than (2.12), while O. Mejane [Mej02] shows (2.11). Fluctuation exponents similar to the above are also discussed in a number of related models. For the crossing Brownian motion in a soft Poissonian potential, M. Wüthrich proves in [Wut98a] upper and lower bounds supporting the scaling identity (2.7), he shows (2.12) in [Wut98b] with a lower bound $\xi(1) \geq 3/5$, (2.11) in [Wut98c], and $\chi(1) \geq 1/5$ in [Wut01]. For first passage percolation, similar results are obtained by C. Licea, M. Piza and C. Newman [NePi95, LiNePi96]. K. Johansson, in some particular models of oriented first passage percolation [Joh00a, Joh00b], proves not only (2.8), but also the scaling limits, and also in the model of maximal increasing subsequences in a paper with J. Baik and P. Deift [BDJ99].

3 Proofs

3.1 Proof of Theorem 2.1.1

In this subsection, we prove Theorem 2.1.1. The proof is based on the L^2 analysis of certain martingales on $(\mathcal{M}, \mathcal{G}, Q)$. This approach was introduced by E. Bolthausen [Bol89] and then investigated further by R. Song and X. Y. Zhou [SoZh96]. The following lemma [CY03, Proposition 4.2.1] is an important technical step in proving Theorem 2.1.1:

Lemma 3.1.1 *For $d \geq 3$, there exists $\beta_0(d) > 0$ with $\lim_{d \nearrow \infty} \beta_0(d) = \infty$ such that for $\beta \in (-\infty, \beta_0(d))$,*

$$\sup_{t \geq 0} Q[W_t^2] \leq P\left[\exp\left(2\lambda^2 \int_0^\infty \chi_{s,0} ds\right)\right] < \infty. \tag{3.1}$$

We define

$$\overline{\zeta}_t = e^{-\lambda t}\zeta_t \tag{3.2}$$

We consider a process $(M_t)_{t \geq 0}$ on $(\mathcal{M}, \mathcal{G}, Q)$ of the form;

$$M_t = P[\varphi(t, \omega_t)\overline{\zeta}_t]. \tag{3.3}$$

Here, $\overline{\zeta}_t$ has been introduced by (3.2) and $\varphi \in C(\mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R})$ is a function for which we assume the following properties:

(P1) There are constants $C_i, p \in [0, \infty)$, $i = 0, 1, 2$ such that

$$|\varphi(t, x)| \leq C_0 + C_1|x|^p + C_2 t^{p/2} \quad \text{for all } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d. \tag{3.4}$$

(P2) The process:

$$\Phi_t(\omega) \stackrel{\text{def.}}{=} \varphi(t, \omega_t), \quad t \geq 0 \tag{3.5}$$

is a martingale on $(\Omega, \mathcal{F}, P)$ with respect to the filtration $\mathcal{F}_t = \sigma[\omega_s \ ; \ s \leq t]$.

It is easy to see from (P2) that $(M_t)_{t \geq 0}$ is a $(\mathcal{G}_t)$-martingale on $(\mathcal{M}, \mathcal{G}, Q)$.

Proposition 3.1.2 *Suppose that $d \geq 3$, and that (3.1), (P1), (P2) are satisfied.*

(a) *For the process $(M_t)_{t \geq 0}$ defined by (3.3), there exists $\kappa \in [0, p)$ such that*

$$\max_{0 \leq s \leq t} |M_s| = \mathcal{O}(t^{\kappa/2}), \quad \text{as } t \nearrow \infty, \ Q\text{-a.s.} \tag{3.6}$$

If in addition, $1 + p < \frac{1}{2}d$, then

$$\lim_{t \nearrow \infty} M_t \text{ exists } Q\text{-a.s. and in } L^2(Q). \tag{3.7}$$

(b) *For the process $(\Phi_t)_{t \geq 0}$ defined by (3.5), there exists C such that*

$$P^{\otimes 2}\left[\Phi_t(\omega)\Phi_t(\widetilde{\omega})|U(\omega_t) \cap U(\widetilde{\omega}_t)| \exp\left(\lambda^2 |V_t(\omega) \cap V_t(\widetilde{\omega})|\right)\right] \leq C(1+t)^{p-d/2} \quad \text{for all } t > 0. \tag{3.8}$$

Let us first complete the proof of Theorem 2.1.1 by assuming Proposition 3.1.2.

Proof of Theorem 2.1.1 (a): We let $a = (a_j)_{j=1}^d$ and $b = (b_j)_{j=1}^d$ denote multi indices in what follows. We will use standard notation $|a|_1 = a_1 + ... + a_d$, $x^a = x_1^{a_1} \cdots x_d^{a_d}$ and $(\frac{\partial}{\partial x})^a = \left(\frac{\partial}{\partial x_1}\right)^{a_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{a_d}$ for $x \in \mathbb{R}^d$. It is enough to prove (2.1) for any monomial of the form $f(x) = x^a$. We will do this by induction on $|a|_1$. The statement is clear for $|a|_1 = 0$. We introduce the Hermite polynomials $\{\varphi_a\}_{a \in \mathbb{N}^d}$ by

$$\varphi_a(t, x) = \left(\frac{\partial}{\partial \theta}\right)^a \exp(\theta \cdot x - t|\theta|^2/2)\Big|_{\theta=0}.$$

Clearly, the function φ satisfies (P1) and (P2) with $p = |a|_1$. On the other hand, we see from the definition of φ_a that

$$(2\pi)^{-d/2} \int_{\mathbb{R}^d} \varphi_a(1, x) e^{-|x|^2/2} dx = 0. \tag{3.9}$$

Moreover, it is well-known that $\varphi_a(t, x) = x^a + \psi_a(t, x)$, where

$$\psi_a(t, x) = \sum_{\substack{|b|_1 + 2j = |a|_1 \\ j \geq 1}} A_a(b, j) x^b t^j,$$

for some $A_a(b, j) \in \mathbb{R}$. We now write $\mu_t[(\omega_t/\sqrt{t})^a]$ as

$$\mu_t[(\omega_t/\sqrt{t})^a] = \frac{1}{W_t} P[\varphi_a(t, \omega_t)\overline{\zeta}_t] t^{-|a|_1/2} - \frac{1}{W_t} P[\psi_a(1, \omega_t/\sqrt{t})\overline{\zeta}_t].$$

As $t \nearrow \infty$, the second term converges to $(2\pi)^{-d/2} \int_{\mathbb{R}^d} x^a e^{-|x|^2/2} dx$ by the induction hypothesis and (3.9). The first term vanishes by Proposition 3.1.2 (a).

The second statement (2.2) is obtained from (2.1) just by noting that the set of bounded, uniformly continuous functions on $\mathbb{R}^d$ is separable with respect to the sup-norm.

Proof of Theorem 2.1.1 (b): We write

$$Q\{t^{d/2} I_t \geq \gamma\} \leq Q\{W_t \leq \gamma^{-1/4}\} + Q\{t^{d/2} I_t \geq \gamma,\ W_t \geq \gamma^{-1/4}\}$$

Since W_t^{-1} converges Q-a.s., its distribution is tight:

$$\lim_{\gamma \nearrow \infty} \sup_{t > 0} Q(W_t \leq \gamma^{-1/4}) = 0. \tag{3.10}$$

On the other hand,

$$\begin{aligned}
& Q\{t^{d/2} I_t \geq \gamma,\ W_t \geq \gamma^{-1/4}\} \\
& \leq Q\{t^{d/2} W_t^2 I_t \geq \gamma^{1/2}\} \\
& \leq \gamma^{-1/2} t^{d/2} Q[W_t^2 I_t] \\
& = \gamma^{-1/2} t^{d/2} P^{\otimes 2} \left[|U(\omega_t) \cap U(\widetilde{\omega}_t)| \exp\left(\lambda^2 |V_t(\omega) \cap V_t(\widetilde{\omega})|\right) \right] \\
& \leq C \gamma^{-1/2},
\end{aligned} \tag{3.11}$$

where we have used Proposition 3.1.2 (b) on the last line. We now conclude the desired tightness from (3.10) and (3.11). □

We now turn to the proof of Proposition 3.1.2. We owe the following general observation to M. Takeda [Tak03].

Lemma 3.1.3 *For $d \geq 3$, define*

$$\Phi(x) = P^x \exp\left(\int_0^\infty v(\omega_s) ds \right)$$

where $v : \mathbb{R}^d \longrightarrow \mathbb{R}$ is a bounded compactly supported measurable function. Suppose that

$$0 < \inf_{x \in \mathbb{R}^d} \Phi(x) \leq \sup_{x \in \mathbb{R}^d} \Phi(x) < \infty. \tag{3.12}$$

Then, there exists a constant $C \in (0,\infty)$ such that

$$\sup_{x\in\mathbb{R}^d} P^x\left[\exp\left(\int_0^t v(\omega_s)ds\right)|f(\omega_t)|\right] \le Ct^{-d/2}\int_{\mathbb{R}^d}|f(x)|dx, \tag{3.13}$$

for all $f \in L^1(\mathbb{R}^d)$ and $t > 0$.

Proof: We will abbreviate $\int_{\mathbb{R}^d} f(x)dx$ by $\int_{\mathbb{R}^d} f$. Let us recall the Sobolev inequality:

$$\int_{\mathbb{R}^d}|f|^{\frac{2d}{d-2}} \le c_1\left(\int_{\mathbb{R}^d}|\nabla f|^2\right)^{\frac{d}{d-2}} \quad \text{for all } f \in H^1, \tag{3.14}$$

where $c_1 = c_1(d) \in (0,\infty)$ and $H^1 = \{f \in L^2(\mathbb{R}^d)\ ;\ |\nabla f| \in L^2(\mathbb{R}^d)\}$. For a measurable function f on $\mathbb{R}^d$, we introduce

$$(P_t^v f)(x) = P^x\left[\exp\left(\int_0^t v(\omega_s)ds\right)f(\omega_t)\right],\quad x\in\mathbb{R}^d,$$

whenever the expectation on the right-hand-side makes sense. Then, $(P_t^v)_{t\ge 0}$ is a symmetric, strongly continuous semi-group on $L^2(\mathbb{R}^d)$. On the other hand, we define a symmetric, strongly continuous semi-group on $L^2(\mathbb{R}^d, \Phi^2 dx)$ by

$$P_t^\Phi f = \frac{1}{\Phi}P_t^v[f\Phi].$$

Then the associated quadratic form and its domain is given respectively by

$$\mathcal{E}^\Phi(f,f) = \tfrac{1}{2}\int_{\mathbb{R}^d}|\nabla f|^2\Phi^2 \quad\text{and}\quad \mathrm{Dom}(\mathcal{E}^\Phi) = H^1. \tag{3.15}$$

Now, assuming (3.15) whose proof is standard and will be reproduced later, we see from (3.12) and (3.14) that

$$\int_{\mathbb{R}^d}|f|^{\frac{2d}{d-2}}\Phi^2 \le c_2\mathcal{E}^\Phi(f,f)^{\frac{d}{d-2}} \quad\text{for all } f\in H^1.$$

It is well-known that this implies that there is a constant C such that

$$\|P_t^\Phi\|_{\Phi,2\to\infty} \le Ct^{-d/4} \quad\text{for all } t>0,$$

e.g.,[Dav89, page 75, Theorem 2.4.2], where $\|\cdot\|_{\Phi,p\to q}$ denotes the operator norm from $L^p(\mathbb{R}^d,\Phi^2dx)$ to $L^q(\mathbb{R}^d,\Phi^2dx)$. Note that $\|P_t^\Phi\|_{\Phi,1\to 2} = \|P_t^\Phi\|_{\Phi,2\to\infty}$ by duality. We therefore have via semi-group property that

$$\|P_t^\Phi\|_{\Phi,1\to\infty} \le \|P_{t/2}^\Phi\|_{\Phi,2\to\infty}^2 \le C^2t^{-d/2} \quad\text{for all } t>0. \tag{3.16}$$

Since $P_t^v f = \Phi P_t^\Phi[f/\Phi]$, the desired bound (3.13) follows from (3.12) and (3.16).

We now turn to the proof of (3.15). We first check that $\Phi\in C^1(\mathbb{R}^d)$ and that

$$\int_{\mathbb{R}^d}\left(\tfrac{1}{2}f^2|\nabla\Phi|^2 + f\Phi\nabla\Phi\cdot\nabla f - v\Phi^2f^2\right) = 0,\quad\text{for all } f\in C_c^\infty(\mathbb{R}^d). \tag{3.17}$$

By differentiating $\exp\left(\int_0^t v(\omega_s)ds\right)$ with respect to t and then integrating, we have

$$\Phi(x) = 1 + \int_{\mathbb{R}^d} G(x-y)v(y)\Phi(y)dy,$$

where $G(x) = \frac{\Gamma(d/2)}{(d-2)\pi^{d/2}|x|^{d-2}}$, the Green function. We see from this expression that $\Phi \in C^1(\mathbb{R}^d)$ [PoSt78, page 115, Theorem 6.3] and that

$$\int_{\mathbb{R}^d} \left(\tfrac{1}{2}\nabla f \cdot \nabla\Phi - vf\Phi\right) = 0, \quad \text{for all } f \in C_c^\infty(\mathbb{R}^d). \tag{3.18}$$

It is clear that (3.18) remains true for all $f \in C_c^1(\mathbb{R}^d)$. Thus, plugging $f^2\Phi$ ($f \in C_c^\infty(\mathbb{R}^d)$) into (3.18) in place of f, we obtain (3.17).

We are now ready to conclude (3.15). The quadratic form associated to $(P_t^v)_{t\geq 0}$ and its domain is given respectively by

$$\mathcal{E}^v(f,f) = \int_{\mathbb{R}^d} \left(\tfrac{1}{2}|\nabla f|^2 - vf^2\right) \quad \text{and} \quad \mathrm{Dom}(\mathcal{E}^v) = H^1,$$

e.g.,[Szn98, pages 16 and 26]. Therefore, for $f \in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned}
\mathcal{E}^\Phi(f,f) &= \lim_{t\searrow 0}\frac{1}{t}\int_{\mathbb{R}^d} f\Phi^2\left(f - P_t^\Phi[f]\right) \\
&= \lim_{t\searrow 0}\frac{1}{t}\int_{\mathbb{R}^d} f\Phi\left(f\Phi - P_t^v[f\Phi]\right) \\
&= \mathcal{E}^v(f\Phi, f\Phi) \\
&= \int_{\mathbb{R}^d} \left(\tfrac{1}{2}|\nabla(f\Phi)|^2 - vf^2\Phi^2\right) \\
&= \tfrac{1}{2}\int_{\mathbb{R}^d} |\nabla f|^2\Phi^2,
\end{aligned}$$

where we have used (3.17) on the last line. Since $C_c^\infty(\mathbb{R}^d)$ is dense in H^1, we have proved (3.15). □

Lemma 3.1.4 *Suppose that $d \geq 3$ and that (3.1) holds. Then, there exists a constant $C \in (0,\infty)$ such that*

$$\sup_{x\in\mathbb{R}^d} P^x\left[\exp\left(2\lambda^2\int_0^t \chi_{0,s}ds\right)|f(\omega_t)|\right] \leq Ct^{-d/2}\int_{\mathbb{R}^d}|f(x)|dx, \tag{3.19}$$

for all $f \in L^1(\mathbb{R}^d)$ and $t > 0$.

Proof: We have

$$\sup_{x\in\mathbb{R}^d} P^x\left[\exp\left(2\lambda^2\int_0^\infty \chi_{s,0}ds\right)\right] = P\left[\exp\left(2\lambda^2\int_0^\infty \chi_{s,0}ds\right)\right].$$

This can be seen either from explicit formula for the expectation [BoSa02, page 376] or from a general comparison theorem [IkWa89, pages 437–438] applied to the d-dimensional Bessel process. Thus, we can apply Lemma 3.1.3 to $v = 2\lambda^2 1_{U(0)}$. □

Lemma 3.1.5 *Suppose that $d \geq 3$ and that (3.1), (P1), (P2) are satisfied. Then,*

$$Q[M_t^2] = \mathcal{O}(b_t), \quad \text{as } t \nearrow \infty, \ Q\text{-a.s.} \tag{3.20}$$

where $b_t = 1$ if $p < \frac{d}{2} - 1$, $b_t = \ln t$ if $p = \frac{d}{2} - 1$, and $b_t = t^{p-\frac{d}{2}+1}$ if $p > \frac{d}{2} - 1$.

Proof: We write M_t^2 in terms of the independent copy:

$$\begin{aligned} M_t^2 &= P[\Phi_t \overline{\zeta}_t]^2 \\ &= P^{\otimes 2}[\Phi_t(\omega)\Phi_t(\widetilde{\omega})\overline{\zeta}_t(\omega, \eta)\overline{\zeta}_t(\widetilde{\omega}, \eta)]. \end{aligned} \tag{3.21}$$

It follows from (3.21) and [CY03, proof of Proposition 4.2.1] that

$$\begin{aligned} &Q[M_t^2] \\ &= P^{\otimes 2}\left[\Phi_t(\omega)\Phi_t(\widetilde{\omega})Q[\overline{\zeta}_t(\omega, \eta)\overline{\zeta}_t(\widetilde{\omega}, \eta)]\right] \\ &= P^{\otimes 2}\left[\Phi_t(\omega)\Phi_t(\widetilde{\omega}) \exp\left(\lambda^2 |V_t(\omega) \cap V_t(\widetilde{\omega})|\right)\right] \\ &= P^{\otimes 2}\left[\Phi_t(\omega)\Phi_t(\widetilde{\omega})\right] \\ &\quad + \lambda^2 \int_0^t P^{\otimes 2}\left[\Phi_t(\omega)\Phi_t(\widetilde{\omega})|U(\omega_s) \cap U(\widetilde{\omega}_s)| \exp\left(\lambda^2 |V_s(\omega) \cap V_s(\widetilde{\omega})|\right)\right] ds \\ &= \Phi_0(\omega)^2 \\ &\quad + \lambda^2 \int_0^t P^{\otimes 2}\left[\Phi_s(\omega)\Phi_s(\widetilde{\omega})|U(\omega_s) \cap U(\widetilde{\omega}_s)| \exp\left(\lambda^2 |V_s(\omega) \cap V_s(\widetilde{\omega})|\right)\right] ds, \end{aligned} \tag{3.22}$$

where we have used the martingale property on the last line. We now introduce independent Brownian motions $\widehat{\omega}$ and $\breve{\omega}$ by

$$\widehat{\omega}_t = \frac{\omega_t - \widetilde{\omega}_t}{\sqrt{2}}, \quad \breve{\omega}_t = \frac{\omega_t + \widetilde{\omega}_t}{\sqrt{2}}.$$

Observe that $U(\omega_s) \cap U(\widetilde{\omega}_s) \neq \emptyset$ if and only if $\widehat{\omega}_s \in \sqrt{2}U(0)$ and hence that

$$|\Phi_s(\omega)\Phi_s(\widetilde{\omega})||U(\omega_s) \cap U(\widetilde{\omega}_s)| \leq \left(c_1 + c_1|\breve{\omega}_s|^{2p} + c_1 s^p\right) 1_{\sqrt{2}U(0)}(\widehat{\omega}_s),$$

for some $c_1 = c_1(p) \in (0, \infty)$. Therefore,

$$\begin{aligned} &P^{\otimes 2}\left[\Phi_s(\omega)\Phi_s(\widetilde{\omega})|U(\omega_s) \cap U(\widetilde{\omega}_s)| \exp\left(\lambda^2 |V_s(\omega) \cap V_s(\widetilde{\omega})|\right)\right] \\ &\quad \leq c_2(1 + s^p)P^{\otimes 2}\left[1_{\sqrt{2}U(0)}(\widehat{\omega}_s) \exp\left(\lambda^2 \int_0^s 1_{\sqrt{2}U(0)}(\widehat{\omega}_u) du\right)\right] \\ &\quad = c_2(1 + s^p)P\left[1_{\sqrt{2}U(0)}(\omega_s) \exp\left(2\lambda^2 \int_0^s \chi_{0,u} du\right)\right] \\ &\quad \leq c_3(1 + s^p)s^{-d/2}. \end{aligned} \tag{3.23}$$

where we have used Lemma 3.1.4 on the last line. Plugging this into (3.22), we get the desired estimate. □

It is now easy to complete the proof of Proposition 3.1.2. Part (b) has already been proven by (3.23). To show part (a), we set $M_t^* = \max_{0 \leq s \leq t} |M_s|$. For (3.20), it is sufficient to prove that for any $\delta > 0$,

$$M_t^* = \mathcal{O}(t^\delta \sqrt{b_t}) \quad \text{as } t \nearrow \infty, \ Q\text{-a.s.,} \tag{3.24}$$

where b_t is the L^2-bound in Lemma 3.1.5. Moreover, by the monotonicity of M_t^* and the polynomial growth of $t^\delta\sqrt{b_t}$, it is enough to prove (3.24) along a subsequence $t = n^k$, $n = 1, 2, \ldots$ for some power $k \geq 2$. Now, take $k > 1/\delta$. We then have by Chebychev's inequality, Doob's inequality and Lemma 3.1.5 that

$$\begin{aligned} Q\{M_{n^k}^* > n^{k\delta}\sqrt{b_{n^k}}\} &\leq Q\{M_{n^k}^* > n\sqrt{b_{n^k}}\} \\ &\leq Q[(M_{n^k}^*)^2]/(n^2 b_{n^k}) \\ &\leq 4Q[M_{n^k}^2]/(n^2 b_{n^k}) \\ &\leq Cn^{-2}. \end{aligned}$$

Then, it follows from the Borel-Cantelli lemma that

$$Q\{M_{n^k}^* \leq n^{k\delta}\sqrt{b_{n^k}} \text{ for large enough } n\text{'s}\} = 1.$$

This ends the proof of (3.6).

The second statement (3.7) in Proposition 3.1.2 follows from Lemma 3.1.5 and the martingale convergence theorem. This completes the proof of Proposition 3.1.2. □

3.2 Proof of Theorem 2.2.1(b)

We will prove (2.5) in the following form.

Proposition 3.2.1 *Let* $d = 1$.

(a) *Suppose that a number* $0 < \xi < 1$ *and a sequence* $t_n \nearrow \infty$ *satisfy* $\varliminf_{t\nearrow\infty}(t_n/t_{n+1}) > 0$ *and*

$$\lim_{n\nearrow\infty} Q\mu_{t_n}\{|\omega_{\delta t_n}| \geq (\delta t_n)^\xi\} = 0 \quad \textit{for all } 0 < \delta < 1. \tag{3.25}$$

Then, the variance of the free energy diverges at least with the power $1 - \xi$*:*

$$\varliminf_{t\nearrow\infty} t^{-(1-\xi)}\mathrm{Var}_Q(\ln Z_t) > 0. \tag{3.26}$$

(b) *The power divergence estimate (3.26) holds for* $\xi > 3/4$.

Remark 3.2.1 Proposition 3.2.1(a) may be interpreted as $2\chi(1) \geq 1 - \xi(1)$.

We use the following large deviation result for the transversal fluctuation of the Brownian polymer shown in [CY03], where more complete statement and the proof can be found.

Theorem 3.2.2 *Let* t_n *be a positive sequence tending to infinity as* $n \to \infty$, *let* $\chi \geq 0$ *and* $\xi \geq 1/2$ *be such that*

$$\chi < 2\xi - 1 \tag{3.27}$$

and that

$$\sum_{n\geq 1} Q\left(|\ln Z_{t_n} - Q[\ln Z_{t_n}]| > t_n^\chi\right) < \infty\,. \tag{3.28}$$

Then,

(a) *For any* $\varepsilon > 0$,

$$\lim_{n\nearrow\infty} -t_n^{-(2\xi-1)} \ln \mu_{t_n}\left\{|\omega_{t_n}| \geq \varepsilon t_n^{\xi}\right\} = \varepsilon^2/2, \quad Q\text{-a.s.} \tag{3.29}$$

(b) *Assume that* $\lim_{n\nearrow\infty}(t_n^{\chi} \wedge t_n^{2\chi-1})/\ln n = \infty$. *Then, for* $d \geq 1$ *and* $\beta \in \mathbb{R}$, *(3.28) holds true with any* $\chi > 1/2$ *and hence (3.29) holds for all* $\xi > 3/4$.

Remark 3.2.2 Assumptions (3.27) and (3.28) can roughly be interpreted as $\chi(d) < 2\xi - 1$. In this interpretation, Theorem 3.2.2(a) implies (2.10). On the other hand Theorem 3.2.2(b) implies (2.11). Although we have to formulate $\xi(d)$ and $\chi(d)$ appropriately for these relations to be rigorous, at a heuristic level, relations (2.10) and $2\chi(1) \geq 1 - \xi(1)$, cf. Remark 3.2.1, lead to $\chi(1) \geq 1/5$ and then, $\xi(1) \geq 3/5$ by (2.7).

Proof of Proposition 3.2.1: **(a):** We recall that the variance in question has the following upper and lower bounds [CSY03]:

$$\lambda(|\beta|)^{-2}\mathrm{Var}_Q(\ln Z_t) \leq Q\int_{[0,t]\times\mathbb{R}^d} dsdx\, \left(Q^{\mathcal{G}_s}\mu_t[\chi_{s,x}]\right)^2 \leq \lambda(-|\beta|)^{-2}\mathrm{Var}_Q(\ln Z_t). \tag{3.30}$$

We then see from the lower bound and Jensen's inequality that

$$\lambda(-|\beta|)^{-2}\mathrm{Var}_Q(\ln Z_t) \geq Q\int_{[0,t]\times\mathbb{R}^d} dsdx\, \left(Q^{\mathcal{G}_s}\mu_t[\chi_{s,x}]\right)^2 \geq v_t.$$

where

$$v_t = \int_{[0,t]\times\mathbb{R}^d} dsdx\, \left(Q\mu_t[\chi_{s,x}]\right)^2.$$

Therefore, it is enough to prove (3.26) with $\mathrm{Var}_Q(\ln Z_t)$ replaced by v_t. Moreover, it can be seen from (3.37) below that there exists $C = C(\beta) \in (0,\infty)$ such that

$$v_{t+s} \geq \exp(-Ch)v_t$$

for all $t > 0, h \geq 0$ and $0 \leq s \leq h$. Therefore, it is sufficient to prove that

$$\varliminf_{n\nearrow\infty} t_n^{-(1-\xi)}v_{t_n} > 0. \tag{3.31}$$

To do so, we set $\Lambda_s = \{x \in \mathbb{R}^d \ ; \ |x| \leq s^{\xi} + 1\}$ and observe that

$$|\Lambda_s \cap U(\omega_s)| = 1 - |U(\omega_s)\backslash\Lambda_s| \geq 1 - 1\{U(\omega_s) \not\subset \Lambda_s\}$$

and therefore that

$$\begin{aligned}(Q\mu_t[|\Lambda_s \cap U(\omega_s)|])^2 &\geq (1 - Q\mu_t\{U(\omega_s) \not\subset \Lambda_s\})^2 \\ &\geq 1 - 2Q\mu_t\{U(\omega_s) \not\subset \Lambda_s\} \\ &\geq 1 - 2F(t,s),\end{aligned} \tag{3.32}$$

where $F(t,s) = Q\mu_t\{|\omega_s| \geq s^\xi\}$. We then see from Jensen's inequality and (3.32) that

$$\begin{aligned} v_t &\geq \int_0^t ds \int_{\Lambda_s} dx \,(Q\mu_t[\chi_{s,x}])^2 \\ &\geq \int_0^t ds \frac{1}{|\Lambda_s|}(Q\mu_t[|\Lambda_s \cap U(\omega_s)|])^2 \\ &\geq \tfrac{1}{2}\int_0^t \frac{ds}{s^\xi + 1} - \int_0^t s^{-\xi}F(t,s)ds \end{aligned} \tag{3.33}$$

On the other hand, we have by (3.25) and the bounded convergence theorem that

$$\lim_{n\nearrow\infty} t_n^{-(1-\xi)} \int_0^{t_n} s^{-\xi}F(t_n,s)ds = \lim_{n\nearrow\infty}\int_0^1 s^{-\xi}F(t_n, st_n)ds = 0. \tag{3.34}$$

We now get (3.31) by (3.33) and (3.34).

(b): For $\xi > 3/4$, we can choose $1/2 < \chi < 2\xi - 1$ and a sequence $\{t_n\}_{n\geq 1}$ such that $\lim_{n\nearrow\infty}(t_n^\chi \wedge t_n^{2\chi-1})/\ln n = \infty$ and $\varliminf_{t\nearrow\infty}(t_n/t_{n+1}) > 0$. We then see from Theorem 3.2.2 (b) that

$$\lim_{n\nearrow\infty} \mu_{t_n}\{|\omega_{\delta t_n}| \geq (\delta t_n)^\xi\} = 0, \quad Q\text{-a.s.} \tag{3.35}$$

for all $0 < \delta < 1$. (Strictly speaking, only the case $\delta = 1$ is considered in there. However, an inspection of the proof reveals that (3.35) remains true for all $0 < \delta < 1$.) □

Lemma 3.2.3 *There exists $C = C(\beta) \in (0,\infty)$ such that for $t > 0$, $h \geq 0$ and $0 \leq s \leq h$*

$$\exp(-Ch) \leq Q[Z_{t+s}/Z_t|\mathcal{G}_t] \leq \exp(Ch), \quad Q\text{-a.s.} \tag{3.36}$$

In particular, for any $A \in \mathcal{F}$,

$$Q[\mu_{t+s}(A)|\mathcal{G}_t] \geq \exp(-Ch)\,\mu_t(A) \quad Q\text{-a.s.} \tag{3.37}$$

Proof: We set $\delta_t(h) = M_{t+h} - M_t + h$ where $(M_t)_{t\geq 0}$ is a martingale given by

$$M_t = \int \eta_t(dsdx)\mu_{s-}[\chi_{s,x}] - t.$$

It is not difficult to see that [CY03, Lemma 5.3.1] for $0 \leq s \leq h$,

$$\exp(\lambda(-|\beta|)\delta_t(h)) \leq \frac{Z_{t+s}}{Z_t} \leq \exp(\lambda(|\beta|)\delta_t(h)). \tag{3.38}$$

On the other hand, a standard exponential martingale argument gives

$$\exp(-c(\alpha)h) \leq Q[\exp(\alpha M_{t+s} - \alpha M_t)|\mathcal{G}_t] \leq \exp(c(\alpha)h), \tag{3.39}$$

where $c(\alpha) = \alpha^2 e^{|\alpha|}/2$. The desired bound (3.36) follows from (3.38) and (3.39).

Now, (3.37) can be seen as follows. Since $\zeta_{t+s} \geq \zeta_t$,

$$\begin{aligned} Q[\mu_{t+s}(A)|\mathcal{G}_t] &\geq \mu_t(A)Q[Z_t/Z_{t+s}|\mathcal{G}_t] \\ &\geq \mu_t(A)Q[Z_{t+s}/Z_t|\mathcal{G}_t]^{-1} \\ &\geq \exp(-C(\beta)h)\,\mu_t(A). \end{aligned}$$

□

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